Takahashi, T.
Osaka J. Math.
39 (2002), 647–663

GALOIS POINTS ON NORMAL QUARTIC SURFACES

Takeshi TAKAHASHI

(Received October 25, 2000)

1. Introduction

Let k be the field of complex numbers $\mathbb{C}$. We fix it as the ground field of our discussion. Let S be a quartic surface in the projective three space $\mathbb{P}^3 = \mathbb{P}^3(k)$ and $K = k(S)$ the rational function field of S. For each point $P \in S$, let $\pi_P : S \cdots \to H$ be a projection from S to a hyperplane H with the center P. This rational map induces the extension of fields $K/k(H)$ if the multiplicity of P is not four. The structure of this extension does not depend on the choice of H, but on P, so that we write K_P instead of $k(H)$. The field K_P is a maximal rational subfield of K (cf. [5]).

Definition 1. A point $P \in S$ is called a Galois point if the extension K/K_P is Galois.

Note 1.1. If P is a singular point on S with multiplicity two or three, then the degree of the extension K/K_P is two or one. Namely P is a Galois point. Hereafter in this paper, the term "Galois point" means a non-singular point which is Galois.

In the paper [7], Yoshihara studied Galois points on smooth quartic surfaces. As a continuation of his results, in this paper, we consider Galois points and singular points on normal quartic surfaces.

We use the following notation:

- $(X:Y:Z:W)$: homogeneous coordinates on $\mathbb{P}^3$
- T_P: the tangent plane to the surface S at a point P
- $G(X, Y, Z)$: a quartic homogeneous polynomial in three variables over k
- $H(X, Y)$: a quartic homogeneous polynomial in two variables over k
- $l(P, Q)$: the line passing through points P and Q
- ζ: a primitive sixth root of unity
- $\omega := \zeta^2$

2. Statement of results

We use the same notation as is used in Section 1 and restrict ourselves to the case where S is a normal quartic surface.

For a Galois point P, let σ be an element of $\mathrm{Gal}(K/K_P)$. The next proposition is essential in our discussion.

Proposition 2.2. *The birational transformation of S induced by σ is a restriction of a projective transformation of $\mathbb{P}^3$.*

We denote by $M(\sigma) \in PGL(4, k)$ the projective transformation of $\mathbb{P}^3$ induced by σ.

DEFINITION 2. We call σ an automorphism belonging to the Galois point P and $M(\sigma)$ the representation of σ.

Let $\mathrm{GP}(S)$ be the set consisting of Galois points of S and $\delta(S)$ the cardinality of GP(S). Note that $\delta(S)$ is invariant under projective transformations of S.

Theorem 1. *If S is a normal quartic surface and* $\mathrm{GP}(S)$ *is a finite set, then* $\delta(S) = 0$, 1, 2, 4, 5 *or* 8. *Expressing in more detail, we have the following.*
(1) *If $\delta(S) = 1$, then by taking a suitable projective transformation, the defining equation of S can be given by $ZW^3 + G(X, Y, Z) = 0$.*
(2) *If $\delta(S) = 2$, then by taking a suitable projective transformation, the defining equation of S can be given by*
 (a) $XY^3 + ZW^3 + H(X, Z) = 0$ *or*
 (b) $ZY^3 + ZW^3 + H(X, Z) = 0$.
(3) *If $\delta(S) = 4$, then by taking a suitable projective transformation, the defining equation of S can be given by $ZW^3 + Z^4 + H(X, Y) = 0$.*
(4) $\delta(S) = 5$ *if and only if S is projectively equivalent to the surface S_5 given by the equation $XY^3 + ZW^3 + Z^4 = 0$.*
(5) $\delta(S) = 8$ *if and only if S is projectively equivalent to the surface S_8 given by the equation $XY^3 + ZW^3 + X^4 + Z^4 = 0$.*

If S is a smooth quartic surface, then $\mathrm{GP}(S)$ is a finite set (cf. [7]). To the contrary in the case where S is not smooth, the set can be an infinite. Let C be a smooth plane quartic curve with a Galois point. (For the definition of Galois point of a plane curve, see [3].).

Theorem 2. *If S is a normal quartic surface and* $\mathrm{GP}(S)$ *is an infinite set, then S is a cone over C. Expressing in more detail, we have the following.*
(1) *By taking a suitable projective transformation, the defining equation of S can be given by $ZW^3 + H(X, Z) = 0$.*
(2) *Let O be the vertex of the cone S and* $\mathrm{GP}(C)$ *the set consisting of Galois points*

Table 2.1.

Defining equation		$\delta(S)$	$GP(S)$
$ZW^3 + G(X, Y, Z)$	$= 0$	≥ 1	$\supset \{ P_1 \}$
$XY^3 + ZW^3 + H(X, Z)$	$= 0$	≥ 2	$\supset \{ P_1, P_5 \}$
$ZY^3 + ZW^3 + H(X, Z)$	$= 0$	$= 2$	$= \{ P_1, P_5 \}$
$ZW^3 + Z^4 + H(X, Y)$	$= 0$	≥ 4	$\supset \{ P_1, P_2, P_3, P_4 \}$
$XY^3 + ZW^3 + Z^4$	$= 0$	$= 5$	$= \{ P_1, P_2, P_3, P_4, P_5 \}$
$XY^3 + ZW^3 + X^4 + Z^4$	$= 0$	$= 8$	$= \{ P_1, P_2, P_3, P_4,$ $P_5, P_6, P_7, P_8 \}$
$ZW^3 + H(X, Z)$	$= 0$	$= \infty$	$\supset \{ (0 : a : 0 : 1) \mid a \in k \}$
$ZW^3 + X^4 + Z^4$	$= 0$	$= \infty$	$= \{ (0 : a : b : 1) \mid a \in k,$ $b = 0, \zeta, \zeta^3, \zeta^5 \}$

of the base curve C. Then we have that

$$\mathrm{GP}(S) = \bigcup_{P \in \mathrm{GP}(C)} l(P, O) - \{O\}.$$

(3) *The set* $\mathrm{GP}(S) \cup \{O\}$ *consists of at most four lines. Moreover, the maximal number is attained if and only if S is projectively equivalent to the surface $S_{\ast}$ given by the equation* $ZW^3 + X^4 + Z^4 = 0$.

EXAMPLE 2.3. As a kind of converse assertion to the above theorems, we have the examples in Table 2.1. In the table, we use the notation that $P_1 = (0 : 0 : 0 : 1)$, $P_2 = (0 : 0 : \zeta : 1)$, $P_3 = (0 : 0 : \zeta^3 : 1)$, $P_4 = (0 : 0 : \zeta^5 : 1)$, $P_5 = (0 : 1 : 0 : 0)$, $P_6 = (\zeta : 1 : 0 : 0)$, $P_7 = (\zeta^3 : 1 : 0 : 0)$ and $P_8 = (\zeta^5 : 1 : 0 : 0)$.

Note that the surface with $\delta(S) = 8$ has some interesting properties. For example, the number of lines on the surface is 64, this is the maximum number of lines lying on a smooth quartic surface. (For more detail, see [2].) In addition, the surfaces with $\delta(S) = 5$ and ∞ do not appear in [7], because they have a singular point. Let us study them in Section 4 by similar way to [2].

We can make clear what type of singularities S can have, if $\delta(S) \geq 2$. In what follows, to represent types of singularities, we use the same notation as in [1, p. 143, pp. 210–214]. To make sure, we show normal forms of the notation as follows. Let (x, y, z) be a local coordinates.

$$\begin{array}{rlrl}
A_2\colon & x^2 + y^2 + z^3 & A_3\colon & x^2 + y^2 + z^4 \\
D_4\colon & x^2 + y^3 + yz^2 & J_{10}\colon & x^2 + y^3 + z^6 \\
X_9\colon & x^2 + y^4 + z^4 & P_8\colon & x^3 + y^3 + z^3 \\
U_{12}\colon & x^3 + y^3 + z^4 & U_{14}\colon & x^3 + y^3 + yz^3 \\
V'_{18}\colon & x^3 + y^4 + z^4 & &
\end{array}$$

For example, we denote by $A_3^3D_4^2$ the set consisting of three points with A_3-singularity and two points with D_4-singularity.

Theorem 3. *There exists a relation between $\delta(S)$ and the singularities as follows*:

(1) *If $\delta(S) = 2$, then S belongs to one of the following types.*
  (a) *S is smooth.*
  (b) *S is projectively equivalent to the surface given by the equation $XY^3 + ZW^3 + H(X, Z) = 0$ with the singularities D_4, D_4^2, P_8, U_{12}, U_{14}, J_{10}, D_4J_{10}, J_{10}^2, A_2^3, $A_2^3D_4$, $A_2^3P_8$, $A_2^3U_{14}$, $A_2^3J_{10}$, A_2^6 or $A_2^6D_4$.*
  (c) *S is projectively equivalent to the surface given by the equation $ZW^3 + ZY^3 + H(X, Z) = 0$ with the singularities A_3^3, $A_3^3D_4$, $A_3^3D_4^2$, $A_3^3P_8$ or $A_3^3U_{12}$.*

(2) *If $\delta(S) = 4$, then S belongs to one of the following types.*
  (a) *S is smooth.*
  (b) *S is projectively equivalent to the surface given by the equation $ZW^3 + Z^4 + X^2Y^2 = 0$ and has two double points of type X_9.*
  (c) *S is projectively equivalent to the surface given by the equation $ZW^3 + Z^4 + X^2Y(X + Y) = 0$ and has one double point of type X_9.*

(3) *If $\delta(S) = 5$, then S has one triple point of type V'_{18}.*

(4) *If $\delta(S) = 8$, then S is smooth.*

(5) *If* GP(S) *is an infinite set, then S has one singular point O with multiplicity four, and the geometric genus of the singular point O is four.*

Note that in the case where $\delta(S) = 1$, there may exist too many singularities to determine completely.

Finally, we present more concrete examples.

EXAMPLE 2.4. There exist surfaces with the singularities listed in (1) of Theorem 3 as in Table 2.2. We denote by Φ-$(a : b : c : d)$ the singular point of type Φ with coordinates $(a : b : c : d)$.

3. Proofs and some other results

Let $P = (0 : 0 : 0 : 1)$ be a non-singular point on S. First, we give a criterion when the point P becomes Galois. We put $x = X/W$, $y = Y/W$, $z = Z/W$ and $f(x, y, z) = F(X, Y, Z, W)/W^4 = \sum_{i=1}^{4} f_i$, where f_i is a homogeneous part of f with degree i $(i = 1, 2, 3, 4)$.

Lemma 3.5. *Under the notation above, the following assertions are equivalent*:

(1) *P is a Galois point.*

(2) $f_2^2 = 3f_1f_3$

(3) *By taking a suitable projective transformation fixing the point P, the defining*

Table 2.2.

Type	Defining equation / Singular points
D_4	$XY^3 + ZW^3 + (X + Z)(X + 2Z)(X - Z)^2 = 0$
	D_4-$(1 : 0 : 1 : 0)$
D_4^2	$XY^3 + ZW^3 + (X + Z)^2(X - Z)^2 = 0$
	D_4-$(1 : 0 : 1 : 0)$, D_4-$(-1 : 0 : 1 : 0)$
P_8	$XY^3 + ZW^3 + (X + Z)(X - Z)^3 = 0$
	P_8-$(1 : 0 : 1 : 0)$
U_{12}	$XY^3 + ZW^3 + (X - Z)^4 = 0$
	U_{12}-$(1 : 0 : 1 : 0)$
U_{14}	$XY^3 + ZW^3 + X^3(X + Z) = 0$
	U_{14}-$(0 : 0 : 1 : 0)$
J_{10}	$XY^3 + ZW^3 + X^2(X^2 + XZ + Z^2) = 0$
	J_{10}-$(0 : 0 : 1 : 0)$
D_4J_{10}	$XY^3 + ZW^3 + X^2(X - Z)^2 = 0$
	D_4-$(1 : 0 : 1 : 0)$, J_{10}-$(0 : 0 : 1 : 0)$
J_{10}^2	$XY^3 + ZW^3 + X^2Z^2 = 0$
	J_{10}-$(1 : 0 : 0 : 0)$, J_{10}-$(0 : 0 : 1 : 0)$
A_2^3	$XY^3 + ZW^3 + X^4 + XZ(X + Z)(X - Z) = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$
$A_2^3D_4$	$XY^3 + ZW^3 - X(X + Z)(X - Z)^2 = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$, D_4-$(1 : 0 : 1 : 0)$
$A_2^3P_8$	$XY^3 + ZW^3 + X(X - Z)^3 = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$, P_8-$(-1 : 0 : 1 : 0)$
$A_2^3U_{14}$	$XY^3 + ZW^3 - XZ^3 = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$, U_{14}-$(1 : 0 : 0 : 0)$
$A_2^3J_{10}$	$XY^3 + ZW^3 + XZ^2(X - Z) = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$, J_{10}-$(1 : 0 : 0 : 0)$
A_2^6	$XY^3 + ZW^3 - XZ(X^2 + XZ + Z^2) = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$, A_2-$(1 : 0 : 0 : 1)$, A_2-$(1 : 0 : 0 : \omega)$, A_2-$(1 : 0 : 0 : \omega^2)$

$A_2^6D_4$	$XY^3 + ZW^3 - XZ(X - Z)^2 = 0$
	A_2-$(0 : 1 : 1 : 0)$, A_2-$(0 : \omega : 1 : 0)$, A_2-$(0 : \omega^2 : 1 : 0)$, A_2-$(1 : 0 : 0 : 1)$, A_2-$(1 : 0 : 0 : \omega)$, A_2-$(1 : 0 : 0 : \omega^2)$, D_4-$(1 : 0 : 1 : 0)$
A_3^3	$ZY^3 + ZW^3 + X^4 + Z^4 = 0$
	A_3-$(0 : \zeta : 0 : 1)$, A_3-$(0 : \zeta^3 : 0 : 1)$, A_3-$(0 : \zeta^5 : 0 : 1)$
$A_3^3D_4$	$ZY^3 + ZW^3 + X^2(X + Z)(X - Z) = 0$
	A_3-$(0 : \zeta : 0 : 1)$, A_3-$(0 : \zeta^3 : 0 : 1)$, A_3-$(0 : \zeta^5 : 0 : 1)$, D_4-$(0 : 0 : 1 : 0)$
$A_3^3D_4^2$	$ZY^3 + ZW^3 + X^2(X - Z)^2 = 0$
	A_3-$(0 : \zeta : 0 : 1)$, A_3-$(0 : \zeta^3 : 0 : 1)$, A_3-$(0 : \zeta^5 : 0 : 1)$, D_4-$(0 : 0 : 1 : 0)$, D_4-$(1 : 0 : 1 : 0)$
$A_3^3P_8$	$ZY^3 + ZW^3 + X^3(X + Z) = 0$
	A_3-$(0 : \zeta : 0 : 1)$, A_3-$(0 : \zeta^3 : 0 : 1)$, A_3-$(0 : \zeta^5 : 0 : 1)$, P_8-$(0 : 0 : 1 : 0)$
$A_3^3U_{12}$	$ZY^3 + ZW^3 + X^4 = 0$
	A_3-$(0 : \zeta : 0 : 1)$, A_3-$(0 : \zeta^3 : 0 : 1)$, A_3-$(0 : \zeta^5 : 0 : 1)$, U_{12}-$(0 : 0 : 1 : 0)$

equation can be given by $z + g_4(x, y, z) = 0$, *where* g_4 *is a homogeneous polynomial with degree four.*

Proof. First let us prove the implication (1) $\Rightarrow$ (2). Putting $g(x, y, z) = f_2^2 - 3f_1f_3$ and letting H_P be a general hyperplane given by the equation $z = ax + by$, where $a, b \in k$, we infer from Bertini's theorem that $C_P := S \cap H_P$ is a smooth quartic curve given by the equation $f(x, y, ax + by) = 0$. We note that P also becomes a Galois point of the curve C_P. Hence we obtain that $g(x, y, ax + by) = 0$ for general a, b. (see [6, Lemma 11].) Therefore we get $g(x, y, z) = 0$.

Next, we prove the implication (2) $\Rightarrow$ (3). By taking a suitable projective transformation fixing the point P, we may assume that $f_1 = z$. Then from the equation $f_2^2 = 3f_1f_3$, we infer that z is a factor of f_2 and f_3. Hence, f can be expressed as $z + zg_1 + z{g_1}^2/3 + f_4$, where $g_1 = g_1(x, y, z)$ is a homogeneous polynomial with degree one. Let $F = ZW^3 + Zg_1(X, Y, Z)W^2 + Zg_1(X, Y, Z)^2W/3 + f_4(X, Y, Z)$, which is obtained from f. Substituting $W - g_1(X, Y, Z)/3$ for W in the form F, that is, we consider $F(X, Y, Z, W - g_1(X, Y, Z)/3)$. Then we obtain $F = ZW^3 + g_4(X, Y, Z)$, which means that $f = z + g_4$.

Finally let us prove the implication (3) $\Rightarrow$ (1). Putting $s = y/x$ and $t = z/x$, we obtain $K_P = k(s, t)$ and $K = K_P(x)$. The minimal equation of x over K_P is the one $x^3 + t/g_4(1, s, t) = 0$, which implies that the extension K/K_P is Galois. $\square$

From Lemma 3.5, we infer the following readily.

Note 3.6. If P is a Galois point, then $T_P \cap S$ consists of only lines which meet at P.

Let us prove Proposition 2.2. Suppose P is a Galois point of S. Then, by Lemma 3.5, we may assume that S is given by the equation $f = z + f_4(x, y, z) = 0$. Putting $s = y/x$ and $t = z/x$, we obtain $K_P = k(s, t)$ and $K = K_P(x)$. The minimal equation of x over K_P is the one $x^3 + t/f_4(1, s, t) = 0$. Hence, we infer that $\sigma(x) = \omega x$ if $\sigma \in \mathrm{Gal}(K/K_P)$ is not identity. Therefore, a birational transformation of S induced by σ is a restriction of a projective transformation of $\mathbb{P}^3$. Thus, we complete the proof of Proposition 2.2.

REMARK 3.7. Copying the proofs in [7], we obtain the following.
(1) Let F be the homogeneous defining equation of S and $H(F)$ be the Hessian of F. If P is a Galois point, then $H(F)(P) = 0$.
(2) Suppose that P and P' are Galois points, and σ and σ' are automorphisms belonging to P and P', respectively. Then, $\sigma(P')$ is also a Galois point and $\sigma\sigma'\sigma^{-1}$ is an automorphism belonging to $\sigma(P')$.
(3) Suppose that P and P' are two Galois points, and the line l passing through these points does not lie on S. Then in (2) we have that $\sigma(P') \neq P'$, hence there exist two more Galois points $\sigma(P')$ and $\sigma^2(P')$.

To prove Theorem 1 and 2, we show the following two lemmas.

Lemma 3.8. *Let l be a line lying on S. Then $\#(l \cap \mathrm{GP}(S)) = 0$, 1, 2 or ∞. The last case occurs if and only if S is projectively equivalent to the cone given by the equation*

$$ZW^3 + H(X, Z) = 0,$$

especially, $l \cap \mathrm{GP}(S) = l - \{O\}$, where O is the vertex of the cone S.

Proof. Suppose that there exist three Galois points P_1, P_2 and P_3 on l. Then, from Lemma 3.5, we may assume that $P_1 = (0 : 0 : 0 : 1)$, l is given by the equation $X = Z = 0$, and S is given by the equation

$$ZW^3 + XH(X, Y) + ZG(X, Y, Z) = 0.$$

Here, we define linear systems as follows:

$$\mathcal{H} = \{\, H_\lambda \mid l \subset H_\lambda, H_\lambda \text{ is a hyperplane} \,\},$$
$$\mathcal{C} = \{\, C_\lambda := S \cap H_\lambda - l \mid H_\lambda \in \mathcal{H} \,\}.$$

Let H_λ be an element of $\mathcal{H}$ given by the equation $X = \lambda Z$ ($\lambda \in k$). Then, $C_\lambda =$

$S \cap H_\lambda - l$ is given by the equation

$$W^3 + \lambda H(\lambda Z, Y) + G(\lambda Z, Y, Z) = 0,$$

hence, we see that $C_\lambda \cap l$ is given by the equation

$$W^3 + \lambda H(0, Y) + G(0, Y, 0) = 0.$$

If $H(0, Y) \neq 0$, then the linear system $\mathcal{C}$ determines the finite morphism with degree three $\Phi : l \to \mathbb{P}^1$. Note that $T_{P_i} \subset \mathcal{H}$ ($i = 1, 2, 3$) and let us put $C_{P_i} = S \cap T_{P_i} - l$. Then from Note 3.6, we can see easily that $C_{P_i} \cap l = \{P_i\}$, this implies that P_i must be a ramification point of Φ. However, the number of ramification points of Φ are two, this is contradiction.

Assume that $H(0, Y) = 0$. Then, the points of $C_\lambda \cap l$, which are given by the equations $X = Z = 0$ and $W^3 + G(0, Y, 0) = 0$, are singular points of S. Hence, $P_i \notin C_\lambda \cap l$, and we infer from Note 3.6 that $T_{P_1} = T_{P_2} = T_{P_3}$, and $T_{P_i} \cap S$ consists of one line l. So, we may assume that S is given by the equation

$$ZW^3 + X^4 + Z\,(\,G_0 Y^3 + G_1 Y^2 + G_2 Y + G_3\,) = 0,$$

where $G_i = G_i(X, Z)$ is a homogeneous polynomial with degree i. Either of P_2 or P_3 can be represented by $(0 : 1 : 0 : a)$ ($a \in k$, $a \neq 0$), now we assume that P_3 can be so. Then, checking the condition (2) of Lemma 3.5 at P_3, we obtain that $G_0 = G_1 = G_2 = 0$. Namely, we may assume that S is given by the equation $ZW^3 + H(X, Z) = 0$. Note that $O = (0 : 1 : 0 : 0)$ is the vertex of the cone S. Then, using Lemma 3.5, we see easily that $l \cap \mathrm{GP}(S) = l - \{O\}$. □

Lemma 3.9. *Suppose that S has four Galois points P_i ($i = 1, 2, 3, 4$) and these are collinear. In addition, suppose that the line passing through these four points does not lie on S. Then S is projectively equivalent to the surface given by the equation*

$$ZW^3 + Z^4 + H(X, Y) = 0.$$

Proof. Since Lemma 3.5, by taking a suitable projective transformation, we may assume that $P_1 = (0 : 0 : 0 : 1)$ and S is given by the equation $ZW^3 + G(X, Y, Z) = 0$. Let l be the line passing through the Galois points P_i ($i = 1, 2, 3, 4$). Note that $l \not\subset S$, we may assume that l is given by the equation $X = Y = 0$. Then, the points P_i ($i = 1, 2, 3, 4$) are given by the equations $X = Y = 0$ and $Z(W^3 + G(0, 0, 1)Z^3) = 0$. Namely, $P_i = (0 : 0 : 1 : \omega^{i-2}\sqrt[3]{-c})$ ($i = 2, 3, 4$), where $c = G(0, 0, 1)$. Now we put $G = \Sigma_{j=0}^{4} G_j(X, Y) Z^{4-j}$, where $G_j(X, Y)$ is a homogeneous polynomial with degree j ($j = 0, 1, 2, 3, 4$). Then, by checking the condition (2) of Lemma 3.5 at each Galois point P_i, we obtain that $G_1 = G_2 = G_3 = 0$. Therefore, we get the defining equation $ZW^3 + Z^4 + H(X, Y) = 0$. □

Let us prove Theorem 1. The assertion (1) of Theorem 1 is trivial from Lemma 3.5.

We prove the assertion (2) of Theorem 1. Let P and P' be two Galois points. Then, let σ and σ' be automorphisms belonging to P and P', $M(\sigma)$ and $M(\sigma')$ their representations, respectively. Let l be the line passing through the points P and P'. We infer from Remark 3.7 that l is contained in S. Hence, we see easily that $M(\sigma)$ and $M(\sigma')$ have the following properties:

- $M(\sigma)(P) = P$, $M(\sigma)(P') = P'$, $M(\sigma')(P) = P$, $M(\sigma')(P') = P'$
- $M(\sigma)(l_P) = l_P$ [resp. $M(\sigma')(l_{P'}) = l_{P'}$], for any line l_P [resp. $l_{P'}$] passing through P [resp. P'].
- $M(\sigma)^3$ and $M(\sigma')^3$ are identity.

So by taking a suitable projective transformation, we may assume that $P = (0 : 0 : 0 : 1)$, $P' = (0 : 1 : 0 : 0)$,

$$M(\sigma) = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M(\sigma') = \begin{pmatrix} \omega^i & 0 & 0 & 0 \\ a & 1 & b & 0 \\ 0 & 0 & \omega^i & 0 \\ 0 & 0 & 0 & \omega^i \end{pmatrix},$$

where $a, b \in k$, $i = 1$ or 2. Then $M(\sigma)$ and $M(\sigma')$ can be diagonalized simultaneously, since $M(\sigma)M(\sigma') = M(\sigma')M(\sigma)$. Therefore, we may assume that

$$M(\sigma') = \begin{pmatrix} \omega & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}.$$

From the conditions $M(\sigma)(S) = S$ and $M(\sigma')(S) = S$, we obtain the assertion (2) of Theorem 1.

It is easy to see that the surface with $\delta = 3$ cannot exist. Indeed, suppose that $\delta(S) = 3$. Then, let P_1, P_2 and P_3 be three Galois points. From Remark 3.7, we obtain that $l(P_1, P_2)$, $l(P_2, P_3)$, $l(P_3, P_1) \subset S$. Hence we infer from Lemma 3.8 that three points P_1, P_2 and P_3 are not collinear. However, we see that the configuration of three lines l_{ij} contradicts to Note 3.6.

Let us consider the case where $4 \leq \delta(S) \lneq \infty$. Then, we infer from Remark 3.7 and Lemma 3.8 that there exist four Galois points which are collinear. Hence from Lemma 3.9, by taking a suitable projective transformation, the defining equation of S can be given by

$$ZW^3 + Z^4 + H(X, Y) = 0.$$

Now we obtain the assertion (4) of Theorem 1.

If $H(X, Y)$ has only simple factors, then S is smooth. Therefore we have $\delta(S) = 4$ or 8, especially, $\delta(S) = 8$ if and only if S is projectively equivalent to the surface given by the equation $XY^3+ZW^3+X^4+Z^4=0$ (see [7, Theorem 3]). On the other hand, if $H(X,Y)$ has a multiple factor, then S is given by one of the following equations, by taking a suitable projective transformation.

(i) $ZW^3+Z^4+X^3Y=0$ (ii) $ZW^3+Z^4+X^2Y^2=0$

(iii) $ZW^3+Z^4+X^2Y(X+Y)=0$ (iv) $ZW^3+Z^4+X^4=0$

Calculate $\delta(S)$ using Lemma 3.5 and Remark 3.7, we obtain that $\delta(S)=5$ when S is given by the equation (i), $\delta(S)=4$ when S is given by the equation (ii) or (iii), and $\mathrm{GP}(S)$ is not a finite set when S is given by the equation (iv). It is clear that the surface given by the equation (i) is projectively equivalent to the surface given by the equation $XY^3+ZW^3+Z^4=0$. Thus we complete the proof of Theorem 1.

Next, let us prove Theorem 2. Suppose that $\mathrm{GP}(S)$ is an infinite set. Then let P_1, P_2 and P_3 be three Galois points of S. First, we suppose that these are not collinear. Then from Note 3.6, we see that $l(P_1,P_2)\not\subset S$, $l(P_2,P_3)\not\subset S$ or $l(P_3,P_1)\not\subset S$. Now we assume that $l(P_1,P_2)\not\subset S$. Then, from Remark 3.7 and Lemma 3.9, by taking a suitable projective transformation, we may assume that S is given by the equation $ZW^3+Z^4+H(X,Y)=0$. Hence noting that $\mathrm{GP}(S)$ is an infinite set, similarly as the last part of the proof of Theorem 1, we may assume that S is given by the equation $ZW^3+X^4+Z^4=0$. This is the special one of the surfaces given by the equation $ZW^3+H(X,Z)=0$. Next, we suppose that P_1, P_2 and P_3 are collinear. Then let l be the line passing through these points. If $l\not\subset S$, then similarly as above, by taking a suitable projective transformation, we may assume that S is given by the equation $ZW^3+X^4+Z^4=0$. On the other hand, if $l\subset S$, then from Lemma 3.8, we may assume that S is given by the equation $ZW^3+H(X,Z)=0$. Thus we obtain the assertion (1) of Theorem 2.

Suppose that S is a cone and P is a Galois point of S. Then by a suitable projective transformation and by choosing a suitable base curve C, we may assume that $P=(0:0:0:1)$, the vertex $O=(0:1:0:0)$ and $P\in C$. Moreover, from [6, Proposition 5], we may assume that C is given by the equation $ZW^3+H(X,Z)=0$ on the hyperplane given by the equation $Y=0$. For each point P' of $l(P,O)-\{O\}$, using Lemma 3.5, we can check easily that P' is a Galois point of S. Thus we obtain the assertion (2) of Theorem 2.

The assertion (3) of Theorem 2 is clear from [6, Theorem 4 and Proposition 5]

Thus we complete the proof of Theorem 2.

Finally we prove Theorem 3. From Theorem 1, its proof and Theorem 2, we infer readily the assertions (2), (3), (4) and (5). Let us consider the case where S is given by the equation $XY^3+ZW^3+H(X,Z)=0$. We put

$$H(X,Z)=h_0X^4+h_1X^3Z+h_2X^2Z^2+h_3XZ^3+h_4Z^4,$$

where $h_i\in k$ $(i=0,1,2,3,4)$. Let l_1, l_2 and l_3 be lines given by the equations

$X = W = 0$, $Y = Z = 0$ and $Y = W = 0$ respectively. We see easily that any singular point of S must be on $l_1 \cup l_2 \cup l_3$. By calculating local equations, we can decide types of singularities which S can have on l_1, l_2 and l_3 as follows:

1. On l_1. Let Q be the point $(0 : a : 1 : 0)$, where $a \in k$.
   (a) Q is a singular point if and only if $h_4 = 0$ and $h_3 = -a^3$.
   (b) Q is a singular point of type A_2 if and only if $h_4 = 0$ and $h_3 = -a^3 \neq 0$.
   (c) Q is a singular point of type J_{10} if and only if $a = h_3 = h_4 = 0$ and $h_2 \neq 0$.
   (d) Q is a singular point of type U_{14} if and only if $a = h_2 = h_3 = h_4 = 0$.
   Hence we see that the type of singularities on l_1 is A_2^3, J_{10} or U_{14}.
2. On l_2. Let Q be the point $(1 : 0 : 0 : a)$, where $a \in k$.
   (a) Q is a singular point if and only if $h_0 = 0$ and $h_1 = -a^3$.
   (b) Q is a singular point of type A_2 if and only if $h_0 = 0$ and $h_1 = -a^3 \neq 0$.
   (c) Q is a singular point of type J_{10} if and only if $a = h_0 = h_1 = 0$ and $h_2 \neq 0$.
   (d) Q is a singular point of type U_{14} if and only if $a = h_0 = h_1 = h_2 = 0$.
   Hence we see that the type of singularities on l_2 is A_2^3, J_{10} or U_{14}.
3. On $l_3 - (l_1 \cup l_2) \cap l_3$. Let Q be the point $(a : 0 : 1 : 0)$, where $a \in k$ and $a \neq 0$. Now we put $H(X, Z) = (X - aZ)^j H_1(X, Z)$, where $j = 0, 1, 2, 3$ or 4, $H_1(X, Z)$ is a homogeneous polynomial with degree $4 - j$ and $H_1(a, 1) \neq 0$.
   (a) Q is a singular point if and only if $j \geq 2$.
   (b) Q is a singular point of type D_4 if and only if $j = 2$.
   (c) Q is a singular point of type P_8 if and only if $j = 3$.
   (d) Q is a singular point of type U_{12} if and only if $j = 4$.
   Hence we see that the type of singularities on $l_3 - (l_1 \cup l_2) \cap l_3$ is D_4, D_4^2, P_8 or U_{12}.

Let us consider the combinations of above types of singularities as in Table 3.1. In the table, the symbol $\emptyset$ means that there does not exist a singular point of S on l_1, l_2 or $l_3 - (l_1 \cup l_2) \cap l_3$. Moreover, if there exists the surface with the singular points, then we use the symbol $\bigcirc$, otherwise we use $\times$.

Therefore, we infer the assertion (1)-(b) of Theorem 3. Similarly as above, we can prove the assertion (1)-(c) of Theorem 3. Thus we complete the proof of Theorem 3.

From the above discussions, it seems easy to check Example 2.3 and Example 2.4.

4. The surfaces with many Galois points

In the paper [2], it is studied that the structures of the quartic surface which has eight Galois points. So, in this section, let us study the structure of the quartic surfaces which appear in (4) of Theorem 1 and (3) of Theorem 2 similarly as it. We denote by S_5 the surface given by the equation

$$XY^3 + ZW^3 + Z^4 = 0,$$

Table 3.1.

combination of types of singularities			existence of S
on l_1	on l_2	on $l_3 - (l_1 \cup l_2) \cap l_3$	
$\emptyset$	$\emptyset$	$\emptyset$	◯
$\emptyset$	A_2^3	$\emptyset$, D_4 or P_8	◯
$\emptyset$	A_2^3	D_4^2 or U_{12}	×
$\emptyset$	J_{10}	$\emptyset$ or D_4	◯
$\emptyset$	J_{10}	D_4^2, P_8 or U_{12}	×
$\emptyset$	U_{14}	$\emptyset$	◯
$\emptyset$	U_{14}	D_4, D_4^2, P_8 or U_{12}	×
A_2^3	$\emptyset$	$\emptyset$, D_4 or P_8	◯
A_2^3	$\emptyset$	D_4^2 or U_{12}	×
A_2^3	A_2^3	$\emptyset$ or D_4	◯
A_2^3	A_2^3	D_4^2, P_8 or U_{12}	×
A_2^3	J_{10}	$\emptyset$	◯
A_2^3	J_{10}	D_4, D_4^2, P_8 or U_{12}	×
A_2^3	U_{14}	$\emptyset$	◯
A_2^3	U_{14}	D_4, D_4^2, P_8 or U_{12}	×
J_{10}	$\emptyset$	$\emptyset$ or D_4	◯
J_{10}	$\emptyset$	D_4^2, P_8 or U_{12}	×
J_{10}	A_2^3	$\emptyset$	◯
J_{10}	A_2^3	D_4, D_4^2, P_8 or U_{12}	×
J_{10}	J_{10}	$\emptyset$	◯
J_{10}	J_{10}	D_4, D_4^2, P_8 or U_{12}	×
J_{10}	U_{14}	any type of singularities	×
U_{14}	$\emptyset$	$\emptyset$	◯
U_{14}	$\emptyset$	D_4, D_4^2, P_8 or U_{12}	×
U_{14}	A_2^3	$\emptyset$	◯
U_{14}	A_2^3	D_4, D_4^2, P_8 or U_{12}	×
U_{14}	J_{10}	any type of singularities	×
U_{14}	U_{14}	any type of singularities	×

which has five Galois points and one singular point of type V'_{18} (cf. Theorem 1 and 3). Now we put $P_1 = (0:0:0:1)$, $P_2 = (0:0:\zeta:1)$, $P_3 = (0:0:\zeta^3:1)$, $P_4 = (0:0:\zeta^5:1)$ and $P_5 = (0:1:0:0)$, which are five Galois points of S_5, and we put $Q = (1:0:0:0)$, which is the singular point of S_5. Let $\mathcal{L}(S_5)$ be the set of automorphisms of S_5 induced by projective transformations, and let $G(S_5)$ be the group generated by the automorphisms belonging to the five Galois points on S_5. Since $G(S_5)$ has an injective representation in $PGL(4,k)$ (cf. Proposition 2.2), we use the same notation of an element of $G(S_5)$ as the projective transformation induced by it.

We denote by $S_{\ast}$ the surface given by the equation

$$ZW^3 + X^4 + Z^4 = 0.$$

Now, we put $O = (0 : 1 : 0 : 0)$, which is the vertex of $S_{\ast}$, the singular point with multiplicity four and the geometric genus of O is four. Let H_Y be the hyperplane given by the equation $Y = 0$. Then, we put $C_4 = S_{\ast} \cap H_Y$, which is a base curve of the cone $S_{\ast}$. Using [6, Lemma 11], we see that P_1, P_2, P_3 and P_4 are four Galois points of C_4. Hence, we have that

$$\mathrm{GP}(S_{\ast}) = \bigcup_{i=1}^{4} l(P_i, O) - \{O\} = \{\, (0 : a : b : 1) \mid a \in k,\ b = 0, \zeta,\ \zeta^3,\ \zeta^5 \,\}.$$

(cf. Theorem 2 and Example 2.3). Let us define $\mathcal{L}(S_{\ast})$ and $G(S_{\ast})$ similarly as above.

We can prove the following lemma by similar argument to the proof of [2, Lemma 2].

Lemma 4.10. *Under the notation above, we have the following.*
(1) *If σ_i ($\neq$ id) is an automorphism of S_5 belonging to the Galois point P_i ($i = 1, \ldots,$ 5), then σ_i (or ${\sigma_i}^2$) has the following representation*:

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta - 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{-\zeta-1}{3} \\ 0 & 0 & \frac{4\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix},$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{-\zeta+2}{3} \\ 0 & 0 & \frac{-2\zeta+4}{3} & \frac{\zeta+1}{3} \end{pmatrix}, \quad \sigma_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{2\zeta-1}{3} \\ 0 & 0 & \frac{-2\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix},$$

and

$$\sigma_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2) *Let us denote that $P_{a,1} = (0 : a : 0 : 1)$, $P_{a,2} = (0 : a : \zeta : 1)$, $P_{a,3} = (0 : a : \zeta^3 : 1)$ and $P_{a,4} = (0 : a : \zeta^5 : 1)$, where $a \in k$. If $\sigma_{a,i}$ is an automorphism of $S_{\ast}$ belonging to the Galois point $P_{a,i}$ ($a \in k$ and $i = 1, 2, 3$ or 4), then $\sigma_{a,i}$ has the following*

representation:

$$\sigma_{a,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & (\zeta-2)a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \zeta-1 \end{pmatrix}, \quad \sigma_{a,2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{4\zeta-2}{3}a & \frac{\zeta-2}{3}a \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{-\zeta-1}{3} \\ 0 & 0 & \frac{4\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix},$$

$$\sigma_{a,3} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-2\zeta+4}{3}a & \frac{\zeta-2}{3}a \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{-\zeta+2}{3} \\ 0 & 0 & \frac{-2\zeta+4}{3} & \frac{\zeta+1}{3} \end{pmatrix}, \quad \sigma_{a,4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{-2\zeta-2}{3}a & \frac{\zeta-2}{3}a \\ 0 & 0 & \frac{2\zeta-1}{3} & \frac{2\zeta-1}{3} \\ 0 & 0 & \frac{-2\zeta-2}{3} & \frac{\zeta+1}{3} \end{pmatrix}.$$

We can prove the following proposition readily, by the similar method to the proof of [2, Theorem 1] and a elementary consideration of matrices.

Proposition 4.11.
(1) *The order of* $G(S_5)$ *is* $2^3 3^2$. *Moreover, we have the following*:

$$\begin{aligned} G(S_5) = \{\, \sigma_1^{i_1}\sigma_3^{i_2}\sigma_5^{i_3}\tau^{i_4} \mid i_1, i_2, i_3 = 0, 1, 2,\ i_4 = 0, 1 \,\} \cup \\ \{\, \sigma_3^{i_1}\sigma_1^{i_1}\sigma_5^{i_2}\tau^{i_3} \mid i_2 = 0, 1, 2,\ i_1, i_2 = 0, 1 \,\} \cup \\ \{\, \sigma_1\sigma_3\sigma_1\sigma_5^{i_1}\tau^{i_2} \mid i_1 = 0, 1, 2,\ i_2 = 0, 1 \,\}, \end{aligned}$$

where

$$\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(2) $G(S_{\ast})$ *is an infinite group. Moreover, there exists an exact sequence of groups as follows*: (*for the definition of* $G(C_4)$, *see* [2])

$$1 \longrightarrow H_1 \longrightarrow G(S_{\ast}) \xrightarrow{r} G(C_4) \longrightarrow 1,$$

where H_1 *is the subgroup of* $G(S_{\ast})$ *consisting of*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in PGL(4, k) \qquad (a,\ b \in k),$$

and the map r is defined as

$$r\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}\right) = \begin{pmatrix} a_{11} & a_{13} & a_{14} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{pmatrix}$$

(*note that r is a homomorphism, since for any element of $G(S_{\ast})$, $a_{12} = a_{32} = a_{42} = 0$.*).

By the similar way to the proof of [2, Theorem 3] and a elementary consideration of matrices, we have the following proposition.

Proposition 4.12.
(1) *The order of the group $\mathcal{L}(S_5)$ is infinite. In fact, $\mathcal{L}(S_5)$ consists of the following elements*:

$$\begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \alpha & \alpha\beta\gamma \\ 0 & 0 & 2\alpha\beta^2 & \alpha\gamma \end{pmatrix} \quad or \quad \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \alpha' & 0 \\ 0 & 0 & 0 & \alpha'\beta' \end{pmatrix},$$

where $\alpha^4 = 1/9$, $\beta^3 = -1$, $\gamma^3 = -1$, $\alpha'^4 = 1$, $\beta'^3 = 1$ *and* $\lambda\mu^3 = 1$.
(2) *The order of the group $\mathcal{L}(S_{\ast})$ is infinite. In fact, there exists an exact sequence of groups as follows*: (*the definition of $\mathcal{L}(C_4)$, see* [2])

$$1 \longrightarrow H_2 \longrightarrow \mathcal{L}(S_{\ast}) \xrightarrow{r} \mathcal{L}(C_4) \longrightarrow 1,$$

where H_2 is the subgroup of $\mathcal{L}(S_{\ast})$ consisting of

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & b & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in PGL(4, k) \qquad (a, b, c, d \in k),$$

and r is the same homomorphism used in Proposition 4.11.

The number of lines on the surface with $\delta(S) = 8$ is 64, this is the maximum number of lines lying on a smooth quartic surface (cf. [2, Remark 4]). However, the lines on the surface S_5 is not so many, and the lines on the surface $S_{\ast}$ is finitely many since $S_{\ast}$ is a cone.

Proposition 4.13. *The number of lines on the surface S_5 is eight. In fact, they are $l(P_1, P_5)$, $l(P_2, P_5)$, $l(P_3, P_5)$, $l(P_4, P_5)$, $l(P_1, Q)$, $l(P_2, Q)$, $l(P_3, Q)$ and $l(P_4, Q)$.*

Proof. A line in $\mathbb{P}^3$ is given by one of the following equations:

- $X - aZ - bW = Y - cZ - dW = 0$
- $X - aY - bW = Z - cW = 0$
- $X - aY - bZ = W = 0$
- $Y - aW = Z - bW = 0$
- $Y - aZ = W = 0$
- $Z = W = 0$

$(a,\ b,\ c,\ d \in k)$

Therefore, by elementary calculation, we conclude. □

From S_5 has the triple point and $S_{\ast}$ is a cone over a smooth plane quartic curve, We infer the following readily.

Proposition 4.14.
(1) *A non-singular model of S_5 is a rational surface.*
(2) *A non-singular model of $S_{\ast}$ is birationally equivalent to a ruled surface of genus three.*

REMARK 4.15. By similar argument in this section and [2], if the defining equation of a normal quartic surface S is given, then we can find all elements of $G(S)$, $\mathcal{L}(S)$ and the set of lines on S, and then we can calculate these orders. Moreover, by [4, Theorem 1], we can see easily what type of surface a non-singular model of S is.

ACKNOWLEDGEMENT. The author expresses his gratitude to Professor Hisao Yoshihara for suggesting this research and giving valuable advice. And the author expresses his sincere thanks to the referee for giving him valuable suggestions.

References

[1] T. Higuchi, E. Yoshinaga and K. Watanabe: Tahensuu hukusokaiseki nyuumon (in Japanese), Suugaku Raiburarî, 51, Morikita Syuppan, 1980.
[2] M. Kanazawa, T. Takahashi and H. Yoshihara: *The group generated by automorphisms belonging to Galois points of the quartic surface*, Nihonkai Math. J. **12** (2001), 89–99.
[3] K. Miura and H. Yoshihara: *Field theory for function fields of plane quartic curves*, J. Algebra, **226** (2000), 283–294.
[4] Y. Umezu: *On normal projective surfaces with trivial dualizing sheaf*, Tokyo J. Math. **4** (1981), 343–354.
[5] H. Yoshihara: *Degree of irrationality of an algebraic surface*, J. Algebra, **167** (1994), 634–640.
[6] H. Yoshihara: *Function field theory of plane curves by dual curves*, J. Algebra, **239** (2001), 340–355.
[7] H. Yoshihara: *Galois points on quartic surfaces*, J. Math. Soc. Japan, **53** (2001), 731–743.

Graduate School of Science and Technology
Niigata University
Niigata 950-2181
Japan
e-mail: takahashi@melody.gs.niigata-u.ac.jp