Chiang, Y.
Osaka J. Math.
39 (2002), 723–752

ELEMENTARY INTERSECTION NUMBERS ON PUNCTURED SPHERES

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(Received October 6, 2000)

Introduction

According to Thurston, for any analytically finite Riemann surface $\mathcal{R}$, the set $\overline{\mathcal{G}}(\mathcal{R})$ of all projective geodesic laminations in $\mathcal{R}$ can be made into a topological space homeomorphic to a sphere of dimension depending on the topology of $\mathcal{R}$. Understanding the space $\overline{\mathcal{G}}(\mathcal{R})$ is important for various approaches to the Teichmüller space and the mapping class group of $\mathcal{R}$. The space $\overline{\mathcal{G}}(\mathcal{R})$ was then investigated by several authors from many different points of view. See [3–10], [12, 13, 15], and references there in.

In this paper, we consider the space $\overline{\mathcal{G}}_n = \overline{\mathcal{G}}(\Sigma_n)$ for any integer $n \geq 4$, where Σ_n is an n-punctured sphere endowed with a hyperbolic metric. Note that $\overline{\mathcal{G}}_n$ is homeomorphic to a sphere of dimension $2n - 7$.

This work was an attempt to generalize the projective coordinates defined in [3, 4] to an arbitrary $\overline{\mathcal{G}}_n$. This work and that of Keen, Parker and Series [10] are essentially based on cutting sequence technique developed by Birman and Series [2], and complement the works of Masur and Minsky [12, 13].

Let $\mathcal{G}_n$ be the set of all simple closed geodesics on Σ_n. For $n = 4$ or 5, the author has defined a set of projective coordinates for $\mathcal{G}_n$ so that the completion of these coordinates parametrize $\overline{\mathcal{G}}_n$, (see [3, 4]). The coordinates of each $\gamma \in \mathcal{G}_n$ are geometric intersection numbers of γ with $2(n-3)$ fixed geodesics in $\mathcal{G}_n$, and read off directly from the topology of γ. Moreover, these coordinates have three remarkable applications. First, the geometric intersection number of any two geodesics in $\mathcal{G}_n$ can be formulated explicitly in terms of the corresponding coordinates. Secondly, the coordinates of each $\gamma \in \mathcal{G}_n$ determine a canonical expression of γ as a word in a given set of generators for the fundamental group $\pi_1(\Sigma_n)$. Finally, the coordinates of each $\gamma \in \mathcal{G}_n$ are related to trace polynomials of the transformations corresponding to γ in a family of regular B-groups uniformizing Σ_n.

For an arbitrary $n \geq 5$, following [3, 4], we shall choose $n-3$ fixed triples $(\gamma_j^1, \gamma_j^2, \gamma_j^3)$ of geodesics in $\mathcal{G}_n$ $(1 \leq j \leq n-3)$, and compute the geometric intersec-

The work was partially supported by a grant from the National Science Council of the Republic of China.

tion numbers $i(\gamma, \gamma_j^k)$, called the *elementary intersection numbers* of γ. The elementary intersection numbers of γ will determine a set of parameters for γ.

The geodesics γ_j^k are defined explicitly in §1.2. They are chosen intuitively as described below. First, we line up the punctures of Σ_n, say $\zeta_1, \ldots, \zeta_n$. For every j, the geodesic γ_j^1 is chosen to separate $\zeta_1, \ldots, \zeta_{j+1}$ from $\zeta_{j+2}, \ldots, \zeta_n$. These geodesics γ_j^1 determine $n-3$ subsurfaces of Σ_n each of which is homeomorphic to a four punctured sphere. Two of them are isometric to spheres with three punctures and one hole, denoted by $\Sigma_4^{(1)}$ and $\Sigma_4^{(n-3)}$, and the others are isometric to spheres with two punctures and two holes, denoted by $\Sigma_4^{(j)}$, $2 \le j \le n-4$. More explicitly, $\Sigma_4^{(1)}$ is the subsurface containing ζ_1, ζ_2 and ζ_3 with the boundary geodesic γ_2^1; $\Sigma_4^{(n-3)}$ is the subsurface containing ζ_{n-2}, ζ_{n-1} and ζ_n with the boundary geodesic γ_{n-4}^1; $\Sigma_4^{(j)}$ is the subsurface bounded by γ_{j-1}^1 and γ_{j+1}^1 for $2 \le j \le n-4$. For every j, we choose γ_j^2 so that γ_j^1 and γ_j^2 form a marking of a four punctured sphere as γ_∞ and γ_0 given in [3]. The geodesic γ_j^3 plays the role of γ_1 given [3] which is obtained from γ_j^2 by a half-twist along γ_j^1.

The main work of this paper is to find formulas for computing elmentary intersection numbers so that the formulas agree with that given in [4] when $n = 5$. These formulas will be called *elementary intersection formulas*. To derive these formulas, we introduce $2(n-3)$ integers for each $\gamma \in \mathcal{G}_n$, denoted by $I_j(\gamma)$ and $N_j(\gamma)$ for $1 \le j \le n-3$, (see §2.1 and §2.4). These integers are defined analogously to the projective coordinates given in [3, 4]. For $\gamma \in \mathcal{G}_n$, every $I_j(\gamma)$ is defined to be $(1/2)i(\gamma, \gamma_j^1)$, and the sign of every $N_j(\gamma)$ is determined by the symmetry of a fundamental domain for a Fuchsian representation of $\pi_1(\Sigma_n)$ acting on the upper half plane. With these integer valued functions I_j and N_j, we prove in §2.5 the elementary intersection formulas (Theorem 2.10) by applying induction to the number n of punctures. In this paper, we develop a new idea that makes the induction work for $n \ge 5$, (cf. Remark 2.3).

As an application of elementary intersection formulas, at the end of the paper, we construct a continuous map Ψ from $\overline{\mathcal{G}}_n$ into a sphere $\Delta_n \subset \mathbb{R}^{3(n-3)}$ of dimension $2n-7$ whose restriction to $\mathcal{G}_n$ is written explicitly in terms of I_j and N_j.

It would be very interesting to derive a geometric intersection formula as given in [3, Theorem 2.6] and [4, Theorem 3.1] for any two geodesics in $\mathcal{G}_n$. With the formula, one proves easily the injectivity of Ψ. To prove that the integers $I_j(\gamma)$ and $N_j(\gamma)$ form a set of projective coordinates for $\gamma \in \mathcal{G}_n$, one also need the surjectivity of the map $\Psi\colon \overline{\mathcal{G}}_n \longrightarrow \Delta_n$. This will follow if $\Psi(\mathcal{G}_n)$ is dense in Δ_n. For the proof, one may consider π_1-train tracks introduced by Birman and Series [1], (cf. [3, 4]). The work will appear elsewhere.

1. Preliminaries

1.1. The space of complete simple geodesics. For any integer $n \ge 4$, a loop on Σ_n with no self intersections will be called a *simple loop*. An *essential simple loop* on Σ_n is a simple loop which is neither homotopically trivial nor homotopic to a simple closed curve around to a puncture of Σ_n. A finite union of mutually disjoint essential simple loops on Σ_n will be called a *multiple simple loop*. The set of all free homotopy classes of non-oriented essential simple loops on Σ_n is denoted by $\mathcal{G}_n$, while the set of all free homotopy classes of non-oriented multiple simple loops is denoted by $\mathcal{GL}_n$. Obviously, $\mathcal{G}_n \subset \mathcal{GL}_n$.

In general, we shall use $[\alpha]$ for the free homotopy class represented by a curve α lying on Σ_n. Every element of $\mathcal{G}_n$ contains a unique geodesic γ on Σ_n. By abuse of notation, we shall also use γ for the free homotopy class containing γ.

We shall write every element of $\mathcal{GL}_n$ as an integral combination of elements of $\mathcal{G}_n$. For every integer $m > 1$, we use $\mathcal{Z}_+^m$ for the set of m-tuples $(k_1, \ldots, k_m)$ of integers $k_j \geq 0$ with $\sum_{j=1}^m k_j > 0$, and Λ_n^m for the set of m-tuples $(\gamma_1, \ldots, \gamma_m)$ of mutually disjoint geodesics in $\mathcal{G}_n$.

Let α be an arbitrary multiple simple loop on Σ_n. All connected components of α fall into at most $n-3$ distinct free homotopy classes. There exist $(k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$ and $(\gamma_1, \ldots, \gamma_{n-3}) \in \Lambda_n^{n-3}$ such that, for every j, α has exactly k_j connected components freely homotopic to γ_j. We shall write:

$$[\alpha] = k_1\gamma_1 \oplus \cdots \oplus k_{n-3}\gamma_{n-3} = \bigoplus_{j=1}^{n-3} k_j\gamma_j\,.$$

Let $[\mathcal{G}_n, \mathbb{R}_+]$ be the set of all functions from $\mathcal{G}_n$ into the set $\mathbb{R}_+$ of all non-negative real numbers. We provide $\mathcal{G}_n$ with the discrete topology, and provide $[\mathcal{G}_n, \mathbb{R}_+]$ with the compact-open topology.

Two elements f and g of $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ are called *projectively equivalent* if there is a positive number t such that $f = tg$. Let $\mathrm{P}[\mathcal{G}_n, \mathbb{R}_+]$ be the set of all projective equivalence classes in $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ provided with the quotient topology. Let π_n be the quotient map of $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ onto $\mathrm{P}[\mathcal{G}_n, \mathbb{R}_+]$.

Following [5], we embed $\mathcal{GL}_n$ into $[\mathcal{G}_n, \mathbb{R}_+]$ by using geometric intersection numbers of elements of $\mathcal{GL}_n$. For any two curves α_1 and α_2 on Σ_n, let $\#(\alpha_1 \cap \alpha_2)$ denote the cardinality of the intersection $\alpha_1 \cap \alpha_2$. The *geometric intersection number* $i([\alpha_1], [\alpha_2])$ of $[\alpha_1]$ with $[\alpha_2]$ is defined by

$$i([\alpha_1], [\alpha_2]) = \min\{\#(\alpha_1' \cap \alpha_2') : [\alpha_j'] = [\alpha_j] \quad \text{for } j = 1,\ 2\}\,.$$

It follows immediately from the definition that, for any curve β on Σ_n,

$$i\left(\bigoplus_{j=1}^{n-3} k_j\gamma_j, [\beta]\right) = \sum_{j=1}^{n-3} k_j \cdot i(\gamma_j, [\beta])\,.$$

Every $\alpha \in \mathcal{GL}_n$ induces a function $\mathcal{I}_\alpha^{(n)} : \mathcal{G}_n \longrightarrow \mathbb{R}_+$ given by

$$\mathcal{I}_\alpha^{(n)}(\gamma) = i(\alpha\,,\,\gamma) \quad \text{for all} \quad \gamma \in \mathcal{G}_n .$$

Let $\mathcal{I}^{(n)} : \mathcal{GL}_n \longrightarrow [\mathcal{G}_n, \mathbb{R}_+]$ be defined by

$$\mathcal{I}^{(n)}(\alpha) = \mathcal{I}_\alpha^{(n)} \quad \text{for all} \quad \alpha \in \mathcal{GL}_n .$$

When there is no risk of confusion, we shall simply write π_n as π, write $\mathcal{I}_\alpha^{(n)}$ as $\mathcal{I}_\alpha$, and write $\mathcal{I}^{(n)}$ as $\mathcal{I}$.

It is well known that the composition $\pi\mathcal{I}$ is injective [5, Exposé 3], and that $\overline{\pi\mathcal{I}(\mathcal{GL}_n)} = \overline{\pi\mathcal{I}(\mathcal{G}_n)}$ [5, Exposé 4, Theorem 4], where $\overline{\pi\mathcal{I}(\mathcal{GL}_n)}$ and $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ denote the closures of $\pi\mathcal{I}(\mathcal{GL}_n)$ and $\pi\mathcal{I}(\mathcal{G}_n)$ in $\mathrm{P}[\mathcal{G}_n, \mathbb{R}_+]$, respectively. These results are original due to Thurston [15].

Note that an element $\mathcal{L}$ of $\mathrm{P}[\mathcal{G}_n, \mathbb{R}_+]$ is in $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ if and only if for any l in $[\mathcal{G}_n, \mathbb{R}_+] - \{0\}$ with $\pi(l) = \mathcal{L}$ there exist a sequence $\{t_k\}_{k=1}^\infty$ of positive numbers and a sequence $\{\gamma_k\}_{k=1}^\infty$ of geodesics in $\mathcal{G}_n$ such that the sequence $\{t_k \mathcal{I}_{\gamma_k}\}_{k=1}^\infty$ converges to l. A sequence $\{l_k\}_{k=1}^\infty$ in $[\mathcal{G}_n, \mathbb{R}_+]$ is called *convergent* to $l \in [\mathcal{G}_n, \mathbb{R}_+]$ if for every $\gamma \in \mathcal{G}_n$ the sequence $\{l_k(\gamma)\}_{k=1}^\infty$ converges in $\mathbb{R}$ to $l(\gamma)$.

1.2. Cyclic reduced words. It is well known that every free homotopy class in $\mathcal{G}_n$ corresponds to a unique conjugacy class in the fundamental group of Σ_n. Now, we consider a Fuchsian representation G_n of the fundamental group of Σ_n acting on the upper half plane $\mathcal{U} = \{z \in \mathbf{C} : \operatorname{Im} z > 0\}$, and find a representative for each conjugacy class in G_n by using Birman and Series' cutting sequence techenique [2].

Let G_n be the subgroup of $PSL(2, \mathbb{R})$ generated by the following transformations

$$S_1 = \begin{pmatrix} 1 & 2(n-2) \\ 0 & 1 \end{pmatrix}, \; S_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \; T_{n-j-2} = \begin{pmatrix} 2j+1 & 2j(j+1) \\ 2 & 2j+1 \end{pmatrix},$$

where $1 \le j \le n-3$ are integers.

For every integer j with $1 \le j \le n-3$, let C_j be the isometric circle of T_j, and C'_j be the isometric circle of T_j^{-1}. Let C_{n-2} be the isometric circle of S_2, and C'_{n-2} be the isometric circle of S_2^{-1}. Let

$$C'_0 = \{z \in \mathbf{C} : \operatorname{Re} z = -(n-2)\} \quad \text{and} \quad C_0 = \{z \in \mathbf{C} : \operatorname{Re} z = n-2\}\,.$$

Note that $S_1(C'_0) = C_0$, and that the polygon $\mathcal{D}_n \subset \mathcal{U}$ bounded by C_j and C'_j, $0 \le j \le n-2$, is a fundamental domain for G_n acting on $\mathcal{U}$.

For simplicity, we shall schematically draw $\mathcal{D}_n$ as a rectangular region. See Fig. 1 for $n = 4, 5, 6$, where the points on the boundary of $\mathcal{D}_n$ marked by "$\times$" correspond to punctures of Σ_n.

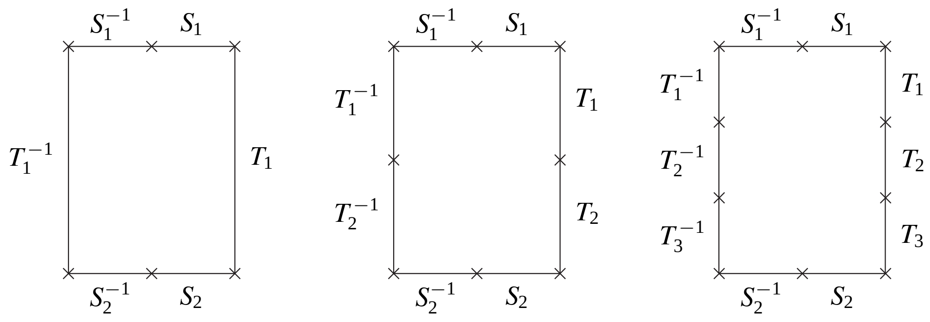

Fig. 1. The fundamental domain $\mathcal{D}_n$ for $n = 4, 5, 6$.

Let Γ_n denote the set of all side pairings of $\mathcal{D}_n$, i.e.,

$$\Gamma_n = \{S_1, S_1^{-1}, S_2, S_2^{-1}, T_j, T_j^{-1} : j = 1, \dots, n-3\}\,.$$

For every $X \in \Gamma_n$, we label the common side s of $\mathcal{D}_n$ and $X(\mathcal{D}_n)$ by X^{-1} on the side inside $\mathcal{D}_n$ and by X on the side inside $X(\mathcal{D}_n)$, (cf. Fig. 1). This side s will be called the *X-side* of $\mathcal{D}_n$.

For every $g \in G_n$, the image $g(\mathcal{D}_n)$ will be called a *G_n-translate* of $\mathcal{D}_n$. We transport the above side labelling to all G_n-translates of $\mathcal{D}_n$.

For any closed curve γ in Σ_n, let $\tilde{\gamma}$ be a lift of γ to $\mathcal{U}$ which starts on a side of a G_n-translate of $\mathcal{D}_n$ and projects to γ bijectively, except the endpoints of $\tilde{\gamma}$. Let $z_0 \in \mathcal{U}$ be an endpoint of $\tilde{\gamma}$, and we orient $\tilde{\gamma}$ so that its initial point is z_0. The arc $\tilde{\gamma}$ cuts in order the G_n-translates $g_0(\mathcal{D}_n)$, $g_1(\mathcal{D}_n), \dots, g_k(\mathcal{D}_n)$ of $\mathcal{D}_n$. For every integer j with $1 \le j \le k$, let $X_j \in \Gamma_n$ be the label of the common side of $g_{j-1}\mathcal{D}_n)$ and $g_j(\mathcal{D}_n)$, interior to $g_j(\mathcal{D}_n)$. Then $X_j = g_{j-1}^{-1} \circ g_j$ for every j, and γ is represented by

$$(g_0^{-1} \circ g_1) \circ (g_1^{-1} \circ g_2) \circ \cdots \circ (g_{k-1}^{-1} \circ g_k) = X_1 \circ X_2 \circ \cdots \circ X_k = \prod_{j=1}^{k} X_j\,.$$

Such an expression is called a *Γ_n-word* representing γ. See [4, §1.2] for a full discussion.

A Γ_n-word $\prod_{j=1}^k X_j$ will be called *reduced* if $X_j \neq X_{j+1}^{-1}$ for $1 \le j \le k-1$. It is called *cyclically reduced* if in addition $X_1 \neq X_k^{-1}$.

Let γ be a simple loop on Σ_n represented by a Γ_n-word given above. For every $1 \le j \le k$, let l_j be the image of the intersection of $\tilde{\gamma}$ with $g_j(\overline{\mathcal{D}}_n)$ mapped by g_j^{-1}, where $\overline{\mathcal{D}}_n$ is the relative closure of $\mathcal{D}_n$ in $\mathcal{U}$. Note that each l_j is a simple arc in $\overline{\mathcal{D}}_n$ connecting the X_j^{-1}-side to the X_{j+1}-side, where $X_{k+1} = X_1$. Each l_j will be called a *strand* of γ.

Let α be a multiple simple loop on Σ_n. A strand of a connected component of α will be also called a *strand* of α.

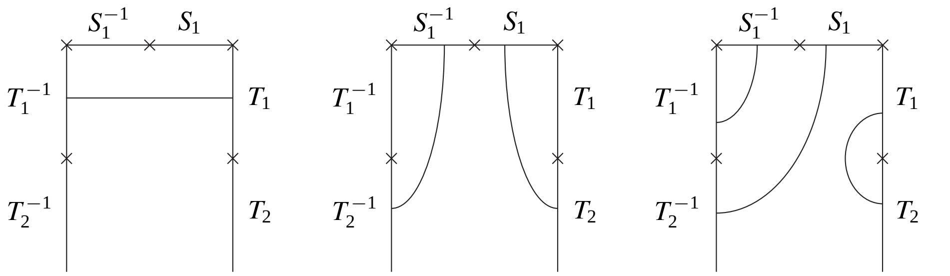

Fig. 2. From the left to the right : $\gamma_1^1,\ \gamma_1^2,\ \gamma_1^3$.

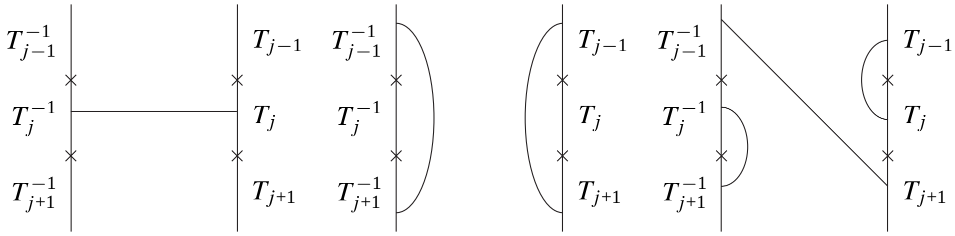

Fig. 3. From the left to the right : $\gamma_j^1,\ \gamma_j^2,\ \gamma_j^3,\ 2 \le j \le n-4$.

A loop on Σ_n is called *reduced* if it is represented by a cyclically reduced Γ_n-word. A multiple simple loop α on Σ_n is called *reduced* if every connected component of α is reduced. Note that a simple loop or a multiple simple loop on Σ_n is reduced if and only if every strand of the loop connects two different sides of $\mathcal{D}_n$. It is easy to see that every simple closed geodesic on Σ_n is a reduced loop. Thus every free homotopy class of multiple simple loops on Σ_n contains a reduced one.

If $\gamma \in \mathcal{G}_n$ is a geodesic represented by a reduced Γ_n-word W, then γ is also represented by an arbitrary cyclic permutation of W. If γ' is a geodesic which has the same underlying set with γ but opposite orientation, then γ' is represented by W^{-1}. Because we are only interested in non-oriented simple loops, we shall identify all reduced Γ_n-words which are cyclic permutations of W or cyclic permutations of W^{-1}, and call any one of them a *cyclic reduced* Γ_n*-word* representing γ. Every cyclic reduced Γ_n-word is cyclically reduced.

As examples, let $\gamma_j^k \in \mathcal{G}_n$ be the geodesics given in Fig. 2, Fig. 3 and Fig. 4, where j and k are integers with $1 \le j \le n-3$ and $1 \le k \le 3$. See introduction for a geometric interpretation of γ_j^k. Each γ_j^k is represented by a cyclic reduced Γ_n-word W_j^k as given below:

(i) $W_1^1 = T_1,\ W_1^2 = S_1 T_2^{-1},\ W_1^3 = S_1 T_1^{-1} T_2$;

(ii) $W_j^1 = T_j,\ W_j^2 = T_{j-1} T_{j+1}^{-1},\ W_j^3 = T_{j+1} T_j^{-1} T_{j-1}$ for $2 \le j \le n-4$;

(iii) $W_{n-3}^1 = T_{n-3},\ W_{n-3}^2 = S_2 T_{n-4}^{-1}$ and $W_{n-3}^3 = S_2 T_{n-3}^{-1} T_{n-4}$.

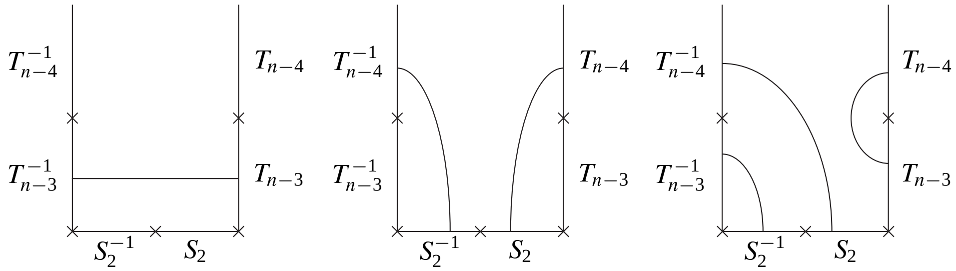

Fig. 4. From the left to the right : γ_{n-3}^1, γ_{n-3}^2, γ_{n-3}^3.

1.3. Subwords and admissible subarcs. Let $\hat{\mathcal{G}}_n = \mathcal{G}_n - \{\gamma_j^1 : 1 \leq j \leq n-3\}$. Let $\gamma \in \hat{\mathcal{G}}_n$ be a geodesic represented by a cyclic reduced Γ_n-word $W = \prod_{j=1}^k X_j$. Note that $k > 1$. For any two integers $1 \leq j \leq k$ and $1 \leq l \leq k$, the reduced Γ_n-word

$$W' = X_j \cdots X_{j+l-1} \tag{1}$$

will be called a *subword* of W, where $X_{j+i} = X_{j+i-k}$ whenever $1 \leq i \leq l$ and $i+j > k$.

We shall relate W' to γ geometrically. For every i, let l_i be the strand of γ connecting the X_{i-1}^{-1}-side to the X_i-side, where $X_{i-1} = X_k$ if $i = 1$. Assume that $1 \leq l < k$, i.e., $W' \neq W$. We think that W' "represents" a subarc γ' of γ. We choose γ' to be the projection to Σ_n of the union $\cup_{i=j}^{j+l-1} l_i$. Each of the arcs $l_1, \ldots, l_{j+l-1}$ is called a strand of γ'.

This arc γ' has two distinct endpoints. One of the two endpoints is the projection of the endpoint of l_j on the X_{j-1}^{-1}-side, and the other one is the projection of the endpoint of l_{j+l-1} on the X_{j+l-1}-side. The word given in (1) is not clear enough to indicate the endpoint on the X_{j-1}^{-1}-side. To distinguish it from cyclic reduced words representing simple closed geodesics, we write the reduced Γ_n-word representing γ' as

$$\vec{X}_{j-1} W' = \vec{X}_{j-1} X_j \cdots X_{j+l-1}\,, \tag{2}$$

where $\vec{X}_{j-1}$ is to indicate that $\vec{X}_{j-1} W'$ is not cyclic, and to indicate that one of the endpoints of γ' is the projection of a point on the X_{j-1}^{-1}-side.

A subarc of a geodesic $\gamma \in \mathcal{G}_n$ will be called *admissible* if either it is γ itself, or is represented by a reduced Γ_n-word as given in (2).

REMARK 1.1. For $\varepsilon = \pm 1$, $X \in \Gamma_n$, X_1, $X_2 \in \Gamma_n - \{X^{\pm 1}\}$, and an integer $k > 1$, we shall write

$$X_1 \underbrace{X^{\varepsilon} \cdots X^{\varepsilon}}_{k\,\text{times}} X_2 = X_1 X^{k\varepsilon} X_2\,.$$

Let $\gamma \in \hat{\mathcal{G}}_n$ be a geodesic represented by a cyclic reduced Γ_n-word $W(\gamma)$. By the same reasoning as that in [3, §3], there are no admissible subarcs of $\gamma \in \mathcal{G}_n$ represented by any one of the following words:

$$\vec{S}_1^{\varepsilon} S_1^{\varepsilon}\,, \qquad \vec{S}_2^{\varepsilon} S_2^{\varepsilon}\,, \qquad \vec{T}_1^{\delta} S_1^{\varepsilon} T_1^{\delta}\,, \qquad \vec{T}_{n-3}^{\delta} S_2^{\varepsilon} T_{n-3}^{\delta}\,,$$
$$\vec{S}_1^{\varepsilon} T_1^{k} S_1^{\delta}\,, \qquad \vec{S}_2^{\varepsilon} T_{n-3}^{k} S_2^{\delta}\,, \qquad \vec{T}_j^{\varepsilon} T_{j+1}^{\delta} T_j^{\varepsilon}\,, \qquad \vec{T}_{j+1}^{\varepsilon} T_j^{\delta} T_{j+1}^{\varepsilon}\,,$$

where $\varepsilon,\ \delta \in \{1, -1\}$, $k \neq 0$ is an integer, and $1 \leq j \leq n-4$. Therefore, none of the words $S_1^{\varepsilon} S_1^{\varepsilon}$, $S_2^{\varepsilon} S_2^{\varepsilon}$, $T_1^{\delta} S_1^{\varepsilon} T_1^{\delta}$, $T_{n-3}^{\delta} S_2^{\varepsilon} T_{n-3}^{\delta}$, $S_1^{\varepsilon} T_1^{k} S_1^{\delta}$, $S_2^{\varepsilon} T_{n-3}^{k} S_2^{\delta}$, $T_j^{\varepsilon} T_{j+1}^{\delta} T_j^{\varepsilon}$ and $T_{j+1}^{\varepsilon} T_j^{\delta} T_{j+1}^{\varepsilon}$ is a subword of $W(\gamma)$.

1.4. The free homotopy relative to $\partial\mathcal{D}_n$. To be able to relate the geometric intersection number of two geodesics in $\mathcal{G}_n$ to the intersection of their admissible subarcs, we shall define the *free homotopy relative to* $\partial\mathcal{D}_n$ on a family of curves on Σ_n which contains all admissible subarcs of geodesics in $\mathcal{G}_n$.

The union of a finite number of mutually disjoint simple curves on Σ_n will be called a *multiple simple curve*. Let $\mathcal{A}$ be the family of all multiple simple curves β on Σ_n satisfying the following three properties.

(i) β lifts to a finite number of mutually disjoint simple arcs in $\mathcal{D}_n$, called the *strands* of β.

(ii) Except the endpoints, each strand of β lies in the interior of $\mathcal{D}_n$.

(iii) Each strand of β connects two different sides of $\mathcal{D}_n$.

Note that $\mathcal{A}$ contains all reduced multiple simple loops on Σ_n, and contains all admissible subarcs of geodesics in $\mathcal{G}_n$.

Two multiple simple curves β_1 and β_2 in $\mathcal{A}$ will be called *freely homotopic relative to* $\partial\mathcal{D}_n$, written by $\beta_1 \sim \beta_2$ (rel. $\partial\mathcal{D}_n$), if for any two distinct $X,\ X' \in \Gamma_n$

$$\begin{aligned} &\#(\text{strands of } \beta_1 \text{ connecting the } X\text{-side and the } X'\text{-side}) \\ =\ &\#(\text{strands of } \beta_2 \text{ connecting the } X\text{-side and the } X'\text{-side}). \end{aligned}$$

Note that two reduced multiple simple loops on Σ_n are freely homotopic if and only if they are freely homotopic relative to $\partial\mathcal{D}_n$. For $\beta \in \mathcal{A}$, let

$$[\beta]_{\partial\mathcal{D}_n} = \{\beta' \in \mathcal{A} \,:\, \beta' \sim \beta\,(\text{rel. } \partial\mathcal{D}_n)\}\,,$$

and we shall call a strand of β a *strand* of $[\beta]_{\partial\mathcal{D}_n}$.

Now, we may define the *strands* of a free homotopy class $\alpha \in \mathcal{GL}_n$ as follows. Write $\alpha = \bigoplus_{j=1}^{m} k_j \gamma_j$, where $(\gamma_1, \ldots, \gamma_m) \in \Lambda_n^m$, and $m,\ k_1, \ldots, k_m$ are positive integers with $m \leq n-3$. A strand of some γ_j is called a *strand* of α. Similarly, an admissible subarc of some γ_j is called an *admissible subarc* of α.

For $\beta_1,\ \beta_2 \in \mathcal{A}$, we define

$$i([\beta_1]_{\partial\mathcal{D}_n}, [\beta_2]_{\partial\mathcal{D}_n}) = \min\{\#(\beta_1' \cap \beta_2') : \beta_1' \sim \beta_1 \ \text{ and } \ \beta_2' \sim \beta_2 \ (\text{rel. } \partial\mathcal{D}_n)\}\,.$$

where $\#(\beta_1' \cap \beta_2')$ denotes the cardinality of the intersection $\beta_1' \cap \beta_2'$. For simplicity, from now on we shall write

$$i([\beta_1]_{\partial\mathcal{D}_n}, [\beta_2]_{\partial\mathcal{D}_n}) = i([\beta_1], [\beta_2])_{\partial\mathcal{D}_n}$$

for $\beta_1, \beta_2 \in \mathcal{A}$. Note that if β_1 and β_2 are reduced multiple simple loops, then $i([\beta_1], [\beta_2])_{\partial\mathcal{D}_n} = i([\beta_1], [\beta_2])$.

1.5. Four automorphisms of $\mathcal{GL}_n$. We have set up a very symmetric fundamental domain $\mathcal{D}_n$ for G_n. As we did in [4], by use of the symmetry of $\mathcal{D}_n$, we may cut down our discussion to fewer cases by introducing four automorphisms Θ_1, Θ_2, $\mathcal{T}_1$ and $\mathcal{T}_2$ of G_n defined by

$$\begin{aligned}
\Theta_1(X) &= X^{-1} \text{ for } X \in \{S_1, S_2, T_j : 1 \le j \le n-3\};\\
\Theta_2(S_1) &= S_2,\ \Theta_2(S_2) = S_1, \text{ and } \Theta_2(T_j) = T_{n-j-2} \text{ for } 1 \le j \le n-3.\\
\mathcal{T}_1(S_1) &= S_1^{-1}T_1 \text{ and } \mathcal{T}_1(X) = X \text{ for } X \in \{S_2, T_j : 1 \le j \le n-3\};\\
\mathcal{T}_2(S_2) &= S_2^{-1}T_{n-3} \text{ and } \mathcal{T}_2(X) = X \text{ for } X \in \{S_1, T_j : 1 \le j \le n-3\}.
\end{aligned}$$

It follows from Nielsen's isomorphism theorem ([14] or [11, Theorem V.H.1]) that for $j = 1$ or 2, each of the automorphisms Θ_j and $\mathcal{T}_j$ induces a homeomorphism of Σ_n onto itself, still denoted by Θ_j and $\mathcal{T}_j$.

Let φ be any one of the four homeomorphisms Θ_1, Θ_2, $\mathcal{T}_1$ and $\mathcal{T}_2$. The action of φ on $\mathcal{GL}_n$ is defined as follows. For every geodesic $\gamma \in \mathcal{G}_n$, let $\varphi(\gamma)$ denote the free homotopy class containing the homeomorphic image of γ under φ. As before, let $\varphi(\gamma)$ also denote the geodesic in the free homotopy class $\varphi(\gamma)$. Thus φ extends naturally to $\mathcal{GL}_n$ such that

$$\varphi\left(\bigoplus_{j=1}^{n-3} k_j\gamma_j\right) = \bigoplus_{j=1}^{n-3} k_j\varphi\left(\gamma_j\right),$$

where $(k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$ and $(\gamma_1, \ldots, \gamma_{n-3}) \in \Lambda_n^{n-3}$.

Note that if $\gamma \in \mathcal{G}_n$ is represented by a cyclic reduced Γ_n-word W, then $\varphi(\gamma)$ is represented by $\varphi(W)$.

2. Elementary Intersection Numbers

In this section, we generalize elementary intersection numbers of elements of $\mathcal{GL}_5$ [4, §2.1] to elements of $\mathcal{GL}_n$, and prove the elementary intersection formulas.

2.1. The integer valued functions I_j. Let $\gamma_j^k \in \mathcal{G}_n$ be the geodesics given in §1.2. For any $\alpha \in \mathcal{GL}_n$, the geometric intersection numbers $i(\alpha, \gamma_j^k)$ are called the *elementary intersection numbers* of α.

Note that if β_1 and β_2 are two simple closed curves on a 2-sphere, and if they intersect transversally at every point of intersection, then $\#(\beta_1 \cap \beta_2)$ is an even integer. Thus $i(\alpha_1, \alpha_2)$ is an even integer for any two $\alpha_1, \alpha_2 \in \mathcal{GL}_n$. We shall write

$$I_j(\alpha) = \frac{i(\alpha, \gamma_j^1)}{2}$$

for $\alpha \in \mathcal{GL}_n$, and for $1 \le j \le n-3$. Note that if $\gamma \in \mathcal{G}_n$ is represented by a cyclic reduced Γ_n-word $W(\gamma) = W$, then

$$\begin{aligned} I_1(\gamma) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_1\text{-side}) \\ &= \text{the total number of the letters } S_1 \text{ and } S_1^{-1} \text{ appearing in } W; \\ I_{n-3}(\gamma) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_2\text{-side}) \\ &= \text{the total number of the letters } S_2 \text{ and } S_2^{-1} \text{ appearing in } W. \end{aligned}$$

Thus for $\alpha \in \mathcal{GL}_n$ we have

$$\begin{aligned} I_1(\alpha) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_1\text{-side}); \\ I_{n-3}(\alpha) &= \#(\text{strands of } \alpha \text{ with an endpoint on the } S_2\text{-side}). \end{aligned}$$

Since $\Theta_1(\gamma_j^1) = \gamma_j^1$ and $\Theta_2(\gamma_j^1) = \gamma_{n-j-2}^1$, the following proposition is an immediate consequence of the definition.

Proposition 2.1. *If $\alpha \in \mathcal{GL}_n$, then*

$$I_j(\alpha) = I_j(\Theta_1(\alpha)) \quad \textit{and} \quad I_j(\alpha) = I_{n-j-2}(\Theta_2(\alpha)) \quad \textit{for } 1 \le j \le n-3.$$

By an argument similar to the one in the proof of [4, Proposition 2.2 (i), (ii)], we obtain:

Proposition 2.2. *Let $\alpha \in \mathcal{GL}_n$. For any integer m, $\mathcal{T}_1^m(\alpha) = \alpha$ when $I_1(\alpha) = 0$, while $\mathcal{T}_2^m(\alpha) = \alpha$ if $I_{n-3}(\alpha) = 0$.*

Proposition 2.3. *If $\alpha \in \mathcal{GL}_n$, and if m is an integer, then*

$$I_1(\alpha) = I_1(\mathcal{T}_j^m(\alpha)) \quad \textit{and} \quad I_{n-3}(\alpha) = I_{n-3}(\mathcal{T}_j^m(\alpha)) \quad \textit{for } j = 1, 2.$$

Proof. Since γ_1^1 and γ_{n-3}^1 are invariant under each $\mathcal{T}_j$, the proof is straightforward. □

Proposition 2.4. *If $\alpha \in \mathcal{GL}_n$, and if j, k and m are integers with $1 \le k \le 3$, then*

$$i(\alpha, \gamma_j^k) = i(\mathcal{T}_1^m(\alpha), \gamma_j^k) \quad \textit{for} \quad 1 < j \le n-3, \textit{ and}$$

$$i(\alpha, \gamma_j^k) = i(\mathcal{T}_2^m(\alpha), \gamma_j^k) \quad \textit{for} \quad 1 \leq j < n-3.$$

Proof. By Proposition 2.2, we have $\mathcal{T}_1(\gamma_j^k) = \gamma_j^k$ for $1 < j \leq n-3$, and $\mathcal{T}_2(\gamma_j^k) = \gamma_j^k$ for $1 \leq j < n-3$. The proof is complete. □

2.2. Cyclic semi-reduced words. To compute elementary intersection numbers, we associate to geodesics in $\mathcal{G}_n$ *cyclic semi-reduced* Γ_n*-words*, which are defined analogously to those in [4, §2.2].

Let $\gamma \in \mathcal{G}_n$ with $I_{n-3}(\gamma) > 0$. Assume that γ is represented by a cyclic reduced Γ_n-word $W(\gamma)$. If $S_2^\varepsilon X$ or XS_2^ε is a subword of $W(\gamma)$ with $\varepsilon = \pm 1$ and $X \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$, we shall write

$$S_2^\varepsilon X = S_2^\varepsilon T_{n-3}^0 X \quad \text{and} \quad XS_2^\varepsilon = XT_{n-3}^0 S_2^\varepsilon .$$

Similarly, for a geodesic $\gamma \in \mathcal{G}_n$ with $I_1(\gamma) > 0$, if $X \in \Gamma_n - \{S_1^{\pm 1}, T_1^{\pm 1}\}$, and if $S_1^\varepsilon X$ or XS_1^ε is a subword of $W(\gamma)$, then we write $S_1^\varepsilon X = S_1^\varepsilon T_1^0 X$ and $XS_1^\varepsilon = XT_1^0 S_1^\varepsilon$. The resulting cyclic Γ_n-word is called a *semi-reduced*, still denoted by $W(\gamma)$.

As in [4, §2.5], we shall write cyclic semi-reduced Γ_n-words in two canonical forms. First, we subdivide $\mathcal{GL}_n$ into four classes.

Note that every geodesic in $\mathcal{G}_n$ can not simultaneously have a strand joining the S_2^ε-side to the T_{n-3}-side and a strand joining the S_2^ε-side to the T_{n-3}^{-1}-side for $\varepsilon = 1$ or -1, (see Remark 1.1).

Let $\mathcal{GL}_n^+(T_{n-3})$ be the set of elements of $\mathcal{GL}_n$ which have no strands connecting the T_{n-3}-side to the S_2^ε-side for $\varepsilon = \pm 1$. Let

$$\mathcal{GL}_n^-(T_{n-3}) = \Theta_1\big(\mathcal{GL}_n^+(T_{n-3})\big) ,$$

and let

$$\mathcal{GL}_n^+(T_1) = \Theta_2\big(\mathcal{GL}_n^+(T_{n-3})\big) \quad \text{and} \quad \mathcal{GL}_n^-(T_1) = \Theta_2\big(\mathcal{GL}_n^-(T_{n-3})\big) .$$

Consequently, $\mathcal{GL}_n^-(T_1) = \Theta_1(\mathcal{GL}_n^+(T_1))$. We remark that $\alpha \in \mathcal{GL}_n^+(T_1)$ if and only if α has no strands connecting the T_1-side to the S_1^ε-side for $\varepsilon = \pm 1$. The set $\mathcal{G}_n$ is then subdivided into four subclasses as:

$$\mathcal{G}_n^+(T_1) = \mathcal{G}_n \cap \mathcal{GL}_n^+(T_1) \quad \text{and} \quad \mathcal{G}_n^-(T_1) = \Theta_1\big(\mathcal{G}_n^+(T_1)\big) .$$
$$\mathcal{G}_n^+(T_{n-3}) = \mathcal{G}_n \cap \mathcal{GL}_n^+(T_{n-3}) \quad \text{and} \quad \mathcal{G}_n^-(T_{n-3}) = \Theta_1\big(\mathcal{G}_n^+(T_{n-3})\big) .$$

Now, by the same reasoning as in [4, §2.5], every $\gamma \in \mathcal{G}_n$ with $I_1(\gamma) > 0$ or $I_{n-3}(\gamma) > 0$ is represented by a cyclic semi-reduced Γ_n-word W as given below.

First, assume that $I_{n-3}(\gamma) = m > 0$. There exist m triples $(\varepsilon_j, p_j, q_j)$ of integers with $\varepsilon_j = \pm 1$, $p_j \geq 0$ and $q_j \geq 0$, and there exist m reduced Γ_n-words $W_j = \prod_{i=1}^{\nu_j} X_{ji}$

with $X_{j1},\ X_{j\nu_j} \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$, and $X_{ji} \in \Gamma_n - \{S_2^{\pm 1}\}$ when $1 < i < \nu_j$ such that

$$\gamma \in \mathcal{G}_n^-(T_{n-3}) \Longrightarrow W = \prod_{j=1}^m T_{n-3}^{-p_j} S_2^{\varepsilon_j} T_{n-3}^{q_j} W_j\,; \tag{3}$$

$$\gamma \in \mathcal{G}_n^+(T_{n-3}) \Longrightarrow W = \prod_{j=1}^m T_{n-3}^{p_j} S_2^{\varepsilon_j} T_{n-3}^{-q_j} W_j\,. \tag{4}$$

If $I_1(\gamma) = m > 0$, by considering $\Theta_2(\gamma)$, then γ is represented by

$$W = \prod_{j=1}^m T_1^{-p_j} S_1^{\varepsilon_j} T_1^{q_j} W_j\,,$$

where $(\varepsilon_j, p_j, q_j)$ are integers with $\varepsilon_j = \pm 1$ and $p_j q_j \geq 0$, and where $W_j = \prod_{i=1}^{\nu_j} X_{ji}$ are reduced Γ_n-words with $X_{j1},\ X_{j\nu_j} \in \Gamma_n - \{S_1^{\pm 1}, T_1^{\pm 1}\}$, and $X_{ji} \in \Gamma_n - \{S_1^{\pm 1}\}$ when $1 < i < \nu_j$. Moreover, $\gamma \in \mathcal{G}_n^-(T_1)$ if and only if $p_j \geq 0$ and $q_j \geq 0$ for all j, while $\gamma \in \mathcal{G}_n^+(T_1)$ if and only if $p_j \leq 0$ and $q_j \leq 0$ for all j.

We remark that any word given above is reduced if each $p_j q_j > 0$.

2.3. Essential blocks and puncture-like blocks. We shall compute elementary intersection numbers by applying mathematical induction to the number n of punctures. From now on, we assume that $n \geq 5$.

To be able to apply mathematical induction to n, we first embed $\mathcal{GL}_{n-1}$ into $\mathcal{GL}_n$. Let $\Phi_n : G_{n-1} \longrightarrow G_n$ be the monomorphism defined by

$$\Phi_n(S_1) = S_1\,, \quad \Phi_n(S_2) = T_{n-3} \quad \text{and} \quad \Phi_n(T_j) = T_j \quad \text{for } 1 \leq j \leq n-4\,.$$

The monomorphism Φ_n induces an injective map of $\mathcal{G}_{n-1}$ into $\mathcal{G}_n$, also denoted by Φ_n. If $\gamma \in \mathcal{G}_{n-1}$ is represented by a cyclic reduced (or semi-reduced) Γ_{n-1}-word W, then $\Phi_n(\gamma)$ is represented by $\Phi_n(W)$.

Let $\mathcal{G}_n^{(n-1)}$ be the image of $\mathcal{G}_{n-1}$ mapped by Φ_n, and let $\mathcal{GL}_n^{(n-1)}$ be the set of all elements of $\mathcal{GL}_n$ of the form

$$\bigoplus_{j=1}^{n-4} k_j \gamma_j\,,$$

where $(k_1, \ldots, k_{n-4}) \in \mathcal{Z}_+^{n-4}$, and $\gamma_j \in \mathcal{G}_n^{(n-1)}$ are mutually disjoint geodesics. Note that if $\gamma \in \mathcal{GL}_n^{(n-1)}$, then $2I_{n-3}(\gamma) = i(\gamma, \gamma_{n-3}^1) = 0$.

The geodesic γ_{n-3}^1 divides Σ_n into two connected components. One of them is a sphere with $n-2$ punctures and one hole, denoted by $\Sigma_n^{(n-1)}$, and the other one is a sphere with two punctures and one hole, denoted by $\Sigma_n^{(3)}$. Note that the punctures of $\Sigma_n^{(3)}$ correspond to the fixed points of the transformations S_2 and $S_2 T_{n-3}^{-1}$. Also note that $\Sigma_n^{(n-1)}$ is homeomorphic to Σ_{n-1}, and $\Sigma_n^{(3)}$ is homeomorphic to a 3-punctured sphere. It follows from the definition that every $\gamma \in \mathcal{GL}_n^{(n-1)}$ contains a representative lying on $\Sigma_n^{(n-1)}$. Thus we can do an induction after we relate free homotopy classes in $\mathcal{GL}_n$ to free homotopy classes in $\mathcal{GL}_n^{(n-1)}$.

To relate free homotopy classes in $\mathcal{GL}_n$ to that in $\mathcal{GL}_n^{(n-1)}$, we consider the set $\mathcal{GL}_n^0$ of all free homotopy classes in $\mathcal{GL}_n$ which have no strands connecting the S_2^ε-side to the X-side, where $\varepsilon = \pm 1$, and where $X \in \Gamma_n - \{S_2^{\pm 1}, T_{n-3}^{\pm 1}\}$. Let $\mathcal{G}_n^0 = \mathcal{G}_n \cap \mathcal{GL}_n^0$.

It follows immediately from the definition that if $\gamma \in \mathcal{GL}_n$ with $I_{n-3}(\gamma) = 0$, then $\gamma \in \mathcal{GL}_n^0$. In particular, $\mathcal{GL}_n^{(n-1)} \subset \mathcal{GL}_n^0$.

Let $\gamma \in \mathcal{G}_n$ with $I_{n-3}(\gamma) = m > 0$. Then $\gamma \in \mathcal{G}_n^-(T_{n-3}) \cap \mathcal{G}_n^0$ if and only if it is represented by a cyclic reduced Γ_n-word as given in (3), while $\gamma \in \mathcal{G}_n^+(T_{n-3}) \cap \mathcal{G}_n^0$ if and only if it is represented by a cyclic reduced Γ_n-word as given in (4) with $p_j > 0$ and $q_j > 0$ for all j.

The admissible subarcs of every $\gamma \in \mathcal{GL}_n^0$ fall into two classes. One contains admissible subarcs of γ which are freely homotopic relative to $\partial\mathcal{D}_n$ to simple curves lying on $\Sigma_n^{(n-1)}$. The other class contains admissible subarcs of γ which are freely homotopic relative to $\partial\mathcal{D}_n$ to simple curves lying on $\Sigma_n^{(3)}$. We shall relate γ to free homotopy classes in $\mathcal{GL}_n^{(n-1)}$ by relating the admissible subarcs of γ in the first class to elements of $\mathcal{GL}_n^{(n-1)}$.

Any $\gamma \in \mathcal{GL}_n$ can be related to an element of $\mathcal{GL}_n^0$ as follows.

Proposition 2.5. *Let $\gamma \in \mathcal{GL}_n$.*
(i) *If $\gamma \in \mathcal{GL}_n^+(T_{n-3})$, then $\mathcal{T}_2^{-2}(\gamma) \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$.*
(ii) *If $\gamma \in \mathcal{GL}_n^-(T_{n-3})$, then $\mathcal{T}_2^{2}(\gamma) \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^-(T_{n-3})$.*

Proof. It suffices to prove (i) for $\gamma \in \mathcal{G}_n^+(T_{n-3})$. There is nothing to prove if $I_{n-3}(\gamma) = 0$. If $I_{n-3}(\gamma) = m > 0$, then γ is represented by the cyclic semi-reduced Γ_n-word given in (4). Now, the assertion follows since $\mathcal{T}_2^{-2}(W) = \prod_{j=1}^m T_{n-3}^{p_j+1} S_2^{\varepsilon_j} T_{n-3}^{-q_j-1} W_j$. □

With Proposition 2.5, we may restrict our attention to the subclass $\mathcal{GL}_n^0$ of $\mathcal{GL}_n$.

Before continuing our discussion, we choose once for all an orientation for the X-side of $\mathcal{D}_n$, where $X \in \{T_{n-3}, T_{n-3}^{-1}, T_{n-4}, T_{n-4}^{-1}\}$. Note that $T_{n-4}T_{n-3}^{-1}$ is parabolic since the trace of $T_{n-4}T_{n-3}^{-1}$ is -2. Let ζ be the fixed point of the transformation $T_{n-4}T_{n-3}^{-1}$. For $X = T_{n-3}$ or T_{n-4}, if P_1 and P_2 are two points lying on the X-side, and if P_1 lies between ζ and P_2, then we write $P_1 \prec P_2$. If Q_1 and Q_2 are two points lying on the X^{-1}-side, we write $Q_1 \prec Q_2$ whenever $X(Q_1) \prec X(Q_2)$.

Proposition 2.6. *Let $\gamma \in \mathcal{G}_n^+(T_{n-3})$ with $I_{n-3}(\gamma) = m > 0$, and let γ be represented by the cyclic reduced Γ_n-word given in* (4).

If γ has a strand joining the T_{n-3}-side to the T_{n-4}-side, and has a strand joining *the T_{n-3}-side to the T_{n-4}^{-1}-side, then $W_j = T_{n-4}$ or $W_j = T_{n-4}^{-1}$ for some j.*

Proof. Let $P_1 \prec \cdots \prec P_k$ be the points where the strands of γ meet the T_{n-4}-side. For every integer l with $1 \leq l \leq k$, let P'_l be the point on the T_{n-4}^{-1}-side identified with P_l by the transformation T_{n-4}. Let l be the strand of γ with an endpoint at P_1, and let l' be the strand of γ with an endpoint at P'_1.

By assumption, l must connect the T_{n-3}-side and the T_{n-4}-side, and l' must connect the T_{n-3}-side and the T_{n-4}^{-1}-side. The union $l \cup l'$ projects to an admissible subarc γ' of γ rerpresented by $\vec{T}_{n-3}^{-1} T_{n-4} T_{n-3}$ or $\vec{T}_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$.

Assume that γ' is represented by $\vec{T}_{n-3}^{-1} T_{n-4} T_{n-3}$. Let Q be the endpoint of l on the T_{n-3}-side. We orient γ' so that the projection of Q to Σ_n is the initial point of γ'. Since $\gamma \in \mathcal{G}_n^0$, then the subword $W' = T_{n-3}^{-1} T_{n-4} T_{n-3}$ of W must be followed by a subword of the form $T_{n-3}^{p} S_2^{\varepsilon} T_{n-3}^{-q}$ for some integers $\varepsilon = \pm 1$, $p \geq 0$ and $q > 0$.

On the other hand, consider the subarc γ'' of γ which has the same underlying set with γ' but with the opposite orientation. Then γ'' is represented by $\vec{T}_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$. Thus $T_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$ is a subword of W^{-1} and is followed by a subword of the form $T_{n-3}^{p'} S_2^{\varepsilon'} T_{n-3}^{-q'}$ for some integers $\varepsilon' = \pm 1$, $p' \geq 0$ and $q' > 0$. We conclude that

$$T_{n-3}^{q'} S_2^{-\varepsilon'} T_{n-3}^{-p'} \cdot W' \cdot T_{n-3}^{p} S_2^{\varepsilon} T_{n-3}^{-q} = T_{n-3}^{q'} S_2^{-\varepsilon'} T_{n-3}^{-p'-1} T_{n-4} T_{n-3}^{p+1} S_2^{\varepsilon} T_{n-3}^{-q}$$

is a subword of W. This proves that $W_j = T_{n-4}$ for some j.

Similarly, if γ' is represented by $\vec{T}_{n-3}^{-1} T_{n-4}^{-1} T_{n-3}$, then there is an integer j such that $W_j = T_{n-4}^{-1}$. □

Blocks of simple closed geodesics. Let γ be given in Proposition 2.6. For every integer j with $1 \leq j \leq m = I_{n-3}(\gamma)$, let γ_j be the admissible subarc of γ represented by $\vec{T}_{n-3}^{-1} W_j T_{n-3}$. Every γ_j will be called a *block* of γ.

Let $l_0^{(j)}$ be the strand of γ_j joining the T_{n-3}-side to the X_{j1}-side with P the endpoint on the T_{n-3}-side, and let $l_1^{(j)}$ be the strand of γ_j joining the $X_{j\nu_j}^{-1}$-side to the T_{n-3}-side with Q the endpoint on the $X_{j\nu_j}^{-1}$-side. Let P' be the point on the T_{n-3}^{-1}-side which is identified with P by the transformation T_{n-3}.

Now, we replace $l_1^{(j)}$ by a simple arc $\tilde{l}_1^{(j)}$ joining Q to P' so that $\tilde{l}_1^{(j)}$ is disjoint from all strands of γ_j except possibly $l_1^{(j)}$. Let $\mathcal{L}_j$ be the union of all strands of γ_j other than $l_1^{(j)}$. The union $\mathcal{L}_j \cup \tilde{l}_1^{(j)}$ projects to a simple closed curve $\tilde{\gamma}_j$ on Σ_n. See the proof of [4, Theorem 5.3].

If $W_j = T_{n-4}$ or T_{n-4}^{-1}, then $\tilde{\gamma}_j$ is a simple loop around the puncture corresponding to the fixed point of $T_{n-3} T_{n-4}^{-1}$. In this case, we shall call γ_j a *puncture-like block* of γ.

We call γ_j an *essential block* of γ if γ_j is not a puncture-like block. Thus γ_j is an essential block if and only if $\tilde{\gamma}_j \in \mathcal{G}_n^{(n-1)}$.

Next, let $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^-(T_{n-3})$ with $I_{n-3}(\gamma) > 0$. An admissible subarc γ' of γ is called a *puncture-like block* if $\Theta_1(\gamma')$ is a puncture-like block of $\Theta_1(\gamma)$, and is called an *essential block* if $\Theta_1(\gamma')$ is a essential block of $\Theta_1(\gamma)$. By Proposition 2.6, γ' is a puncture-like block of γ if and only if it is represented by $\vec{T}_{n-3}T_{n-4}^{\varepsilon}T_{n-3}^{-1}$ with $\varepsilon = \pm 1$.

Blocks of free homotopy classes in $\mathcal{GL}_n^0$. For $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$, there are positive integers $k_1, \ldots, k_m$, and mutually disjoint geodesics $\beta_1, \ldots, \beta_m$ in $\mathcal{G}_n^0$ such that

$$\gamma = \bigoplus_{i=1}^{m} k_i \beta_i ,$$

where m is a positive integer with $m \leq n-3$. An admissible subarc γ' of γ is called a *block* of γ if it is either a connected component of γ with $I_{n-3}(\gamma') = 0$, or is a block of some β_i. A block γ' of γ is called *puncture-like* if it is a puncture-like block of some β_i, and is called *essential* if it is not a puncture-like block. Note that if γ' is a connected component of γ with $I_{n-3}(\gamma') = 0$, then $\gamma' \in \mathcal{G}_n^{(n-1)}$. Such an essential block will be called of the *second kind*. An essential block of γ will be called of the *first kind* if it is not of the second kind.

REMARK 2.1. It follows from Proposition 2.6 that if $\gamma \in \mathcal{GL}_n^+(T_{n-3})$ has a strand joining the T_{n-3}-side to the T_{n-4}-side, and has a strand joining the T_{n-3}-side to the T_{n-4}^{-1}-side, then γ has a puncture-like block. Similarly, if $\gamma \in \mathcal{GL}_n^-(T_{n-3})$ has a strand joining the T_{n-3}^{-1}-side to the T_{n-4}-side, and has a strand joining the T_{n-3}^{-1}-side to the T_{n-4}^{-1}-side, then γ has a puncture-like block.

REMARK 2.2. Let $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$. If γ has no essential blocks, then $I_1(\gamma) = 0$ and $I_{n-3}(\Theta_2(\gamma)) = 0$. Note that $\Theta_2(\gamma) \in \mathcal{GL}_n^{(n-1)}$. Thus the elementary intersection numbers of γ will be obtained from that of $\Theta_2(\gamma)$ by applying induction to n. Therefore, we shall only consider the case where γ has essential blocks.

The following theorem plays an important role in the sequel.

Theorem 2.7. *Let $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$. If γ has essential blocks, then there is an $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ such that*

$$i(\alpha_\gamma, \gamma_{n-4}^1) = i(\gamma, \gamma_{n-4}^1) \quad \textit{and} \quad i(\alpha_\gamma, \gamma_j^k) = i(\gamma, \gamma_j^k)$$

for $1 \leq j < n-4$ and $1 \leq k \leq 3$. Furthermore, α_γ can be chosen so that $\Theta_1(\alpha_{\Theta_1(\gamma)}) = \alpha_\gamma$.

Since for all j, k,

$$i\left(\Theta_1(\alpha_{\Theta_1(\gamma)}),\ \gamma_j^k\right) = i\left(\alpha_{\Theta_1(\gamma)},\ \Theta_1(\gamma_j^k)\right) = i\left(\Theta_1(\gamma),\ \Theta_1(\gamma_j^k)\right) = i(\gamma,\ \gamma_j^k),$$

α_γ can be chosen so that $\Theta_1(\alpha_{\Theta_1(\gamma)}) = \alpha_\gamma$ since for all j, k. Thus, we may assume that $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$.

First, we prove Theorem 2.7 for γ which has no puncture-like blocks. Let $\mathcal{E}$ be the set of all essential blocks of γ, and for every $\gamma' \in \mathcal{E}$ let

$$t(\gamma') = \text{the number of strands of } \gamma' \text{ meeting the } T_{n-3}\text{-side.}$$

If γ has no essential blocks of the first kind, then any essential block of γ serves as α_γ.

Now, we assume that γ has exactly $e > 0$ essential blocks of the first kind, say $\gamma_1, \dots, \gamma_e$. Let $\tilde{\gamma}_j$ be the geodesic in $\mathcal{G}_n^{(n-1)}$ corresponding to γ_j (see the definition of blocks), and let t_j be the number of strands of $\tilde{\gamma}_j$ meeting the T_{n-3}-side. Note that $t(\gamma_j) = t_j + 1$, and the strands of γ_j meet the T_{n-3}^{-1}-side in exactly $t_j - 1$ points. Then the strands of $\cup_{j=1}^e \gamma_j$ meet the T_{n-3}^{-1}-side in exactly $t_0 = \sum_{j=1}^e (t_j - 1)$ points, and meet the T_{n-3}-side in exactly $t_0 + 2e$ points.

We consider the disjoint union $\mathcal{L}$ of strands of all essential blocks of γ. Let $Q_1 \prec Q_2 \prec \cdots \prec Q_q$ be the points where $\mathcal{L}$ meets the T_{n-3}-side, where q is an integer with $q \geq t_0 + 2e$.

CLAIM 1. For every integer j with $q-2e+1 \leq j \leq q$, the point Q_j is an endpoint of a strand L_j of $\cup_{j=1}^e \gamma_j$.

We shall show that Claim 1 implies Theorem 2.7 when γ has no puncture-like blocks. For every integer j with $q-e+1 \leq j \leq q$, let P_j be the endpoint of L_j other than Q_j, and let Q'_{j-e} be the point lying on the T_{n-3}^{-1}-side which is identified with Q_{j-e} by the transformation T_{n-3}. There are mutually disjoint simple arcs L'_j, $q-e+1 \leq j \leq q$, in $\mathcal{D}_n$ satisfying the following two properties:

(i) Each L'_j connects P_j to Q'_{j-e}.

(ii) Each L'_j is disjoint from the strands of any essential block of γ except possibly the strands $L_{q-e+1}, \dots, L_q$.

The set $\mathcal{L}' = (\mathcal{L} - \cup_{j=q-e+1}^q L_j) \cup (\cup_{j=q-e+1}^q L'_j)$ projects to a multiple simple loop α_γ in $\mathcal{GL}_n^{(n-1)}$, and the free homotopy class represented by α_γ, still denoted by α_γ, satisfies the required conditions since $i(\alpha_\gamma, \gamma_{n-4}^1) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n}$ and $i(\alpha_\gamma, \gamma_j^k) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_j^k)_{\partial \mathcal{D}_n}$ for $1 \leq j < n-4$ and $1 \leq k \leq 3$.

Proof of Claim 1. There is nothing to prove if γ has no essential blocks γ' of the second kind with $t(\gamma') > 0$. Assume that γ has exactly $p > 0$ essential blocks $\gamma_{e+1}, \dots, \gamma_{e+p}$ of the second kind with $t(\gamma_{e+j}) > 0$, $1 \leq j \leq p$.

For every j with $1 \leq j \leq e$, the block γ_j is represented by a reduced Γ_n-word $\vec{T}_{n-3}^{-1} W_j T_{n-3}$, where $W_j \neq T_{n-4}^{\pm 1}$ is of the form $W_j = \prod_{i=1}^{\nu_j} X_{ji}$ with $X_{j1}, X_{j\nu_j} \in \Gamma_n - \{T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$ and $X_{ji} \in \Gamma_n - \{S_2^{\pm 1}\}$ for $1 < i < \nu_j$. Let

$l_j^{(1)}$ be the strand of γ_j joining the T_{n-3}-side to the X_{j1}-side,

$l_j^{(2)}$ be the strand of γ_j joining the $X_{j\nu_j}^{-1}$-side to the T_{n-3}-side,

Q_{jk} be the endpoint of $l_j^{(k)}$ on the T_{n-3}-side for $k = 1, 2$, and

Q'_{jk} be the point on the T_{n-3}^{-1}-side identified with Q_{jk} by the transformation T_{n-3}.

By the definition of γ_j, the point Q'_{jk} is an endpoint of a strand $L_j^{(k)}$ of γ joining the T_{n-3}^{-1}-side to the X-side with $X \in \{T_{n-3}, S_2^{\pm 1}\}$.

Suppose that there is an integer m with $q-2e+1 \leq m \leq q$ such that Q_m is an endpoint of a strand of $\cup_{j=1}^{p} \gamma_{e+j}$. Then there is a Q_{jk} such that $Q_{jk} \prec Q_m$. Let Q'_m be the point on the T_{n-3}^{-1}-side identified with Q_m by the transformation T_{n-3}. It follows from the definition of γ_{e+j} that Q'_m is an endpoint of a strand L of γ joining the T_{n-3}^{-1}-side to some X-side with $X \in \Gamma_n - \ T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$. Since $Q_{jk} \prec Q_m$, then $Q'_{jk} \prec Q'_m$, and thus $L_j^{(k)}$ must intersect L transversally. This contradiction completes the proof of the claim. □

In the following, we prove Theorem 2.7 for γ which has puncture-like blocks. For this case, we need the following two lemmas.

Lemma 2.8. *If $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ with $I_{n-3}(\gamma) > 0$, and if γ has a puncture-like block, then every essential block of γ has no strands meeting the T_{n-3}^{-1}-side.*

Proof. Let γ_0 be a puncture-like block of γ. There is a strand l_0 of γ_0 connecting the T_{n-4}-side and the T_{n-3}-side. Let Q_0 be the endpoint of l_0 on the T_{n-3}-side. We may choose γ_0 so that $Q_0 \prec Q$ whenever Q is an endpoint of a strand of γ on the T_{n-3}-side. Let Q'_0 be the point on the T_{n-3}^{-1}-side which is identified with Q_0 by the transformation T_{n-3}. Note that Q'_0 is an endpoint of a strand L_0 of γ joining the T_{n-3}^{-1}-side to the X-side with $X \in \{T_{n-3}, S_2, S_2^{-1}\}$. Also note that if Q' is an endpoint of a strand of γ on the T_{n-3}^{-1}-side, then $Q'_0 \prec Q'$ by the definition of Q_0.

Now, suppose that there is an essential bolck γ' of γ such that γ' has a strand l' meeting the T_{n-3}^{-1}-side at a point Q'. Since $Q'_0 \prec Q'$, and since γ' has no strands joining the T_{n-3}^{-1}-side to the X-side with $X \in \{T_{n-3}, S_2, S_2^{-1}\}$, then l' must intersect L_0 transversally. This is a contradiction. □

Lemma 2.9. *Let $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$ with $I_{n-3}(\gamma) > 0$, and for an arbitrary block γ' of γ, let $t(\gamma')$ be the number of strands of γ' meeting the T_{n-3}-side. If γ has* *puncture-like blocks, then*

$$t(\gamma') = \begin{cases} 2 & \textit{if } \gamma' \textit{ is a puncture-like block,} \\ 2 & \textit{if } \gamma' \textit{ is an essential block of the first kind,} \\ 0 & \textit{if } \gamma' \textit{ is an essential block of the second kind.} \end{cases}$$

Proof. It follows immediately from the definition that $t(\gamma') = 2$ whenever γ' is a puncture-like block of γ.

Let γ' be an essential block of the second kind, i.e., γ' is a simple closed geodesic in $\mathcal{GL}_n^{(n-1)}$. If γ' has a strand meeting the T_{n-3}-side, then γ' must have a strand meeting the T_{n-3}^{-1}-side. This contradicts to Lemma 2.8. Therefore, $t(\gamma') = 0$.

If γ' is an essential block of the first kind, then γ' is represented by a reduced Γ_n-word $\vec{T}_{n-3}^{-1} W T_{n-3}$, where $W \neq T_{n-4}^{\pm 1}$ is of the form $W = \prod_{j=1}^{m} X_j$ with X_1, $X_m \in \Gamma_n - \{T_{n-3}^{\pm 1}, S_2^{\pm 1}\}$, and $X_j \in \Gamma_n - \{S_2^{\pm 1}\}$ for $1 < j < m$. There is a strand l_0 of γ' joining the T_{n-3}-side to the X_1-side, and there is another strand l_1 of γ' joining the X_m-side to the T_{n-3}-side. Thus $t(\gamma') \geq 2$.

Suppose that $t(\gamma') > 2$. There is a $k \in \{2, \ldots, m-1\}$ such that $X_k = T_{n-3}$ or $X_k = T_{n-3}^{-1}$. If $X_k = T_{n-3}$, then γ' has a strand joining the T_{n-3}^{-1}-side to the X_{k+1}-side. This is a contradiction to Lemma 2.8. If $X_k = T_{n-3}^{-1}$, then γ' has a strand joining the X_{k-1}-side to the T_{n-3}^{-1}-side. This is a contradiction to Lemma 2.8 again. Therefore, $t(\gamma') = 2$. □

Now, we complete the proof of Theorem 2.7 as follows. Let $\gamma_1, \ldots, \gamma_e$ be all the first kind essential blocks of γ, and assume that γ has exactly $p > 0$ puncture-like blocks, say $\gamma_{e+1}, \ldots, \gamma_{e+p}$. Note that $t(\gamma_j) = 2$ for all j by Lemma 2.9.

Let $Q_1 \prec \cdots \prec Q_k$ be the points where the strands of γ meet the T_{n-3}-side. Note that $k \geq 2p + 2e$. Since $\gamma \in \mathcal{GL}_n^0$, and since $t(\gamma') = 0$ whenever γ' is an essential block of γ of the second kind, then $Q_1, \ldots, Q_{2p+2e}$ are endpoints of strands of $\cup_{j=1}^{p+e} \gamma_j$, and, for $2p + 2e + 1 \leq j \leq k$, each Q_j is an endpoint of a strand of γ connecting the T_{n-3}-side and the T_{n-3}^{-1}-side whenever $2p + 2e + 1 \leq j \leq k$.

CLAIM 2. $Q_{p+1}, \ldots, Q_{p+2e}$ are the points where the strands of $\cup_{j=1}^{e} \gamma_j$ meet the T_{n-3}-side.

Now, for every integer j with $1 \leq j \leq e$, let L_j be the strand of $\cup_{j=1}^{e} \gamma_j$ with Q_{p+e+j} an endpoint, let P_j be the other endpoint of L_j. Let Q'_{p+j} be the point on the T_{n-3}^{-1}-side which is identified with Q_{p+j} by the transformation T_{n-3}.

There are e mutually disjoint simple arcs L'_j in $\mathcal{D}_n$ connecting P_j to Q'_{p+j} for every j such that every L'_j is disjoint from the strands of any essential block of γ except possibly the strands $L_1, \ldots, L_e$. As before, let $\mathcal{E}$ be the set of all essential blocks of $\mathcal{L}' = (\mathcal{L} - \cup_{j=1}^{e} L_j) \cup (\cup_{j=1}^{e} L'_j)$ projects to Σ_n a multiple simple loop α_γ

in $\mathcal{GL}_n^{(n-1)}$. Let α_γ also denote the corresponding free homotopy class. Note that if γ' is a puncture-like block of γ, then $i(\gamma', \gamma_{n-4}^1)_{\partial\mathcal{D}_n} = 0 = i(\gamma', \gamma_j^k)_{\partial\mathcal{D}_n}$ for $1 \le j < n-4$ and $1 \le k \le 3$. This completes the proof of Theorem 2.7.

Proof of Claim 2. It suffices to prove that if Q is the endpoint of a strand of $\cup_{j=1}^e \gamma_j$ lying on the T_{n-3}-side, then $Q_j \prec Q \prec Q_{p+2e+j}$ for all j with $1 \le j \le p$.

Let γ' be the essential block of γ of the first kind such that Q is one of the two points where the strands of γ' meet the T_{n-3}-side, and let L be the strand of γ' with Q as an endpoint.

If $Q \in \{Q_1, \ldots, Q_p\}$, then there is an integer m with $p < m \le 2p + 2e$ such that Q_m is the endpoint of a strand l of $\cup_{j=1}^p \gamma_{e+j}$ connecting the T_{n-3}-side to the T_{n-4}-side. Thus the other endpoint P of L must lie on the T_{n-4}-side with $P \prec P_m$, where P_m is the endpoint of l other than Q_m. Let P' and P'_m be the points lying on the T_{n-4}^{-1}-side which are identified with P and P_m respectively by the transformation T_{n-4}. Let L' be the strand of γ' with P' as an endpoint. Since $P' \prec P'_m$, then L' must connect the T_{n-4}-side to the T_{n-3}-side. This implies that γ' is a puncture-like block of γ, which is a contradiction. Therefore, $Q_j \prec Q$ for all j with $1 \le j \le p$.

By a similar argument, one proves that $Q \prec Q_{p+2e+j}$ for $1 \le j \le p$. □

2.4. The integer valued functions N_j. To formulate elementary intersection numbers, in addition to the integer valued functions I_j defined in §2.1, we shall need other $n-3$ integer valued functions N_j, $1 \le j \le n-3$. These functions N_j are analogues of the integer valued functions N_T and N_S defined in [4].

We shall define an integer valued function $N_j^{(n)}$ on $\mathcal{GL}_n$ for any given integer $j > 0$ with $j \le n-3$ so that

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}\big(\Phi_n^{-1}(\gamma)\big)$$

whenever $\gamma \in \mathcal{GL}_n^{(n-1)}$ and $j \le n-4$, where Φ_n is defined in §2.3. This means that $N_j^{(n-1)}$ can be regarded as the restriction of $N_j^{(n)}$ to $\mathcal{GL}_n^{(n-1)}$ whenever $1 \le j \le n-4$. Thus $N_j^{(n)}$ can be simply written as N_j. Furthermore, this allows us to define N_j inductively by using Theorem 2.7.

First, we define the functions $N_1^{(n)}$ and $N_{n-3}^{(n)}$. If $\gamma = \bigoplus_{j=1}^{n-3} k_j\gamma_j^1$ with $(k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$, we define

$$N_j^{(n)}(\gamma) = k_j = \#(\text{strands of } \gamma \text{ connecting the } T_j\text{-side and the } T_j^{-1}\text{-side}),$$

for $j = 1$ or $n-3$.

Now, we define $N_1^{(n)}(\gamma)$ and $N_{n-3}^{(n)}(\gamma)$ for $\gamma \in \widehat{\mathcal{GL}}_n$, where

$$\widehat{\mathcal{GL}}_n = \mathcal{GL}_n - \left\{ \bigoplus_{j=1}^{n-3} k_j\gamma_j^1 : (k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3} \right\}.$$

If $\gamma \in \mathcal{GL}_n^+(T_1)$, let

$$\begin{aligned} N_1^{(n)}(\gamma) &= \#(\text{strands of } \gamma \text{ joining the } T_1^{-1}\text{-side to the } S_1^{\varepsilon}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } T_1\text{-side to the } T_1^{-1}\text{-side}), \end{aligned}$$

where $\varepsilon = \pm 1$. If $\gamma \in \mathcal{GL}_n^+(T_{n-3})$, let

$$\begin{aligned} N_{n-3}^{(n)}(\gamma) &= \#(\text{strands of } \gamma \text{ joining the } T_{n-3}^{-1}\text{-side to the } S_2^{\varepsilon}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } T_{n-3}\text{-side to the } T_{n-3}^{-1}\text{-side}). \end{aligned}$$

For $j = 1$ or $n-3$, and for $\gamma \in \mathcal{GL}_n^-(T_j) \cap \widehat{\mathcal{GL}}_n$, let

$$N_j^{(n)}(\gamma) = -N_j^{(n)}\big(\Theta_1(\gamma)\big).$$

It is clear that the definition of $N_1^{(n)}$ is independent of n since $n \geq 5$. Thus $N_1^{(n)}$ will be simply written as N_1.

REMARK 2.3. For $n = 5$, let N_T and N_S be the integer valued functions defined in [4], and let N_1 and $N_2 = N_{n-3}^{(n)}$ be the integer valued functions defined above. Then for $\gamma \in \mathcal{GL}_5$ we have

$$N_1(\gamma) = N_T(\gamma) \quad \text{and} \quad N_2(\gamma) = -N_S(\gamma).$$

Note that the geodesic γ_{23} defined in [4] and the geodesic γ_2^3 defined in this article are imgaes of each other under Θ_1. Thus, the following equations are also valid for $\gamma \in \mathcal{GL}_5$ (see [4, Corollary 3.4]):

$$\begin{aligned} i(\gamma, \gamma_1^2) &= 2|N_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma) \\ i(\gamma, \gamma_1^3) &= 2|N_1(\gamma) - I_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma) \\ i(\gamma, \gamma_2^2) &= 2|N_2(\gamma)| + |I_1(\gamma) - I_2(\gamma)| + I_1(\gamma) - I_2(\gamma) \\ i(\gamma, \gamma_2^3) &= 2|N_2(\gamma) - I_2(\gamma)| + |I_1(\gamma) - I_2(\gamma)| + I_1(\gamma) - I_2(\gamma) \end{aligned}$$

In §2.5, we shall prove similar formulas for elementary intersection numbers of $\gamma \in \mathcal{GL}_n$ for an arbitrary integer $n \geq 5$.

For integers n and j with $1 < j \leq n-4$, the integer valued functions $N_j^{(n)}$ on $\mathcal{GL}_n$ are defined as follows. We first define $N_j^{(n)}(\gamma)$ for $\gamma \in \mathcal{GL}_n^0$.
(i) If $I_{n-3}(\gamma) = 0$, then there exist $(k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$ and mutually disjoint geodesics $\gamma_1, \ldots, \gamma_{n-4}$ in $\mathcal{GL}_n^{(n-1)}$ such that

$$\gamma = \bigoplus_{i=1}^{n-4} k_i \gamma_i \oplus k_{n-3} \gamma_{n-3}^1. \tag{5}$$

Let

$$\alpha_\gamma = \bigoplus_{i=1}^{n-4} k_i \gamma_i \,, \tag{6}$$

and we define

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}\big(\Phi_n^{-1}(\alpha_\gamma)\big)\,.$$

In particular, if $\gamma \in \mathcal{GL}_n^{(n-1)}$, then $k_{n-3} = 0$, $\alpha_\gamma = \gamma$, and

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}\big(\Phi_n^{-1}(\gamma)\big)\,.$$

(ii) If $I_{n-3}(\gamma) > 0$, and if γ has essential blocks, we define

$$N_j^{(n)}(\gamma) = N_j^{(n-1)}\big(\Phi_n^{-1}(\alpha_\gamma)\big)\,,$$

where $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ is given in Theorem 2.7.
(iii) If $I_{n-3}(\gamma) > 0$, and if γ has no essential blocks, we define

$$N_j^{(n)}(\gamma) = 0\,.$$

From (i), we know that $N_j^{(n-1)}$ is the restriction of $N_j^{(n)}$ to $\mathcal{GL}_n^{(n-1)} \equiv \mathcal{GL}_{n-1}$ for any two integers j and n with $1 < j \le n-4$. Note that $N_{n-4}^{(n-1)} = N_{\nu-3}^{(\nu)}$, where $\nu = n-1$. From now on, we shall write $N_j^{(n)}$ as N_j for $1 \le j \le n-3$.

Now, for an arbitrary $\gamma \in \mathcal{GL}_n$ and for an arbitrary integer j with $1 < j \le n-4$, we define

$$N_j(\gamma) = \begin{cases} N_j\big(\mathcal{T}_2^{-2}(\gamma)\big) & \text{if } \gamma \in \mathcal{GL}_n^{+}(T_{n-3}),\\ N_j\big(\mathcal{T}_2^{2}(\gamma)\big) & \text{if } \gamma \in \mathcal{GL}_n^{-}(T_{n-3}).\end{cases}$$

To prove that N_j is well-defined, we have to show that

$$N_j(\gamma) = \begin{cases} N_j\big(\mathcal{T}_2^{-2}(\gamma)\big) & \text{for all } \gamma \in \mathcal{GL}_n^{+}(T_{n-3}) \cap \mathcal{GL}_n^{0},\\ N_j\big(\mathcal{T}_2^{2}(\gamma)\big) & \text{for all } \gamma \in \mathcal{GL}_n^{-}(T_{n-3}) \cap \mathcal{GL}_n^{0}.\end{cases}$$

Without loss of generality, we may assume that $\gamma \in \mathcal{G}_n^0$. There is nothing to prove if $I_{n-3}(\gamma) = 0$ since in this case $\mathcal{T}_2(\gamma) = \gamma$. Assume that $\gamma \in \mathcal{G}_n^{-}(T_{n-3})$ with $I_{n-3}(\gamma) = m > 0$. Then γ is represented by a cyclic reduced Γ_n-word as given in (3), say $W = \prod_{i=1}^m T_{n-3}^{-p_i} S_2^{\varepsilon_i} T_{n-3}^{q_i} W_i$ with $p_i > 0$ and $q_i > 0$ for all i. Since

$$\mathcal{T}_2^2(W) = \prod_{i=1}^m T_{n-3}^{-p_i-1} S_2^{\varepsilon_i} T_{n-3}^{q_i+1} W_i\,,$$

γ has essential blocks if and only if $\mathcal{T}_2^2(\gamma) = \tilde{\gamma}$ has essential blocks. Thus $N_j(\gamma) = 0 = N_j(\tilde{\gamma})$ whenever γ has no essential blocks. When γ has essential blocks, α_γ is completely determined by the subwords $T_{n-3}W_iT_{n-3}^{-1}$, $1 \leq i \leq m$, and so is $\alpha_{\tilde{\gamma}}$. This proves that $N_j(\gamma) = N_j(\tilde{\gamma})$ since $\alpha_\gamma = \alpha_{\tilde{\gamma}}$.

If $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^+(T_{n-3})$, then γ is represented by a cyclic reduced Γ_n-word as given in (4). A similar argument as above, one proves easily that $N_j(\gamma) = N_j(\mathcal{T}_2^{-2}(\gamma))$. Therefore, N_j is well-defined.

Note that since $N_{n-4}^{(n)} \equiv N_{\nu-3}^{(\nu)}$ with $\nu = n-1$, from the definition of N_{n-3}, we may interpretate N_{n-4} geometrically. This gives N_j a geometric interpretation for every integer j with $1 < j \leq n-4$. From Proposition 2.5, we assume that $\gamma \in \mathcal{GL}_n^0$.

Let $\mathcal{GL}_n^+(T_{n-4})$ be the set of all γ in $\mathcal{GL}_n^0$ which satisfy either one of the following two conditions:

(i) If $I_{n-3}(\gamma) = 0$, then γ has no strands connecting the T_{n-4}-side to the T_{n-3}^{ε}-side, where $\varepsilon = \pm 1$.

(ii) If $I_{n-3}(\gamma) > 0$, then every essential block of γ has no strands connecting the T_{n-4}-side to the T_{n-3}^{ε}-side, where $\varepsilon = \pm 1$.

Let $\mathcal{GL}_n^-(T_{n-4}) = \Theta_1(\mathcal{GL}_n^+(T_{n-4}))$. If $\gamma = \bigoplus_{j=1}^{n-3} k_j\gamma_j^1$ with $(k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$, then

$$N_{n-4}^{(n)}(\gamma) = k_{n-4} = \#(\text{strands of } \gamma \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}).$$

Let $\varepsilon = \pm 1$. If $\gamma \in \mathcal{GL}_n^+(T_{n-4}) \cap \widehat{\mathcal{GL}}_n$ with $I_{n-3}(\gamma) = 0$, then

$$\begin{aligned} N_{n-4}^{(n)}(\gamma) &= \#(\text{strands of } \gamma \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma \text{ joining the } T_{n-3}^{\varepsilon}\text{-side to the } T_{n-4}^{-1}\text{-side}). \end{aligned}$$

If $\gamma \in \mathcal{GL}_n^+(T_{n-4}) \cap \widehat{\mathcal{GL}}_n$ with $I_{n-3}(\gamma) > 0$, then

$$N_{n-4}^{(n)}(\gamma) = \sum_{\gamma' \in \mathcal{E}} N_{n-4}^{(n)}(\gamma'), \tag{7}$$

where $\mathcal{E}$ is the set of all essential blocks of γ, and where

$$\begin{aligned} N_{n-4}^{(n)}(\gamma') &= \#(\text{strands of } \gamma' \text{ joining the } T_{n-4}\text{-side to the } T_{n-4}^{-1}\text{-side}) \\ &\quad + \#(\text{strands of } \gamma' \text{ joining the } T_{n-3}^{\varepsilon}\text{-side to the } T_{n-4}^{-1}\text{-side}) \end{aligned}$$

for $\gamma' \in \mathcal{E}$. When $\mathcal{E}$ is empty, the integer on the right of (7) is defined to be zero. If $\gamma \in \mathcal{GL}_n^-(T_{n-4}) \cap \widehat{\mathcal{GL}}_n$, then $N_{n-4}^{(n)}(\gamma) = -N_{n-4}^{(n)}(\Theta_1(\gamma))$.

2.5. Elementary intersection formulas. This subsection is devoted to proving the main theorem:

Theorem 2.10 (Elementary intersection formulas). *For an arbitrary integer $n \geq 6$, if $\gamma \in \mathcal{GL}_n$, then*

$$\begin{aligned} i(\gamma, \gamma_1^2) &= 2|N_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma)\,, \\ i(\gamma, \gamma_1^3) &= 2|N_1(\gamma) - I_1(\gamma)| + |I_2(\gamma) - I_1(\gamma)| + I_2(\gamma) - I_1(\gamma)\,, \\ i(\gamma, \gamma_{n-3}^2) &= 2|N_{n-3}(\gamma)| + |I_{n-4}(\gamma) - I_{n-3}(\gamma)| + I_{n-4}(\gamma) - I_{n-3}(\gamma)\,, \\ i(\gamma, \gamma_{n-3}^3) &= 2|N_{n-3}(\gamma) - I_{n-3}(\gamma)| + |I_{n-4}(\gamma) - I_{n-3}(\gamma)| + I_{n-4}(\gamma) - I_{n-3}(\gamma)\,, \end{aligned}$$

and for every integer j with $1 < j < n-3$

$$\begin{aligned} i(\gamma, \gamma_j^2) &= 2|N_j(\gamma)| + |I_{j-1}\gamma) - I_j(\gamma)| + I_{j-1}(\gamma) - I_j(\gamma) \\ &\quad + |I_{j+1}(\gamma) - I_j(\gamma)| + I_{j+1}(\gamma) - I_j(\gamma)\,, \\ i(\gamma, \gamma_j^3) &= 2|N_j(\gamma) - I_j(\gamma)| + |I_{j-1}(\gamma) - I_j(\gamma)| + I_{j-1}(\gamma) - I_j(\gamma) \\ &\quad + |I_{j+1}(\gamma) - I_j(\gamma)| + I_{j+1}(\gamma) - I_j(\gamma)\,. \end{aligned}$$

For the proof of Theorem 2.10, we need the following two immediate consequences of the definition of N_j.

Lemma 2.11. *If $\gamma \in \mathcal{GL}_n$, then $N_1(\gamma) = N_{n-3}(\Theta_2(\gamma))$.*

Lemma 2.12. *If $(k_1, \ldots, k_{n-3}) \in \mathcal{Z}_+^{n-3}$ and $(\gamma_1, \ldots, \gamma_{n-3}) \in \Lambda_n^{n-3}$, then*

$$N_j\left(\bigoplus_{i=1}^{n-3} k_i\gamma_i\right) = \sum_{j=i}^{n-3} k_i N_j(\gamma_j) \qquad \textit{for every integer } j \textit{ with } 1 \leq j \leq n-3.$$

For $k = 2$ or 3, the elementary intersection numbers $i(\gamma\,,\,\gamma_1^k)$ and $i(\gamma\,,\,\gamma_{n-3}^k)$ are related as follows:

$$i(\gamma, \gamma_{n-3}^k) = i\left(\Theta_2(\gamma), \Theta_2(\gamma_{n-3}^k)\right) = i\left(\Theta_2(\gamma), \gamma_1^k\right).$$

From Proposition 2.1, we obtain $I_1(\Theta_2(\gamma)) = I_{n-3}\gamma)$ and $I_2(\Theta_2(\gamma)) = I_{n-4}(\gamma)$. Now, by Lemma 2.11, the elementary intersection formulas for $i(\gamma\,,\,\gamma_{n-3}^2)$ and $i(\gamma\,,\,\gamma_{n-3}^3)$ follow immediately from those for $i(\gamma\,,\,\gamma_1^2)$ and $i(\gamma\,,\,\gamma_1^3)$.

On the other hand, $i(\gamma, \gamma_1^3) = i(\mathcal{T}_1(\gamma), \gamma_1^2)$ since $\gamma_1^3 = \mathcal{T}_1^{-1}(\gamma_1^2)$. Thus, by Proposition 2.3, one derives easily the elementary intersection formula for $i(\gamma, \gamma_1^3)$ from that for $i(\gamma, \gamma_1^2)$ if

$$N_1\left(\mathcal{T}_1(\gamma)\right) = N_1(\gamma) - I_1(\gamma)\,.$$

By use of the word given in (3), one proves easily the following more general results by a similar argument as that in [4, Proposition 2.8].

Lemma 2.13. *Let $\gamma \in \mathcal{GL}_n$, and let ν be an arbitrary integer. Then*

$$N_1\big(\mathcal{T}_2^{\nu}(\gamma)\big) = N_1(\gamma)\,, \qquad N_1\big(\mathcal{T}_1^{\nu}(\gamma)\big) = N_1(\gamma) - \nu I_1(\gamma)\,,$$
$$N_{n-3}\big(\mathcal{T}_1^{\nu}(\gamma)\big) = N_{n-3}(\gamma)\,, \qquad N_{n-3}\big(\mathcal{T}_2^{\nu}(\gamma)\big) = N_{n-3}(\gamma) - \nu I_{n-3}(\gamma)\,.$$

For the proof of Theorem 2.10, it remains to prove the elementary intersection formulas for $i(\gamma,\gamma_1^2)$, $i(\gamma,\gamma_j^2)$ and $i(\gamma,\gamma_j^3)$ for $1 < j < n-3$.

First, we prove the elementary intersection formula for $i(\gamma,\gamma_1^2)$ by applying induction to n for $n \geq 5$. For the case of $n = 5$, the assertion is proved in [4, Corollary 3.4]. Assume that $n > 5$, and that the equation holds for $\gamma \in \mathcal{GL}_n^{(n-1)}$.

Now, let $\gamma \in \mathcal{GL}_n$. If $I_{n-3}(\gamma) = 0$, write γ as given in (5), and let $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ be given in (6). By the definition, $N_1(\gamma) = N_1(\alpha_\gamma)$. Since $i(\gamma_{n-3}^1\,, \beta) = 0$ for $\beta \in \mathcal{GL}_n^{(n-1)}$, then $I_j(\gamma) = I_j(\alpha_\gamma)$ for $j = 1,\,2$. The assertion follows for the case since $i(\gamma\,,\,\gamma_1^2) = i(\alpha_\gamma\,,\,\gamma_1^2)$.

Assume that $I_{n-3}(\gamma) > 0$. Since $i(\gamma\,,\,\gamma_1^2) = i(\Theta_1(\gamma)\,,\,\gamma_1^2)$, we may assume that $\gamma \in \mathcal{GL}_n^+(T_{n-3})$. Moreover, by considering $\mathcal{T}_2^{-2}(\gamma)$, from Proposition 2.4, Proposition 2.5 and Lemma 2.13 we may assume that $\gamma \in \mathcal{GL}_n^0 \cap \mathcal{GL}_n^+(T_{n-3})$.

If γ has no essential blocks, we have $I_1(\gamma) = I_2(\gamma) = 0 = i(\gamma,\gamma_1^2)$. By the definition of N_1, we have $N_1(\gamma) = 0$ since $I_1(\gamma) = 0$. Now, the intersection formula for $i(\gamma,\gamma_1^2)$ holds trivially in this case.

If γ has essential blocks, then $I_j(\alpha_\gamma) = I_j(\gamma)$ for $j = 1,\,2$, and $i(\alpha_\gamma,\gamma_1^2) = i(\gamma,\gamma_1^2)$, where α_γ is given in Theorem 2.7. Note that $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ and $N_1(\gamma) = N_1(\alpha_\gamma)$. The proof of the intersection formula for $i(\gamma,\gamma_1^2)$ is then completed by induction hypothesis.

In the rest of this subsection, we prove the intersection formulas for $i(\gamma,\gamma_j^2)$ and $i(\gamma,\gamma_j^3)$ with $1 < j < n-3$, by applying induction to $n \geq 6$. If $n = 6$, then the formulas are exactly the same as given below.

Lemma 2.14. *If $n \geq 6$, and if $\gamma \in \mathcal{GL}_n$, then*

$$\begin{aligned}
i(\gamma,\gamma_{n-4}^2) &= 2|N_{n-4}(\gamma)| + |I_{n-5}\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma)\\
&\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma)\\
i(\gamma,\gamma_{n-4}^3) &= 2|N_{n-4}(\gamma) - I_{n-4}(\gamma)|\\
&\quad + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma)\\
&\quad + |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma)
\end{aligned}$$

With Lemma 2.14, we first complete induction step as follows. Assume that $n > 6$. From Lemma 2.14, we may assume that $1 < j < n-4$. If $I_{n-3}(\gamma) = 0$, then we write γ and $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$, respectively, as in (5) and (6). Since $N_j(\alpha_\gamma) = N_j(\gamma)$ and $I_k(\alpha_\gamma) = I_k(\gamma)$ for $0 < j-1 \leq k \leq j < n-4$, the assertions hold for this case by induction hypothesis.

Assume that $I_{n-3}(\gamma) > 0$. If γ has no essential blocks, then $I_j(\gamma) = 0 = N_j(\gamma)$ for $1 < j < n-3$, and $i(\gamma, \gamma_j^k) = 0$ for $1 < j < n-4$ and for $k = 2, 3$. If γ has essential blocks, we may assume that $\gamma \in \mathcal{GL}_n^0$. Let $\alpha_\gamma \in \mathcal{GL}_n^{(n-1)}$ be given in Theorem 2.7. By the induction hypothesis again, the proof is complete.

For the proof of Lemma 2.14, we need:

Lemma 2.15. *If $\gamma \in \mathcal{GL}_n^0$ with $I_{n-3}(\gamma) > 0$, then γ has exactly*

$$\frac{|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma)}{2}$$

puncture-like blocks.

Proof. Without loss of generality, we assume that $\gamma \in \mathcal{GL}_n^+(T_{n-3})$. Let $\mathcal{E}$ denote the set of all essential blocks of γ. If γ' is a puncture-like block of γ, then $i(\gamma', \gamma_{n-4}^1) = 0$ and $2I_{n-4}(\gamma) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n}$.

Let $I_{n-3}(\gamma) = m$, and let $p \geq 0$ be the number of puncture-like blocks of γ. Then γ has exactly $e = m - p$ essential blocks of the first kind. If $p = 0$, then $2I_{n-4}(\gamma) \geq \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} \geq 2m = 2I_{n-3}(\gamma)$, and $|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 0 = 2p$.

Now, assume that $p > 0$. It follows from Lemma 2.9 that

$$i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = \begin{cases} 2 & \text{if } \gamma' \text{ is an essential block of } \gamma \text{ of the first kind,} \\ 0 & \text{if } \gamma' \text{ is an essential block of } \gamma \text{ of the second kind.} \end{cases}$$

If $p = m$, then γ has no essential blocks of the first kind, and

$$2I_{n-4}(\gamma) = \sum_{\gamma' \in \mathcal{E}} i(\gamma', \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = 0\,.$$

Thus $|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 2m = 2p$.

If $0 < p < m$, let $\gamma_1, \ldots, \gamma_e$ be the essential blocks of γ of the first kind. Then $2I_{n-4}(\gamma) = \sum_{j=1}^{e} i(\gamma_j, \gamma_{n-4}^1)_{\partial \mathcal{D}_n} = 2e = 2I_{n-3}(\gamma) - 2p$, and

$$2p = 2\{I_{n-3}(\gamma) - I_{n-4}(\gamma)\} = |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma)\,. \qquad \square$$

Proof of Lemma 2.14. It suffices to prove the lemma for $\gamma \in \mathcal{G}_n$. We shall prove the lemma for $\gamma \in \mathcal{G}_n^+(T_{n-3})$. By a similar argument, one proves the lemma for $\gamma \in \mathcal{G}_n^-(T_{n-3})$.

If $\gamma \in \mathcal{G}_n^+(T_{n-3})$, then $N_{n-4}(\mathcal{T}^{-2}(\gamma)) = N_{n-4}(\gamma)$ by the definition of N_{n-4}. Note that $i(\gamma, \gamma_{n-4}^k) = i(\mathcal{T}_2^{-2}(\gamma), \gamma_{n-4}^k)$ for $k = 2, 3$, and that $2I_j(\gamma) = 2I_j(\mathcal{T}_2^{-2}(\gamma))$ for $n-5 \leq j \leq n-3$. By Proposition 2.5, we may assume that $\gamma \in \mathcal{G}_n^0 \cap \mathcal{G}_n^+(T_{n-3})$.

If $I_{n-3}(\gamma) = 0$, then $\gamma \in \mathcal{G}_n^{(n-1)}$, and

$$|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) = 0\,.$$

By letting $\nu = n-1$, we have

$$\begin{aligned} i(\gamma\,,\,\gamma^2_{n-4}) &= 2|N_{\nu-3}(\gamma)| + |I_{\nu-4}(\gamma) - I_{\nu-3}(\gamma)| + I_{\nu-4}(\gamma) - I_{\nu-3}(\gamma) \\ &= 2|N_{n-4}(\gamma)| + |I_{n-5}(\gamma) - I_{n-4}(\gamma)| + I_{n-5}(\gamma) - I_{n-4}(\gamma) \\ &\quad +|I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) \end{aligned}$$

Similarly, we obtain the intersection formula for $i(\gamma\,,\,\gamma^3_{n-4})$.

If $I_{n-3}(\gamma) = m > 0$, then γ is represented by a cyclic reduced Γ_n-word W as given in (3). Note that $p_j > 0$ and $q_j > 0$ for $1 \le j \le m$. For every j, let γ_j be the block of γ represented by $\vec{T}^{-1}_{n-3} W_j T_{n-3}$, and let $\beta(\gamma_j)$ be the admissible subarc of γ represented by

$$\vec{T}_{n-3} T^{p_j-1}_{n-3} S_2^{\varepsilon_j} T^{-q_j}_{n-3} W_j T_{n-3}\,.$$

Note that every γ_j is a subarc of $\beta(\gamma_j)$, and that $i(\beta(\gamma_j)\,,\,\gamma^k_{n-4}) = 2$ for $k = 2$ or 3 whenever γ_j is puncture-like. Let $\mathcal{E}$ be the set of all essential blocks of γ. From Lemma 2.15, we have, for $k = 2$ or 3,

$$i(\gamma, \gamma^k_{n-4}) = |I_{n-3}(\gamma) - I_{n-4}(\gamma)| + I_{n-3}(\gamma) - I_{n-4}(\gamma) + \sum_{\gamma_j \in \mathcal{E}} i\left(\beta(\gamma_j), \gamma^k_{n-4}\right)_{\partial \mathcal{D}_n}.$$

If γ has no essential blocks, then the lemma holds trivially for γ since $I_{n-3}(\gamma) = I_{n-4}(\gamma) = N_{n-4}(\gamma) = 0$.

Now, assume that $\mathcal{E}$ is not empty. Note that every essential block of γ is of the first kind since $\gamma \in \mathcal{G}_n$. Let $\mathcal{L}$ be the union of all strands of γ which connect the T^{-1}_{n-3}-side to the X-side with $X \in \{T_{n-3}\,,\, S_2\,,\, S_2^{-1}\}$.

For $k = 2$ or 3, each γ^k_{n-4} has a unique strand l_k meeting the T^{-1}_{n-3}-side. Let Q'_k be the endpoint of l_k lying on the T^{-1}_{n-3}-side, and let Q_k be the point on the T_{n-3}-side which is identified with Q'_k by the transformation T^{-1}_{n-3}.

Since $i(\gamma^k_{n-4}\,,\,\gamma^1_{n-3}) = 0$, we may assume that l_k is disjoint from $\mathcal{L}$. This implies that $Q'_k \prec Q'$ whenever Q' is an endpoint of some strand in $\mathcal{L}$ meeting the T^{-1}_{n-3}-side, and that $Q_k \prec Q$ whenever Q is the endpoint of some strand of γ lying on the T_{n-3}-side. Thus, we have

$$\sum_{\gamma_j \in \mathcal{E}} i\left(\beta(\gamma_j)\,,\,\gamma^k_{n-4}\right)_{\partial\mathcal{D}_n} = \sum_{\gamma_j \in \mathcal{E}} i(\gamma_j\,,\,\gamma^k_{n-4})_{\partial\mathcal{D}_n} = i(\alpha_\gamma\,,\,\gamma^k_{n-4})\,,$$

where $\alpha_\gamma \in \mathcal{GL}^{(n-1)}_n$ is given in Theorem 2.7. By letting $\nu = n-1$, we obtain

$$\begin{aligned} i(\alpha_\gamma\,,\,\gamma^2_{n-4}) &= 2|N_{\nu-3}(\alpha_\gamma)| + |I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| \\ &\quad + I_{\nu-4}\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma) \\ i(\alpha_\gamma\,,\,\gamma^3_{n-4}) &= 2|N_{\nu-3}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| + |I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma)| \\ &\quad + I_{\nu-4}(\alpha_\gamma) - I_{\nu-3}(\alpha_\gamma) \end{aligned}$$

The proof of Lemma 2.14 is complete. □

3. A Mapping of $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ into a Sphere

In this section, we construct a continuous mapping Ψ from $\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ into $\mathbb{R}^{3(n-3)}$ whose image set is a sphere of dimension $2n-7$. The mapping Ψ will be constructed in a similar way as that given in [4] for the case of $n = 5$. We shall first define the restriction of Ψ on $\mathcal{GL}_n$ homogeneously, and extend it to $\pi^{-1}\pi\mathcal{I}(\mathcal{GL}_n)$. Note that $\overline{\pi\mathcal{I}(\mathcal{G}_n)} = \overline{\pi\mathcal{I}(\mathcal{GL}_n)}$. By a continuity argument as in [4, §4.3], one proves that Ψ extends continuously to $\pi^{-1}\overline{\pi\mathcal{I}(\mathcal{G}_n)}$. Since the restriction π to $\pi^{-1}\overline{\pi\mathcal{I}(\mathcal{G}_n)}$ is a quotient map, the required continuous mapping Ψ is then obtained.

For the definition of Ψ on $\mathcal{GL}_n$, we first construct a function ψ_0 from $\mathcal{GL}_n$ into $\mathbb{R}^{3(n-3)}$ whose values are written in terms of elementary intersection numbers. For every $\gamma \in \mathcal{GL}_n$, we write

$$\psi_0(\gamma) = \left(x_1^1(\gamma), x_1^2(\gamma), x_1^3(\gamma), \ldots, x_{n-3}^1(\gamma), x_{n-3}^2(\gamma), x_{n-3}^3(\gamma)\right),$$

where $x_j^k(\gamma) = i(\gamma, \gamma_j^k)/\lambda(\gamma)$ for $1 \le j \le n-3$ and for $1 \le k \le 3$, and where $\lambda(\gamma) = \sum_{j=1}^{n-3}\sum_{k=1}^{3} i(\gamma, \gamma_j^k)$. Note that the image of ψ_0 lies in

$$\Pi' = \Pi \cap \left\{(t_1, t_2, \ldots, t_{3(n-3)}) \in \mathbb{R}^{3(n-3)} : 1 - 2\sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}| > 0\right\},$$

where $\Pi = \{(t_1, t_2, \ldots, t_{3(n-3)}) \in \mathbb{R}^{3(n-3)} : \sum_{j=1}^{3(n-3)} t_j = 1\}$. For later use, we define the function $f : \mathbb{R}^{3(n-3)} \longrightarrow \mathbb{R}$ by

$$f(t_1, t_2, \ldots, t_{3(n-3)}) = 1 - 2\sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}|.$$

Following [4], we define the mapping $\Psi : \mathcal{GL}_n \longrightarrow \mathbb{R}^{3(n-3)}$ by

$$\Psi(\gamma) = \left(\xi_1^1(\gamma), \xi_1^2(\gamma), \xi_1^3(\gamma), \ldots, \xi_{n-3}^1(\gamma), \xi_{n-3}^2(\gamma), \xi_{n-3}^3(\gamma)\right),$$

where for every $1 \le j \le n-3$

$$\xi_j^1(\gamma) = \frac{2I_j(\gamma)}{\rho(\gamma)}, \quad \xi_j^2(\gamma) = \frac{2|N_j(\gamma)|}{\rho(\gamma)} \quad \text{and} \quad \xi_j^3(\gamma) = \frac{2|N_j(\gamma) - I_j(\gamma)|}{\rho(\gamma)},$$

and $\rho(\gamma) = 2\sum_{j=1}^{n-3}\{I_j(\gamma) + |N_j(\gamma)| + |N_j(\gamma) - I_j(\gamma)|\}$. It is easy to see that $\Psi(\gamma) \in \Delta_n = \mathcal{C}^{n-3} \cap \Pi$ for every $\gamma \in \mathcal{GL}_n$, where $\mathcal{C}$ is the set of points $(t_1, t_2, t_3) \in \mathbb{R}_+^3$ satisfying:

$$t_2 + t_3 = t_1, \quad t_1 + t_3 = t_2, \quad \text{or} \quad t_1 + t_2 = t_3.$$

A similar argument to that given in [4, §4.2] proves that Δ_n is homeomorphic to a sphere of dimension $2n-7$.

We shall prove that there is a homeomorphism ψ_1 of Π' onto Π so that $\Psi = \psi_1 \circ \psi_0$. Then we obtain:

Theorem 3.1. *The function Ψ extends to $\overline{\pi\mathcal{I}(\mathcal{G}_n)} = \overline{\pi\mathcal{I}(\mathcal{GL}_n)}$ as a continuous mapping into a sphere of dimension $2n-7$.*

It remains to construct the mapping ψ_1. For $\gamma \in \mathcal{GL}_n$, let

$$\nu(\gamma) = 1 - \frac{4}{\lambda(\gamma)} \sum_{j=1}^{n-4} |I_j(\gamma) - I_{j+1}(\gamma)| = 1 - 2\sum_{j=1}^{n-4} |x_j^1(\gamma) - x_{j+1}^1(\gamma)|\,.$$

A direct computation shows that $\rho(\gamma) = \lambda(\gamma)\nu(\gamma)$, and the followings:

$$\xi_j^1(\gamma) = \frac{x_j^1(\gamma)}{\nu(\gamma)} \qquad \text{for } 1 \le j \le n-3,$$

$$\xi_1^2(\gamma) = \frac{x_1^2(\gamma)}{\nu(\gamma)} - \frac{|x_2^1(\gamma) - x_1^1(\gamma)| + \{x_2^1(\gamma) - x_1^1(\gamma)\}}{2\nu(\gamma)}\,,$$

$$\xi_1^3(\gamma) = \frac{x_1^3(\gamma)}{\nu(\gamma)} - \frac{|x_2^1(\gamma) - x_1^1(\gamma)| + \{x_2^1(\gamma) - x_1^1(\gamma)\}}{2\nu(\gamma)}\,,$$

$$\xi_{n-3}^2(\gamma) = \frac{x_{n-3}^2(\gamma)}{\nu(\gamma)} - \frac{|x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)| + \{x_{n-4}^1(\gamma) - x_{n-4}^1(\gamma)\}}{2\nu(\gamma)}\,,$$

$$\xi_{n-3}^3(\gamma) = \frac{x_{n-3}^3(\gamma)}{\nu(\gamma)} - \frac{|x_{n-4}^1(\gamma) - x_{n-3}^1(\gamma)| + \{x_{n-4}^1(\gamma) - x_{n-4}^1(\gamma)\}}{2\nu(\gamma)}\,,$$

and for $1 < j < n-3$

$$\xi_j^2(\gamma) = \frac{x_j^2(\gamma)}{\nu(\gamma)} - \frac{|x_{j-1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j-1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)} - \frac{|x_{j+1}^1(\gamma) - x_j^1(\gamma)| + \ x_{j+1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)}\,,$$

$$\xi_j^3(\gamma) = \frac{x_j^3(\gamma)}{\nu(\gamma)} - \frac{|x_{j-1}^1(\gamma) - x_j^1(\gamma)| + \{x_{j-1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)} - \frac{|x_{j+1}^1(\gamma) - x_j^1(\gamma)| + \ x_{j+1}^1(\gamma) - x_j^1(\gamma)\}}{2\nu(\gamma)}\,.$$

The above equations motivate the function $\psi_1 \colon \Pi' \longrightarrow \mathbb{R}^{3(n-3)}$ defined by

$\psi_1(r_1, r_2, \ldots, r_{3(n-3)}) = (t_1, t_2, \ldots, t_{3(n-3)})$, where

$$t_j = \frac{r_j}{f(r_1, r_2, \ldots, r_{3(n-3)})} \qquad \text{for } j = 3k-2 \text{ with } 1 \leq k \leq n-3$$

$$t_j = \frac{r_j}{f(r_1, r_2, \ldots, r_{3(n-3)})} - \frac{|r_4 - r_1| + (r_4 - r_1)}{2f(r_1, r_2, \ldots, r_{3(n-3)})} \qquad \text{for } j = 2,\ 3,$$

$$t_j = \frac{r_j}{f(r_1, r_2, \ldots, r_{3(n-3)})} - \frac{|r_{3n-14} - r_{3n-11}| + (r_{3n-14} - r_{3n-11})}{2f(r_1, r_2, \ldots, r_{3(n-3)})}$$

for $j = 3n - 10$ or $3(n-3)$, and

$$\begin{aligned} t_{3k-1} = {} & \frac{r_{3k-1}}{f(r_1, r_2, \ldots, r_{3(n-3)})} - \frac{|r_{3k-5} - r_{3k-2}| + (r_{3k-5} - r_{3k-2})}{2f(r_1, r_2, \ldots, r_{3(n-3)})} \\ & - \frac{|r_{3k+1} - r_{3k-2}| + (r_{3k+1} - r_{3k-2})}{2f(r_1, r_2, \ldots, r_{3(n-3)})} \\ t_{3k} = {} & \frac{r_{3k}}{f(r_1, r_2, \ldots, r_{3(n-3)})} - \frac{|r_{3k-5} - r_{3k-2}| + (r_{3k-5} - r_{3k-2})}{2f(r_1, r_2, \ldots, r_{3(n-3)})} \\ & - \frac{|r_{3k+1} - r_{3k-2}| + (r_{3k+1} - r_{3k-2})}{2f(r_1, r_2, \ldots, r_{3(n-3)})} \qquad \text{for } 1 < k < n-3. \end{aligned}$$

A direct computation proves that ψ_1 maps Π' into Π by showing that

$$\sum_{j=1}^{3(n-3)} t_j = 1 \quad \text{and} \quad 1 + 2\sum_{j=1}^{n-4} |t_{3j-2} - t_{3j+1}| = \frac{1}{f(r_1, r_2, \ldots, r_{3(n-3)})}.$$

From the definition of ψ_1, one proves easily that ψ_1 is indeed a homeomorphism of Π' onto Π.

ACKNOWLEDGEMENT. The author would like to thank Professor L. Keen for her many informative comments and also the referee for reading the paper carefully.

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