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THE STRUCTURE OF ALGEBRAIC EMBEDDINGS OF $\mathbb{C}^2$ INTO $\mathbb{C}^3$ (THE NORMAL QUARTIC HYPERSURFACE CASE. I)

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1. Introduction

A polynomial mapping $f : \mathbb{C}^n \to \mathbb{C}^m$ is called an *algebraic embedding* of $\mathbb{C}^n$ into $\mathbb{C}^m$ for $m > n \geq 1$ if f is injective and if the image of f is a smooth algebraic subvariety of $\mathbb{C}^m$. Let $\mathrm{Aut}(\mathbb{C}^n)$ be the group of algebraic automorphisms of $\mathbb{C}^n$. Here we consider the following conjecture:

Conjecture. Let $f : \mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ be an algebraic embedding. Then f is equivalent to a linear embedding, that is, there exists an algebraic automorphism Φ of $\mathbb{C}^{n+1}$ such that $\Phi \circ f$ is a linear embedding.

For the case $n = 1$, Abhyankar-Moh [1] and Suzuki [16] (cf. [17]) showed that the conjecture is true. For the cases $n \geq 2$, the conjectures are still unsolved, however Russell [14], [15] has obtained some sufficient conditions for the conjectures to be true from a view point of ring theory. On the other hand, our approach in this paper is geometric and different from his. We use a method of compactifications of $\mathbb{C}^2$.

From now on, we will consider the case $n = 2$ only. Let $f : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ be an algebraic embedding. We identify $\mathbb{C}^3$ with an affine part of the complex projective three-space $\mathbb{P}^3$ in the standard way. We denote by X_f the closure of the image of f in $\mathbb{P}^3$ and put $Y_f := X_f \setminus f(\mathbb{C}^2)$. By construction, we see that Y_f is a hyperplane section of X_f and that $X_f \setminus Y_f$ is biregular to $\mathbb{C}^2$, that is, (X_f, Y_f) is a *compactification of* $\mathbb{C}^2$. We call Y_f the *boundary* of the compactification. Our main purpose is, for the cases that the images of f are of low degree, to write down explicitly, up to affine transformations of $\mathbb{C}^3$, defining equations of the images and to construct explicitly algebraic automorphisms of $\mathbb{C}^3$ linearizing the defining equations. This explicit way is very important for us not only to obtain examples but also to find geometric invariants and inductive methods. In this direction, in our previous paper [12] (cf. [4], [5]), we have showed that the conjecture is true when the degree of the image is less than or equal to three. For the case of degree three, we needed a so-called *Nagata automorphism* (cf. [11]) to linearize some embedding.

Next we consider the case of degree four. Then we have the following three possibilities: (1) X_f is normal and it has at least a triple point; (2) X_f is normal and it has no triple points; (3) X_f is non-normal. In this paper, we will treat the case (1). The cases (2) and (3) will be dealt with elsewhere. Thus it suffices to consider compactifications (X, Y) of $\mathbb{C}^2$ such that X is a normal quartic hypersurface with at least a triple point in $\mathbb{P}^3$ and Y is a hyperplane section of X. First we will determine the defining equations of such compactifications (X, Y) by using the classification of minimal normal compactifications of $\mathbb{C}^2$ due to Morrow [10] and the notion of *separation* due to Ishii [6] and Ishii-Nakayama [7], which was introduced to classify normal quartic hypersurfaces in $\mathbb{P}^3$ with irrational singularities (cf. [3], [18]). Finally we will explicitly construct algebraic automorphisms of $\mathbb{C}^3$ which linearize the defining equations of the hypersurfaces $X \setminus Y$ of $\mathbb{C}^3$ by using a proposition of Russell [14]. Then we shall obtain a generalization and an analogue of a Nagata automorphism.

From now on to the end of this paper, we assume the following:

ASSUMPTION. Let X be a normal quartic hypersurface with at least a triple point in $\mathbb{P}^3$ and Y a hyperplane section of X such that $X \setminus Y$ is biholomorphic to $\mathbb{C}^2$. Denote by H the hyperplane in $\mathbb{P}^3$ with $Y = X \cap H$.

We define some notations as follows. Let $Y = \bigcup_{i=1}^{t} Y_i$ be the irreducible decomposition of Y. We put $\mathcal{Y} := H|_X$. We note that $\operatorname{Supp} \mathcal{Y} = Y$ and $\mathcal{O}_H(X|_H) \cong \mathcal{O}_{\mathbb{P}^2}(4)$. We put $x := \operatorname{Sing} X = \{x_1, \ldots, x_m\}$ for $m \geq 1$. We may assume that x_1 is a triple point of X. In §2, we shall see that X has only one triple point. Let $l = \bigcup_i l_i$ be the union of lines in X passing through x_1 which are not contained in Y, where the case $l = \emptyset$ is allowed. Let $\pi : M \to X$ be the minimal resolution of X with exceptional set $E = \bigcup_{i=1}^{s} E_i := \pi^{-1}(x)$, where each E_i is irreducible. We denote by $\widehat{C}$ the proper transform of a curve C in X by π. Let $\sigma : \overline{\mathbb{P}^3} \to \mathbb{P}^3$ be the blowing-up at x_1 with exceptional divisor Δ, which is isomorphic to $\mathbb{P}^2$. Let $\overline{X}$ be the proper transform of X by σ. We put $\overline{\mathcal{E}} := \Delta|_{\overline{X}}$ and $\overline{E} = \bigcup_i \overline{E_i} := \overline{X} \cap \Delta$, where each $\overline{E_i}$ is irreducible. We note that $\mathcal{O}_\Delta(\overline{X}|_\Delta) \cong \mathcal{O}_{\mathbb{P}^2}(3)$. In §2, we shall show that $\overline{X}$ is normal and that there exists a birational morphism $\overline{\pi} : M \to \overline{X}$ such that $\pi = (\sigma|_{\overline{X}}) \circ \overline{\pi}$ and such that $\overline{\pi}$ is the minimal resolution of $\overline{X}$. We may assume that, for each $\overline{E_i}$, E_i is its proper transform by $\overline{\pi}$. Then our main results are the following:

Theorem 1. *Let (X, Y) be a pair satisfying Assumption. Then the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l}$ is one of the* Fig. 1, *where one denotes smooth rational curves with self-intersection numbers* 0, (-1), (-2), (-3) *and* (-4) *by* $\odot$, $\bullet$, $\circ$, $\triangle$ *and* $\square$ *respectively and where each* $\circ$ *is an irreducible component of* E.

Theorem 2. *For each dual graph of $\widehat{Y} \cup E \cup \widehat{l}$ in* Theorem 1, *the defining equation of (X, Y) is, up to automorphisms of $\mathbb{P}^3$, one of the following*:

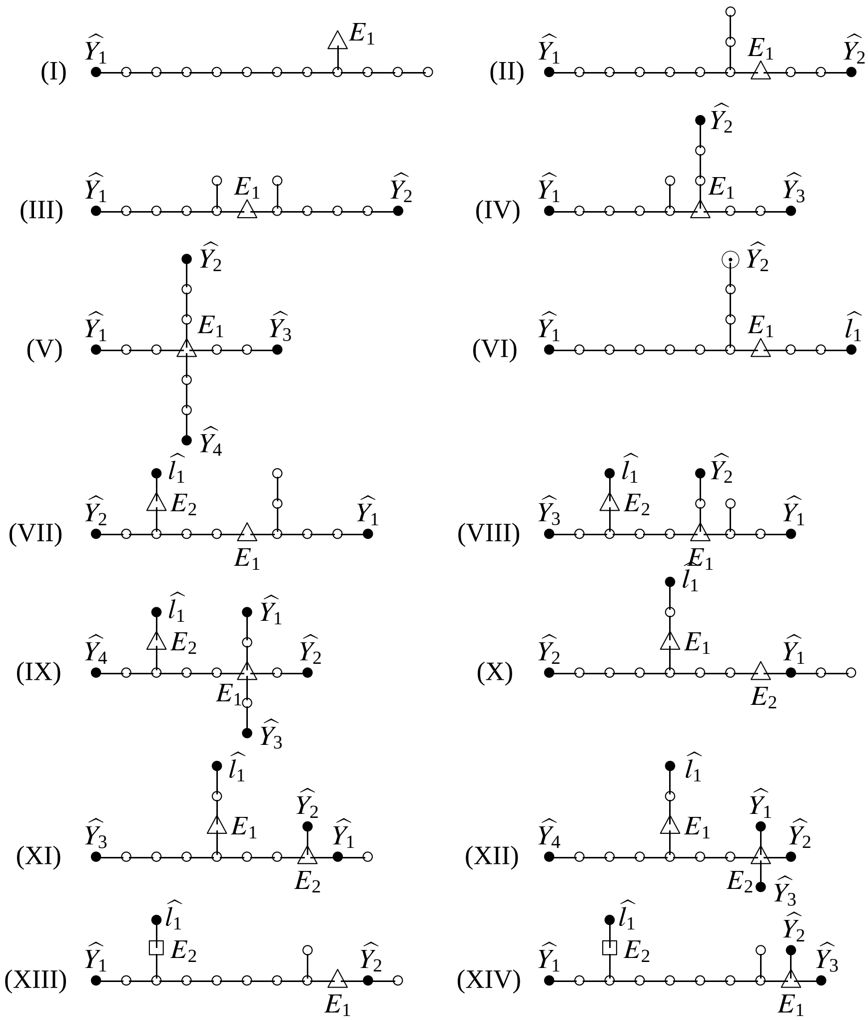

Fig. 1.

(I) $X : (z_2^3)z_3 + z_0^4 + z_2F_1(z'; 1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0.$
(II) $X : (z_2^3)z_3 + z_0^3z_1 + z_2F_1(z'; 1, 0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0.$
(III) $X : (z_2^3)z_3 + z_0^2z_1^2 + z_2F_1(z'; 1, 1, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0.$
(IV) $X : (z_2^3)z_3 + z_0^2z_1(z_0 + z_1) + z_2F_1(z'; 1, 0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0.$
(V) $X : (z_2^3)z_3 + z_0z_1(z_0^2 + \beta z_0z_1 + z_1^2) + z_2F_1(z'; 0, 0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0.$
(VI) $X : z_0(z_0^2z_3 + \gamma z_0z_1^2 + z_1^3) + z_1z_2^3 + z_2F_1(z'; 0, 0, 0, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0.$
(VII) $X : (z_1z_2^2)z_3 + z_0^3z_1 + z_2F_2(z'; 1, 1, \alpha_3, \alpha_4) = 0.$
(VIII) $X : (z_1z_2^2)z_3 + z_0^2z_1(z_0 + z_1) + z_2F_2(z'; 1, \delta, \alpha_3, \alpha_4) = 0.$
(IX) $X : (z_1z_2^2)z_3 + z_0z_1(z_0^2 + \beta z_0z_1 + z_1^2) + z_2F_2(z'; \alpha_1, 1, \alpha_3, \alpha_4) = 0.$

(X) $X : (z_1^2z_2)z_3 + z_0^3z_1 + z_2F_3(z'; 1, \alpha_2, \alpha_3, \alpha_4) = 0.$
(XI) $X : (z_1^2z_2)z_3 + z_0^2z_1(z_0 + z_1) + z_2F_3(z'; 1, \alpha_2, \alpha_3, \alpha_4) = 0.$
(XII) $X : (z_1^2z_2)z_3 + z_0z_1(z_0^2 + \beta z_0z_1 + z_1^2) + z_2F_3(z'; 1, \alpha_2, \alpha_3, \alpha_4) = 0.$
(XIII) $X : (z_0^2 + z_1z_2)z_2z_3 + (z_0^2 + z_1z_2)F_4(z''; \alpha_1, \alpha_2, \delta) + z_0z_2^3 = 0.$
(XIV) $X : (z_0^2 + z_1z_2)z_2z_3 + (z_0^2 + z_1z_2)F_4(z''; \alpha_1, \alpha_2, \delta) + z_0z_2^3 = 0.$

$$\begin{aligned}
F_1(z'; \alpha_1, \dots, \alpha_7) &:= \alpha_1z_1^3 + \alpha_2z_0^3 + \alpha_3z_1^2z_2 + \alpha_4z_0^2z_1 + \alpha_5z_0^2z_2 + \alpha_6z_0z_1^2 + \alpha_7z_0z_1z_2 \\
F_2(z'; \alpha_1, \alpha_2, \alpha_3, \alpha_4) &:= \alpha_1z_1^3 + \alpha_2z_0z_2^2 + \alpha_3z_0^2z_1 + \alpha_4z_0z_1^2 \\
F_3(z'; \alpha_1, \alpha_2, \alpha_3, \alpha_4) &:= \alpha_1z_0z_2^2 + \alpha_2z_1z_2^2 + \alpha_3z_0^2z_1 + \alpha_4z_0z_1z_2 \\
F_4(z''; \alpha_1, \alpha_2, \alpha_3) &:= \alpha_1z_0^2 + \alpha_2z_0z_1 + \alpha_3z_1^2
\end{aligned}$$

where one denotes by $z = (z_0 : z_1 : z_2 : z_3)$ *a homogeneous coordinate system of* $\mathbb{P}^3$ *and puts* $z' := (z_0 : z_1 : z_2)$ *and* $z'' := (z_0 : z_1)$, *where one takes* $\{z_2 = 0\}$ *as* H, *and where* $\alpha_i, \beta, \gamma, \delta$ *are complex parameters with the following conditions*:
(1) $\beta \neq \pm 2$;
(2) $\delta \neq 0$;
(3) $\alpha_2^2 - 4\alpha_1\delta = 0$ *for* (XIII);
(4) $\alpha_2^2 - 4\alpha_1\delta \neq 0$ *for* (XIV).

REMARK. (1) We can obtain the types (II) and (VI) by considering two different hyperplane sections of a common quartic hypersurface. Indeed, we can summarize (II) and (VI) as follows:

$$\text{(II)} + \text{(VI)} \quad X : (z_2^3)z_3 + z_0^3z_1 + z_2F_1(z'; 1, 0, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = 0,$$

where $H = \{\lambda z_0 + \mu z_2 = 0\}$ and $(\lambda : \mu) \in \mathbb{P}^1$ is a parameter. In (II) + (VI), we obtain (II) if $\lambda = 0$ and (VI) if $\lambda \neq 0$. This phenomenon can occur only for the pair of (II) and (VI).
(2) In Theorem 2, the singular loci $x = \operatorname{Sing} X$ are given as follows:

$$\begin{aligned}
\text{(I)} \sim \text{(IX), (XII), (XIV)} \quad & x = \{(0 : 0 : 0 : 1)\}. \\
\text{(X), (XI)} \quad & x = \{(0 : 0 : 0 : 1), (0 : 1 : 0 : 0)\}. \\
\text{(XIII)} \quad & x = \left\{(0 : 0 : 0 : 1), \left(1 : -\frac{\alpha_2}{2\delta} : 0 : 0\right)\right\}.
\end{aligned}$$

For all the cases, the point $(0 : 0 : 0 : 1)$ is the (unique) triple point of X and the rests are rational double points of A_*-type.
(3) In Theorems 1 and 2, the divisors $\overline{\mathcal{E}} = \Delta|_{\overline{X}}$ and $\mathcal{Y} = H|_X$ are given as follows:
(I) $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line). $\mathcal{Y} = 4Y_1$ (Y_1 : line).
(II) $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line). $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i : line).
(III) $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line). $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i : line).
(IV) $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line). $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : line).

(V) $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line). $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i : line).
(VI) $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line). $\mathcal{Y} = Y_1 + Y_2$ (Y_1 : line, Y_2 : cuspidal cubic).
(VII) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line). $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i : line).
(VIII) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line). $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : line).
(IX) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line). $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i : line).
(X) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line). $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i : line).
(XI) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line). $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : line).
(XII) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line). $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i : line).
(XIII) $\overline{\mathcal{E}} = \overline{E_1} + \overline{E_2}$ ($\overline{E_1}$: line, $\overline{E_2}$: conic). $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i : line).
(XIV) $\overline{\mathcal{E}} = \overline{E_1} + \overline{E_2}$ ($\overline{E_1}$: line, $\overline{E_2}$: conic). $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : line).

Here we introduce some special subgroups of $\mathrm{Aut}(\mathbb{C}^n)$. Let $(x_1, \ldots, x_n)$ and $(x'_1, \ldots, x'_n)$ be two coordinate systems of $\mathbb{C}^n$. For an element (a_{ij}) of the general linear group $GL(n, \mathbb{C})$ over $\mathbb{C}$ and $b_1, \ldots, b_n \in \mathbb{C}$, there exists an automorphism of $\mathbb{C}^n$ such that $x'_i = \sum_{j=1}^n a_{ij}x_j + b_i$ $(i = 1, \ldots, n)$. This type of automorphism is called an *affine transformation* of $\mathbb{C}^n$. The set $A(n, \mathbb{C})$ of all affine transformations of $\mathbb{C}^n$ is a subgroup of $\mathrm{Aut}(\mathbb{C}^n)$. For $c_1, \ldots, c_n \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$, $p_i \in \mathbb{C}[x_{i+1}, \ldots, x_n]$ $(i = 1, \ldots, n-1)$ and $p_n \in \mathbb{C}$, there exists an automorphism of $\mathbb{C}^n$ such that $x'_i = c_i x_i + p_i$ $(i = 1, \ldots, n)$. This type of automorphism is called a *de Jonquières automorphism* of $\mathbb{C}^n$. The set $J(n, \mathbb{C})$ of all de Jonquières automorphisms of $\mathbb{C}^n$ is a subgroup of $\mathrm{Aut}(\mathbb{C}^n)$. Let us denote by $J(n, \mathbb{C}) \vee A(n, \mathbb{C})$ the subgroup of $\mathrm{Aut}(\mathbb{C}^n)$ generated by $J(n, \mathbb{C})$ and $A(n, \mathbb{C})$.

Theorem 3. *For each defining equation of* (X, Y) *in* Theorem 2, *there exists an algebraic automorphism* Φ *of* $\mathbb{C}^3$ *which transforms the hypersurface* $X \setminus Y$ *of* $\mathbb{P}^3 \setminus H = \mathbb{C}^3$ *onto a hyperplane of* $\mathbb{C}^3$. *For the type* (VI), *one can take* $\Phi = \Phi_1 \circ \Psi$ *with some* $\Psi \in J(3, \mathbb{C}) \vee A(3, \mathbb{C})$. *For the types* (X), (XI) *and* (XII), *one can take* $\Phi = \Phi_2$. *For the other types, one can take* $\Phi \in J(3, \mathbb{C}) \vee A(3, \mathbb{C})$. *Here, for two coordinate systems* (x, y, z) *and* (x', y', z') *of* $\mathbb{C}^3$, *the automorphisms* Φ_1 *and* Φ_2 *of* $\mathbb{C}^3$ *are defined as follows*:

$$\Phi_1 : \begin{cases} x' = x \\ y' = f_1(x, y) + x^3 z \\ z' = \{g_1(x, f_1(x, y) + x^3 z) - y\}/x^3, \end{cases}$$

$$\Phi_1^{-1} : \begin{cases} x' = x \\ y' = g_1(x, y) - x^3 z \\ z' = \{y - f_1(x, g_1(x, y) - x^3 z)\}/x^3, \end{cases}$$

where $f_1, g_1 \in \mathbb{C}[x, y]$ *with complex parameters* a_i *are defined by*

$$f_1(x, y) := (1 + a_1 x + a_2 x^2)y + (a_3 x + a_4 x^2)y^2 + (x)y^3,$$
$$g_1(x, y) := \{1 - a_1 x + (-a_2 + a_1^2)x^2\}y + \{-a_3 x + (3a_1 a_3 - a_4)x^2\}y^2$$

$$+\{-x + (2a_3^2 + 4a_1)x^2\}y^3 + (5a_3x^2)y^4 + (3x^2)y^5.$$

$$\Phi_2 : \begin{cases} x' = x \\ y' = y + xf_2(x, y, z) \\ z' = z - g_2(x, y, z), \end{cases}$$

$$\Phi_2^{-1} : \begin{cases} x' = x \\ y' = y - xf_2(x, y, z) \\ z' = z + h_2(x, y, z), \end{cases}$$

where $f_2, g_2, h_2 \in \mathbb{C}[x, y, z]$ *with complex parameters* a_{ij} *are defined by*

$$f_2(x, y, z) := xz + \sum_{i,j\geq 0} a_{ij}x^i y^j,$$

$$g_2(x, y, z) := \sum_{i\geq 0, j\geq 1} a_{ij} \left\{ \sum_{k=1}^{j} \binom{j}{k} y^{j-k}x^k f_2^k \right\} x^i \Big/ x,$$

$$h_2(x, y, z) := \sum_{i\geq 0, j\geq 1} a_{ij} \left\{ \sum_{k=1}^{j} \binom{j}{k} (y - xf_2)^{j-k}x^k f_2^k \right\} x^i \Big/ x.$$

REMARK. (1) For the defining equation of Φ_2, putting $a_{02} = 1$ and $a_{ij} = 0$ for $(i, j) \neq (0, 2)$, then we obtain the following automorphism of $\mathbb{C}^3$:

$$\Phi_N : \begin{cases} x' = x \\ y' = y + x(xz + y^2) \\ z' = z - 2y(xz + y^2) - x(xz + y^2)^2, \end{cases}$$

$$\Phi_N^{-1} : \begin{cases} x' = x \\ y' = y - x(xz + y^2) \\ z' = z + 2y(xz + y^2) - x(xz + y^2)^2. \end{cases}$$

The automorphism Φ_N is called a *Nagata automorphism* (cf. [11]). Hence we can regard Φ_2 as a generalization of a Nagata automorphism.
(2) For the defining equation of Φ_1, putting $a_i = 0$ for any $i \geq 0$, we obtain a hypersurface $y + x(x^2z + y^3) = 0$ in $\mathbb{C}^3$ and an automorphism of $\mathbb{C}^3$ which transforms this hypersurface onto a hyperplane of $\mathbb{C}^3$. This hypersurface is analogous to the hypersurface $y + x(xz + y^2) = 0$, which is transformed onto a hyperplane of $\mathbb{C}^3$ by a Nagata automorphism. Thus we can regard Φ_1 as an analogue of a Nagata automorphism.

As a consequence of Theorems 2 and 3, we obtain the following:

Theorem 4. *Let* $f : \mathbb{C}^2 \hookrightarrow \mathbb{C}^3$ *be an algebraic embedding. Assume that* X_f *is a normal quartic hypersurface with at least a triple point. Then there exists an algebraic automorphism* Φ *of* $\mathbb{C}^3$ *such that* $\Phi \circ f$ *is a linear embedding.*

Indeed, if one has such an algebraic embedding f, by Theorem 2, (X_f, Y_f) is, up to automorphisms of $\mathbb{P}^3$, one of the types (I) through (XIV) and, by Theorem 3, there exists an algebraic automorphism of $\mathbb{C}^3$ which transforms the hypersurface $f(\mathbb{C}^2) = X_f \setminus Y_f$ of $\mathbb{C}^3$ onto a hyperplane of $\mathbb{C}^3$. Thus we obtain Theorem 4.

NOTATION.
ω_V: dualizing sheaf of V.
K_V: canonical divisor on V.
$D|_V$: restriction of Cartier divisor D to V.
$\mathfrak{m}_{V,v}$: maximal ideal of $\mathcal{O}_{V,v}$.
$\mathrm{mult}_W V$: multiplicity of V at general point of W.
$\mathrm{Exc}\,\varphi$: exceptional set of birational morphism $\varphi : V \to W$.
$b_i(V) := \dim_{\mathbb{R}} H^i(V;\mathbb{R})$: the i-th Betti number of V.
$(D \cdot C)_{V,v}$: local intersection number of Cartier divisor D and curve C of V at $v \in V$.
$\sim$: linear equivalence.
(V, v): normal two-dimensional singularity.
$p_g(v)$: geometric genus of (V, v).
$p_g(v_1, \cdots, v_m) := \sum_{i=1}^m p_g(v_i)$.
$(-m)$-curve: smooth rational curve with self-intersection number $-m$.
$\odot$: 0-curve.
$\bullet$: (-1)-curve.
$\circ$: (-2)-curve.
$\triangle$: (-3)-curve.
$\square$: (-4)-curve.
$\overset{-m}{\circ}$: $(-m)$-curve.

2. Preliminaries

In this section, we shall describe fundamental properties of a pair (X, Y) satisfying Assumption in §1. We use the same notation as that in §1. Let $Y = \bigcup_{i=1}^t Y_i$ be the irreducible decomposition of Y. We denote by $\deg Y_i$ the degree of Y_i as a plane curve of $H \cong \mathbb{P}^2$. We put $\mathcal{Y} := H|_X = \sum_{i=1}^t k_i Y_i$, where $\sum_{i=1}^t k_i \deg Y_i = 4$. We put $x := \mathrm{Sing}\, X = \{x_1, \ldots, x_m\}$ for $m \geq 1$. We may assume that x_1 is a triple point of X. Let $\pi : M \to X$ be the minimal resolution of X with exceptional set $E = \bigcup_{i=1}^s E_i := \pi^{-1}(x)$, where each E_i is irreducible. Let Γ be a smooth hyperplane section of X with $\Gamma \cap x = \emptyset$ and H_Γ a hyperplane in $\mathbb{P}^3$ such that $\Gamma = X \cap H_\Gamma$. We denote by $\widehat{C}$ the proper transform of a curve C in X by π. We set $A := \widehat{Y} \cup E$. Here we note that $\omega_X = \mathcal{O}_X$ and $x \subset Y$ and that $M \setminus A$ is biholomorphic to $\mathbb{C}^2$. By Kodaira [8] and Ramanujam [13], we see that $X \setminus Y$ and $M \setminus A$ are biregular to $\mathbb{C}^2$ and, in particular, that X and M are rational surfaces. Then we obtain the following:

Proposition 2.1 ([12]). (i) $H_0(X,\mathbb{Z}) \cong H_0(Y,\mathbb{Z}) = \mathbb{Z}$.
(ii) $H_1(X,\mathbb{Z}) \cong H_1(Y,\mathbb{Z}) = 0$.
(iii) $H_2(X,\mathbb{Z}) \cong H_2(Y,\mathbb{Z}) = \bigoplus_{i=1}^{t} \mathbb{Z}.Y_i$.
(iv) $H_3(X,\mathbb{Z}) \cong H_3(Y,\mathbb{Z}) = 0$.
(v) $H^1(X,\mathcal{O}_X) = 0$.
(vi) $p_g(x) = 1$.
(vii) *X is not a cone.*
(viii) $\gcd(\deg Y_1,\dots,\deg Y_t) = 1$.
(ix) $\operatorname{mult}_p X \leq \sum_{i=1}^{t} k_i \operatorname{mult}_p Y_i$ $(\forall p \in Y = X \cap H)$.

REMARK. (1) We note that Y is connected by (i) and that Y cannot have any cycles, that is, each Y_i is a rational curve without nodes by (ii). If Y contains more than two lines, then Y consists of lines which meet only at one point. Indeed, this follows since Y cannot have any cycles and each Y_i is a plane curve.
(2) Since $p_g(x) = 1$ and x_1 is a triple point, we obtain $p_g(x_1) = 1$, that is, x_1 is a minimally elliptic singular point. If x contains at least two points, then $x \setminus \{x_1\}$ consists of rational double points. Hence x_1 is a unique triple point of X. By Artin [2] and Laufer [9], we obtain $K_M \sim \pi^* K_X - Z \sim -Z$ and $Z^2 = -3$, where Z is the fundamental cycle of $\pi^{-1}(x_1)$.

Next we consider the projection from x_1 and the blowing-up at x_1 to investigate the compactification (X,Y). We denote by N the number of lines in X through x_1. Since X is not a cone by Proposition 2.1(vii), we obtain $0 \leq N < +\infty$. Let L be the union of these lines and l the closure of $L \setminus Y$, where the case $l = \emptyset$ is allowed. Let $l = \bigcup_i l_i$ be the irreducible decomposition of l. Let $\sigma : \overline{\mathbb{P}^3} \to \mathbb{P}^3$ be the blowing-up at x_1 with exceptional divisor Δ, which is isomorphic to $\mathbb{P}^2$. Then we have $\sigma|_{\overline{\mathbb{P}^3}\setminus\Delta} : \overline{\mathbb{P}^3}\setminus\Delta \cong \mathbb{P}^3\setminus\{x_1\}$ and $\mathcal{O}_{\overline{\mathbb{P}^3}}(\Delta)|_\Delta \cong \mathcal{O}_{\mathbb{P}^2}(-1)$. We denote by $\overline{V}$ the proper transform of a subvariety V of $\mathbb{P}^3$ by σ. We set $\overline{x} := \operatorname{Sing}\overline{X}$ and $\overline{E} = \bigcup_i \overline{E_i} := \overline{X} \cap \Delta$, where each $\overline{E_i}$ is irreducible. We put $\overline{\mathcal{E}} := \Delta|_{\overline{X}} = \sum_i e_i \overline{E_i}$ with $\sum_i e_i \deg \overline{E_i} = 3$, where $\deg \overline{E_i}$ is the degree of $\overline{E_i}$ as a plane curve of $\Delta \cong \mathbb{P}^2$. Since $\sigma|_{\overline{X}\setminus\overline{E}} : \overline{X}\setminus\overline{E} \cong X\setminus\{x_1\}$, we can easily see that $(\overline{X}, \overline{Y}\cup\overline{E})$ is a compactification of $\mathbb{C}^2$. Then we note that $\overline{x} \subset \overline{Y}\cup\overline{E}$ and $\omega_{\overline{X}} = \mathcal{O}_{\overline{X}}(-\mathcal{E})$ and that $\overline{Y}\cup\overline{E}$ does not have any cycles.

Let $\psi : \mathbb{P}^3 \cdots \to \mathbb{P}^2$ be the projection from x_1 and $\overline{\psi} : \overline{\mathbb{P}^3} \to \mathbb{P}^2$ the resolution of indeterminacy of ψ. We put $\overline{V}^* := \overline{\psi}(\overline{V})$, $\overline{E_i}^* := \overline{\psi}(\overline{E_i})$ and $\overline{E}^* := \bigcup_i \overline{E_i}^*$. Here we note that $\overline{\psi}|_\Delta : \Delta \cong \mathbb{P}^2$ and that $\overline{\Gamma}^*$ is a smooth plane quartic curve. Since $\deg(\overline{\psi}|_{\overline{X}}) = \deg X - \operatorname{mult}_{x_1} X = 1$, we see that $\overline{\psi}|_{\overline{X}} : \overline{X} \to \mathbb{P}^2$ is a birational morphism. In particular, we obtain $\overline{\psi}|_{\overline{X}\setminus\overline{L}} : \overline{X}\setminus\overline{L} \cong \mathbb{P}^2 \setminus \overline{L}^*$, $\overline{x} \subset \overline{L}$ and $x \subset L$. Since $\overline{H}^*$ is a line in $\mathbb{P}^2$ containing $\overline{Y}^*$, we have either that $\overline{Y}^*$ consists of finite points or that $\overline{Y}^*$ is a line. Since $\overline{\Gamma}\cap\overline{E}$ is empty and each irreducible component of $\overline{L}$ meets both of $\overline{\Gamma}$ and $\overline{E}$, we obtain $\overline{\Gamma}^* \cap \overline{E}^* = \overline{L}^* = (\overline{\psi}|_{\overline{X}})(\operatorname{Exc}(\overline{\psi}|_{\overline{X}}))$. Then we obtain the following:

$$\begin{tikzcd}
M \arrow[r, "\overline{\pi}"] \arrow[dr, "\pi"'] & \overline{X} \arrow[d, "\sigma|_{\overline{X}}"'] \arrow[dr, "\overline{\psi}|_{\overline{X}}"] & \\
& X \arrow[r, dashed, "\psi|_X"'] & \mathbb{P}^2
\end{tikzcd}$$

Fig. 2.

Proposition 2.2 ([12]). (i) $0 < N < +\infty$, $N = b_2(\overline{X}) - 1$.
(ii) $\mathrm{mult}_q \overline{X} \leq \sum_i e_i \,\mathrm{mult}_q \overline{E_i}$ $(\forall q \in \overline{E} = \overline{X} \cap \Delta)$.
(iii) $\overline{X}$ *is a normal hypersurface with at most rational double points of* A_**-type.*
(iv) *There exists a birational morphism* $\overline{\pi} : M \to \overline{X}$ *such that it is the minimal resolution of* $\overline{X}$ *and satisfies the commutative diagram in* Fig. 2.
(v) *If* γ *is a line in* X *through* x_1*, then* $\widehat{\gamma}$ *is a* (-1)*-curve in* M *and* $\overline{x} \cap \overline{\gamma}$ *consists of at most one rational double point of* A_**-type. Moreover, if* $\overline{x} \cap \overline{\gamma} \neq \emptyset$*, then the weighted dual graph of* $\widehat{\gamma} \cup \overline{\pi}^{-1}(\overline{x} \cap \overline{\gamma})$ *is a linear tree* ●—○—○—$\cdots$—○ .
(vi) $\overline{Y}^*$ *consists of finite points if and only if each irreducible component of* Y *is a line through* x_1*. Then there exists a line* $\overline{E_{j_1}} \subset \overline{E}$ *in* $\Delta \cong \mathbb{P}^2$ *such that* $\overline{H}^* = \overline{E_{j_1}}^*$ *and* $\mathcal{Y} = \sum_{i=1}^t (\overline{\Gamma}^* \cdot \overline{E_{j_1}}^*)_{\mathbb{P}^2, Y_i{}^*} \cdot Y_i$.
(vii) $\overline{Y}^*$ *is a line if and only if* $Y = Y_1 \cup Y_2$ *where* Y_1 *is a line through* x_1 *and where* Y_2 *is not a line through* x_1*. Then one has the following*:

(a) $\overline{H}^* = \overline{Y}^* = \overline{Y_2}^*$;
(b) $\overline{Y}^* \not\subset \overline{E}^*$ *and* $\overline{Y}^* \cup \overline{E}^*$ *does not have any cycles*;
(c) *if* $\overline{E}^*$ *contains a line in* $\mathbb{P}^2$*, then* $\overline{E}^*$ *consists of lines in* $\mathbb{P}^2$.

Proof. It suffices to show (vi) since we obtain all the assertions except (vi) by Ohta [12]. The first claim in (vi) is obvious. Let us consider the second one in (vi). Since the morphism $\overline{\psi}|_{\overline{X}} : \overline{X} \to \mathbb{P}^2$ is surjective, there exists a line $\overline{E_{j_1}} \subset \overline{E}$ in $\Delta \cong \mathbb{P}^2$ such that $\overline{H}^* = \overline{E_{j_1}}^*$. For the intersection divisor $\mathcal{Y} = H|_X = \sum_{i=1}^t k_i Y_i$ of X, we show that $k_i = (\overline{\Gamma}^* \cdot \overline{E_{j_1}}^*)_{\mathbb{P}^2, Y_i{}^*}$ $(1 \leq i \leq t)$. We put $C := H \cap H_\Gamma$, which is a line in $\mathbb{P}^3$, and $\{P_i\} := Y_i \cap C$ $(1 \leq i \leq t)$. Since $(X|_H)|_C = (X|_{H_\Gamma})|_C$, we obtain $\Gamma \cap C = \{P_1, \ldots, P_t\}$ and $\sum_{i=1}^t (X|_H \cdot C)_{H,P_i} \cdot P_i = \sum_{i=1}^t (X|_{H_\Gamma} \cdot C)_{H_\Gamma, P_i} \cdot P_i$ as Weil divisors of $C \cong \mathbb{P}^1$. By noting that $X|_H = \sum_{i=1}^t k_i Y_i$ and that $(\overline{\psi}|_{\overline{H_\Gamma}}) \circ (\sigma|_{\overline{H_\Gamma}})^{-1} : H_\Gamma \cong \overline{H_\Gamma} \cong \mathbb{P}^2$ and $\overline{C}^* = \overline{H}^* = \overline{E_{j_1}}^*$, we get $k_i = (X|_H \cdot C)_{H,P_i} = (\Gamma \cdot C)_{H_\Gamma, P_i} = (\overline{\Gamma}^* \cdot \overline{E_{j_1}}^*)_{\mathbb{P}^2, Y_i{}^*}$ $(1 \leq i \leq t)$. Thus we obtain (vi). $\square$

REMARK. By (iv), we may assume that, for each $\overline{E_i}$, E_i is the proper transform of $\overline{E_i}$ by $\overline{\pi}$. We also note that, for a curve C in X, $\widehat{C}$ coincides with the proper transform of $\overline{C}$ by $\overline{\pi}$.

Now we introduce a notion of separation from Ishii [6] and Ishii-Nakayama [7].

This notion was introduced to classify normal quartic hypersurfaces in $\mathbb{P}^3$ with irrational singularities (cf. [3], [18]). Here we shall show only the existence and the uniqueness of separation.

DEFINITION 2.3 ([6], [7]). Let (S, C, D) be a triplet consisting of a nonsingular projective surface S, a smooth curve C on S and an effective anti-canonical divisor D of S. Assume that C is not a component of D. Let $\rho : T \to S$ be a birational morphism from a nonsingular projective surface T and C_T, D_T effective divisors of T. (T, C_T, D_T) is said to be a *separation* of (S, C, D), if the following conditions are satisfied:
(i) $K_T + C_T \sim \rho^*(K_S + C)$.
(ii) $K_T + D_T \sim 0$.
(iii) $C_T \le \rho^*(C)$, $D_T \le \rho^*(D)$.
(iv) $\mathrm{Supp}(C_T) \cap \mathrm{Supp}(D_T) = \emptyset$.

Proposition 2.4 ([6], [7]). *Separation exists uniquely.*

Proof. If $\mathrm{Supp}(C) \cap \mathrm{Supp}(D)$ is empty, then the identity $S \to S$ is a separation. Hence we may assume that $\mathrm{Supp}(C) \cap \mathrm{Supp}(D) \neq \emptyset$. Let $\rho_1 : T_1 \to S$ be the blowing-up at a point $P_1 \in \mathrm{Supp}(C) \cap \mathrm{Supp}(D)$ and B_1 the exceptional divisor. We consider $C_{T_1} := \rho_1^*(C) - B_1$ and $D_{T_1} := \rho_1^*(D) - B_1$. We note that C_{T_1} is the proper transform of C and $(C_{T_1} \cdot D_{T_1}) = (C \cdot D) - 1$. If $\mathrm{Supp}(C_{T_1}) \cap \mathrm{Supp}(D_{T_1}) \neq \emptyset$, then we blow up at a point $P_2 \in \mathrm{Supp}(C_{T_1}) \cap \mathrm{Supp}(D_{T_1})$, and similarly we can define C_{T_2} and D_{T_2}. Thus, by continuing this procedure, we finally get a separation.

Conversely, let (T, C_T, D_T) be a separation of (S, C, D) and $\rho : T \to S$ the birational morphism. We note that ρ is a composite of blowing-ups. By Definition 2.3(i) and (iii), C_T is the proper transform of C in T and hence it is a smooth curve isomorphic to C. Let $\mathrm{Exc}\,\rho = \bigcup_{i=1}^r B_i$ be the decomposition into connected components. We denote by n_i the number of irreducible components of B_i and put $P_i := \rho(B_i)$. Let γ be a ρ-exceptional curve. Since C_T and D_T are ρ-nef, by the adjunction formula, γ is either a (-1)-curve with $(C_T \cdot \gamma) = (D_T \cdot \gamma) = 1$ or a (-2)-curve with $(C_T \cdot \gamma) = (D_T \cdot \gamma) = 0$. Hence the weighted dual graph of B_i is one vertex $\bullet$ or a linear tree $\bullet\!-\!\!\circ\!-\!\!\circ\!-\cdots-\!\circ$.

By Definition 2.3(i),(ii),(iii) and the negative definiteness of the intersection matrix of $\mathrm{Exc}\,\rho$, there exists an effective divisor B with $\mathrm{Supp}(B) = \mathrm{Exc}\,\rho$ such that
(1) $K_T \sim \rho^* K_S + B$;
(2) $\rho^* C = C_T + B$;
(3) $\rho^* D = D_T + B$.
Then we obtain $\rho_*(C_T) = C$, $\rho_*(D_T) = D$ and $\mathrm{Supp}(C) \cap \mathrm{Supp}(D) = \{P_1, \dots, P_r\}$. By noting $(C \cdot D) = (\rho^* C \cdot \rho^* D) = (C_T \cdot B)$ and by computing the intersection numbers of B and its irreducible components, we get $(C \cdot D)_{P_i} = n_i$ $(1 \le i \le r)$. Hence ρ is obtained

$$\begin{tikzcd} M \arrow[d, "\tau"'] \arrow[dr, "\pi"] & \\ \mathbb{P}^2 \arrow[r, dashed, "\phi"'] & X \end{tikzcd}$$

Fig. 3.

by $(C \cdot D)_{P_i}$ times blowing-ups at the points which are on the proper transforms of C and are infinitely near P_i for each $1 \leq i \leq r$. Since $(C \cdot D)_{P_i}$ $(1 \leq i \leq r)$ depend only on the triplet (S, C, D), ρ and T are unique. Since C_T is the proper transform of C by ρ, C_T is also unique. Hence, by (2) and (3), B and D_T are also unique. Thus we obtain the uniqueness of separation. $\square$

Here we return to our situation. Since $\pi^*\mathfrak{m}_{X,x_1} \cong \mathcal{O}_M(-Z)$ by Laufer [9], we have that $Z = \overline{\pi}^*(\Delta|_{\overline{X}}) = \overline{\pi}^*(\overline{\mathcal{E}})$ and $Z \leq (\overline{\pi})^*(\overline{\psi}|_{\overline{X}})^*(\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$. From this, we easily see that the triplet $(M, \widehat{\Gamma}, Z)$ with birational morphism $(\overline{\psi}|_{\overline{X}}) \circ \overline{\pi} : M \to \mathbb{P}^2$ is a (unique) separation of the triplet $(\mathbb{P}^2, \overline{\Gamma}^*, (\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}}))$. Since Proposition 2.1(iii) also holds for the compactification (M, A) of $\mathbb{C}^2$, by using the Noether formula, we obtain $b_2(\widehat{Y}) + b_2(E) = b_2(M) = 10 - K_M^2 = 13$. Thus $\widehat{Y} \cup E$ consists of thirteen irreducible components and $(\overline{\psi}|_{\overline{X}}) \circ \overline{\pi} : M \to \mathbb{P}^2$ is a composite of twelve blowing-ups. Hence, if we know the shape of the divisor $(\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$ and the intersection of $\overline{\Gamma}^*$ and $(\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$, then we can obtain the process of the twelve blowing-ups and the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l} \cup \widehat{\Gamma}$ by using the construction of separation in the proof of Proposition 2.4.

In the last part of this section, we prepare a proposition to write down the defining equation of (X, Y). First we put $\tau := (\overline{\psi}|_{\overline{X}}) \circ \overline{\pi}$ and $\phi := \pi \circ \tau^{-1}$. Then we obtain the commutative diagram in Fig. 3. Let $\widehat{\Lambda}$ be the linear system associated to $\pi : M \to X$. Then $\Lambda := \tau_*\widehat{\Lambda}$ is the linear system associated to $\phi : \mathbb{P}^2 \cdots \to X$. Let $\mathbb{M}_\Lambda$ and $\mathbb{M}_{\widehat{\Lambda}}$ be the $\mathbb{C}$-vector spaces associated to Λ and $\widehat{\Lambda}$ respectively. We note that $\dim \Lambda = \dim \widehat{\Lambda} = 3$. Let H_i $(0 \leq i \leq 3)$ be four general hyperplanes in $\mathbb{P}^3$ such that $x_1 \in H_0, H_1, H_2$ and $x_1 \notin H_3$. We can take H_Γ as H_3. We put $L_i := \overline{H_i}^*$ $(0 \leq i \leq 2)$. Let $(w_0 : w_1 : w_2)$ and $(z_0 : z_1 : z_2 : z_3)$ be homogeneous coordinate systems of $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively. By considering suitable automorphisms of $\mathbb{P}^2$ and $\mathbb{P}^3$, we may assume that $x_1 = (0 : 0 : 0 : 1)$, $H_i = \{z_i = 0\}$ $(0 \leq i \leq 3)$ and $L_j = \{w_j = 0\}$ $(0 \leq j \leq 2)$. Then we note that $H = \{c_0z_0 + c_1z_1 + c_2z_2 = 0\}$ and $\overline{H}^* = \{c_0w_0 + c_1w_1 + c_2w_2 = 0\}$ for some $(c_0 : c_1 : c_2) \in \mathbb{P}^2$. Let F and G be the homogeneous polynomials of w_0, w_1, w_2 of degree four and three which define the divisors $\overline{\Gamma}^*$ and $(\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$ respectively.

Proposition 2.5 ([12]). (i) *$\widehat{\Lambda} = |\widehat{\Gamma}|$ and $\mathbb{M}_{\widehat{\Lambda}}$ is spanned by sections corresponding to the divisors $\pi^*(H_i|_X)$ $(0 \leq i \leq 3)$.*
(ii) *$\Lambda \subset |\overline{\Gamma}^*| = |\mathcal{O}_{\mathbb{P}^2}(4)|$ and $\mathbb{M}_\Lambda$ is spanned by sections corresponding to the divisors*

$L_i + (\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$ $(0 \leq i \leq 2)$ *and* $\overline{\Gamma}^*$. *In particular, the birational map* ϕ, *the image* X *and the boundary* Y *are given as follows*:

$$\phi : \begin{cases} z_0 = w_0 G(w_0, w_1, w_2) \\ z_1 = w_1 G(w_0, w_1, w_2) \\ z_2 = w_2 G(w_0, w_1, w_2) \\ z_3 = F(w_0, w_1, w_2). \end{cases}$$

$$\begin{cases} X : F(z_0, z_1, z_2) - z_3 G(z_0, z_1, z_2) = 0. \\ Y : F(z_0, z_1, z_2) - z_3 G(z_0, z_1, z_2) = 0, \quad c_0 z_0 + c_1 z_1 + c_2 z_2 = 0. \end{cases}$$

3. Determination of Boundaries

In this section, we shall give a proof of Theorem 1. Let (X, Y) be a pair satisfying Assumption in §1. We use the same notation as that in §1 and §2. First we obtain classifications of the divisors $\mathcal{Y}$ and $\overline{\mathcal{E}}$ as follows:

Proposition 3.1. *There exist the following seven possibilities for the divisor* $\mathcal{Y}$:
(i) $\mathcal{Y} = 4Y_1$ $(Y_1 : line)$. *In this case,* $x \subset Y_1$ *and* $x = \{x_1\}$ *or* $\{x_1, A_*\}$.
(ii) $\mathcal{Y} = 3Y_1 + Y_2$ $(Y_i : line)$. *In this case,* $x \subset Y_1$ *and* $x = \{x_1\}$ *or* $\{x_1, A_*\}$.
(iii) $\mathcal{Y} = 2Y_1 + 2Y_2$ $(Y_i : line)$. *In this case,* $Y_1 \cap Y_2 = \{x_1\}$ *and* $x \cap Y_i = \{x_1\}$ *or* $\{x_1, A_*\}$ $(i = 1, 2)$.
(iv) $\mathcal{Y} = 2Y_1 + Y_2$ $(Y_1 : line, Y_2 : conic)$. *In this case,* Y_1 *and* Y_2 *meet tangentially to the second order at* x_1, *and* $x \subset Y_1$ *and* $x = \{x_1\}$ *or* $\{x_1, A_*\}$.
(v) $\mathcal{Y} = Y_1 + Y_2$ $(Y_1 : line, Y_2 : cuspidal\ cubic)$. *In this case,* Y_1 *and* Y_2 *meet tangentially to the third order at* x_1, *and* $x = \{x_1\} = \operatorname{Sing} Y_2$.
(vi) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ $(Y_i : line)$. *In this case,* Y_1, Y_2 *and* Y_3 *meet only at* x_1, *and* $x \subset Y_1$ *and* $x = \{x_1\}$ *or* $\{x_1, A_*\}$.
(vii) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ $(Y_i : line)$. *In this case,* Y_1, Y_2, Y_3 *and* Y_4 *meet only at* x_1, *and* $x = \{x_1\}$.

Proof. We note that the divisor $\mathcal{Y}$ is a plane quartic curve. By Proposition 2.1(ii) and (viii), we see that $\mathcal{Y}$ is not an irreducible quartic and that $Y = \operatorname{Supp} \mathcal{Y}$ does not have any cycles. Hence we obtain the above seven cases for the divisor $\mathcal{Y}$. By Proposition 2.1(ix) and 2.2(v), we get the position and the number of elements of the singular locus x. Thus we complete the proof. □

Proposition 3.2. *There exist the following five possibilities for the divisor* $\overline{\mathcal{E}}$:
(i) $\overline{\mathcal{E}} = 3\overline{E_1}$ $(\overline{E_1} : line)$. *In this case,* $\overline{x} \cap \Delta \subset \overline{E_1}$.
(ii) $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ $(\overline{E_i} : line)$. *In this case,* $\overline{x} \cap \Delta \subset \overline{E_1}$.
(iii) $\overline{\mathcal{E}} = \overline{E_1} + \overline{E_2}$ $(\overline{E_1} : line, \overline{E_2} : conic)$. *In this case,* $\overline{E_1}$ *and* $\overline{E_2}$ *meet tangentially to the second order at a point, and* $\overline{x} \cap \Delta \subset \overline{E_1} \cap \overline{E_2}$.
(iv) $\overline{\mathcal{E}} = \overline{E_1} + \overline{E_2} + \overline{E_3}$ $(\overline{E_i} : line)$. *In this case,* $\overline{E_1}, \overline{E_2}$ *and* $\overline{E_3}$ *meet only at one point,*

and $\overline{x} \cap \Delta \subset \overline{E_1} \cap \overline{E_2} \cap \overline{E_3}$.
(v) $\overline{\mathcal{E}} = \overline{E_1}$ ($\overline{E_1}$: *cuspidal cubic*). *In this case,* $\overline{x} \cap \Delta \subset \operatorname{Sing} \overline{E_1}$.

Proof. We note that the divisor $\overline{\mathcal{E}}$ is a plane cubic curve. Since $\overline{E} = \operatorname{Supp} \overline{\mathcal{E}}$ does not have any cycles, we obtain the above five cases for the divisor $\overline{\mathcal{E}}$. By Proposition 2.2(ii), we have the position of $\overline{x} \cap \Delta$ in $\overline{E}$. Thus we have the assertion. □

From now on, we will investigate the five cases for the divisor $\overline{\mathcal{E}}$ in Proposition 3.2. For each case, we will determine the intersection of $\overline{\Gamma}^*$ and $\overline{E}^*$ and get the weighted dual graph of A by using separation. Next we will transform the smooth compactification (M, A) of $\mathbb{C}^2$ into a minimal normal compactification (M', A') of $\mathbb{C}^2$ by blowing-up and blowing-down in the boundary A repeatedly. Then the weighted dual graph of A' must be a linear tree of smooth rational curves by Ramanujam [13] (cf. [10], [12]). Here a smooth compactification (S, C) of $\mathbb{C}^2$ is said to be *minimally normal* if the pair satisfies the following two conditions:
(1) the curve $C = \bigcup_i C_i$, which is the irreducible decomposition, has at most ordinary double points;
(2) if C_j is a (-1)-curve, then there exist at least three irreducible components of C which are different from C_j and intersect C_j.

3.1. The case $\overline{\mathcal{E}}$ = 3line. Let $\overline{\mathcal{E}} = 3\overline{E_1}$ ($\overline{E_1}$: line) be the restriction of Δ to $\overline{X}$. Then we have the following two cases:
(1) $\overline{Y}^*$ consists of finite points;
(2) $\overline{Y}^*$ is a line.

3.1.1. The case $\overline{Y}^*$ consists of finite points.

Lemma 3.3. (i) $\overline{H}^* = \overline{E_1}^*$.
(ii) *Any lines in X passing through x_1 are contained in Y.*
(iii) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y}^*$ (*at most four points*).

Proof. By Proposition 2.2(i) and (vi), we obtain the assertions easily. □

Proposition 3.4. *For the case $\overline{Y}^*$ consists finite points, one has the following five possibilities*:
(i) $\mathcal{Y} = 4Y_1$ (Y_1 : *line*). *In this case,* $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 4$.
(ii) $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i : *line*). *In this case,* $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_2}^*} = 1$.
(iii) $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i : *line*). *In this case,* $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_i}^*} = 2$ $(i = 1, 2)$.
(iv) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : *line*). *In this case,* $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 2$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_i}^*} = 1$ $(i = 2, 3)$.
(v) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i : *line*). *In this case,* $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_i}^*} = 1$ $(i = 1, 2, 3, 4)$.
Moreover, for the cases (i), (ii), (iii), (iv) *and* (v), *one obtains the weighted dual graphs of $\widehat{Y} \cup E$ of type* (I), (II), (III), (IV) *and* (V) *in* Theorem 1 *respectively.*

Proof. By Proposition 2.2(vi) and Lemma 3.3(iii) and by using separation, we have the assertions. □

3.1.2. The case $\overline{Y}^*$ is a line. By Proposition 2.2(vii), we note that $Y = Y_1 \cup Y_2$ where Y_1 is a line through x_1 and where Y_2 is not a line through x_1 and that $\overline{H}^* = \overline{Y}^* = \overline{Y_2}^*$. Then we obtain the following:

Lemma 3.5. (i) $\overline{Y_2}^* \neq \overline{E_1}^*$, $\overline{Y_1}^* = \overline{Y_2}^* \cap \overline{E_1}^*$.
(ii) *There exists only one line l_1 in X through x_1 such that $l_1 \not\subset Y$.*
(iii) $\overline{l_1}^* \subset \overline{E_1}^* \setminus \overline{Y_1}^*$.
(iv) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y_1}^* \cup \overline{l_1}^*$ (*exactly two points*).

Proof. (i) Since $\overline{\psi}|_{\overline{X}} : \overline{X} \to \mathbb{P}^2$ is a birational morphism, we obtain $\overline{Y_2}^* \neq \overline{E_1}^*$. Since Y_1 is a line in H through x_1, we obtain $\overline{Y_1}^* = \overline{H}^* \cap \overline{E_1}^* = \overline{Y_2}^* \cap \overline{E_1}^*$.
(ii) By Proposition 2.2(i), we obtain the assertion easily.
(iii) Since l_1 is a line through x_1 and $l_1 \not\subset H$, we obtain $\overline{l_1}^* \subset \overline{E_1}^* \setminus \overline{H}^* = \overline{E_1}^* \setminus \overline{Y_1}^*$.
(iv) As mentioned before Proposition 2.2, we obtain $\overline{\Gamma}^* \cap \overline{E}^* = (\overline{\psi}|_{\overline{X}})(\mathrm{Exc}(\overline{\psi}|_{\overline{X}})) = \overline{Y_1}^* \cup \overline{l_1}^*$. □

Proposition 3.6. *For the case $\overline{Y}^*$ is a line, one has the following*:
(i) $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = (\overline{\Gamma}^* \cdot \overline{Y_2}^*)_{\overline{Y_1}^*} = 1$.
(ii) $\mathcal{Y} = Y_1 + Y_2$ ($Y_1 : line, Y_2 : cuspidal\ cubic$), *where Y_1 and Y_2 meet tangentially to the third order at x_1 and $x = \{x_1\} = \mathrm{Sing}\, Y_2$.*
Moreover, one obtains the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l}$ of type (VI) *in* Theorem 1.

Proof. Since $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y_1}^* \cup \overline{l_1}^*$, we obtain the following three cases:
(1) $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 1$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 3$.
(2) $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 2$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 2$.
(3) $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$.
(1) In this case, we note that $1 \leq (\overline{\Gamma}^* \cdot \overline{Y_2}^*)_{\overline{Y_1}^*} \leq 3$. By using separation, we obtain the weighted dual graph of $\widehat{Y_2} \cup E_1 \cup \mathrm{Exc}((\overline{\psi}|_{\overline{X}}) \circ \overline{\pi}) = \widehat{Y} \cup E \cup \widehat{l_1}$ in Fig. 4. Since $\widehat{Y} \cup E$ is the simple normal crossing boundary curve of a smooth compactification of $\mathbb{C}^2$, by contracting suitable (-1)-curves in $\widehat{Y} \cup E$ successively, we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction.
(2) In this case, we note that $(\overline{\Gamma}^* \cdot \overline{Y_2}^*)_{\overline{Y_1}^*} = 1$. Similarly to the case (1), we obtain the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1}$ in Fig. 5, and we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction.
(3) In this case, we note that $(\overline{\Gamma}^* \cdot \overline{Y_2}^*)_{\overline{Y_1}^*} = 1$. By using separation, we obtain the

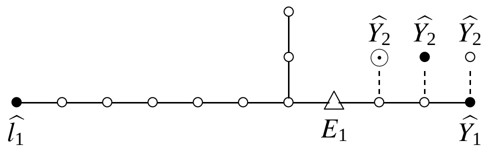

Fig. 4.

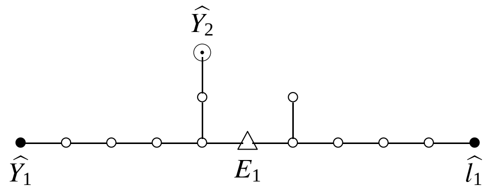

Fig. 5.

weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1}$ of type (VI) in Theorem 1. At the same time, by looking at the process of the separation, we know that $3 = (\widehat{\Gamma} \cdot \widehat{Y_2})_M = (\Gamma \cdot Y_2)_X$ and, in particular, that Y_2 is a cuspidal cubic. $\square$

3.2. The case $\overline{\mathcal{E}}$ = 2line + line. Let $\overline{\mathcal{E}} = 2\overline{E_1} + \overline{E_2}$ ($\overline{E_i}$: line) be the restriction of Δ to $\overline{X}$. We set $\{P\} := \overline{E_1}^* \cap \overline{E_2}^*$. Then we have the following two cases:

(1) $\overline{Y}^*$ consists of finite points;

(2) $\overline{Y}^*$ is a line.

3.2.1. The case $\overline{Y}^*$ consists of finite points.

Lemma 3.7. (i) *$\overline{H}^* = \overline{E_1}^*$ or $\overline{E_2}^*$.*

(ii) *There exists only one line l_1 in X through x_1 such that $l_1 \not\subset Y$.*

(iii) $\overline{l_1}^* \neq \{P\}$.

(iv) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y}^* \cup \overline{l_1}^*$ (*at most five points*).

Proof. Similarly to Lemma 3.5, we obtain the assertions. $\square$

Lemma 3.8. *Assume that $\overline{H}^* = \overline{E_1}^*$. Then one obtains the following*:

(i) $\overline{l_1}^* \subset \overline{E_2}^* \setminus \{P\}$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$.

(ii) *There exists a unique irreducible component Y_{i_1} of Y such that $\overline{Y_{i_1}}^* = \{P\}$ and* $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_{i_1}}^*} = 3$.

(iii) $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_{i_1}}^*} = 1$.

(iv) $\mathcal{Y} = Y_{i_1} + \sum_{i \neq i_1} k_i Y_i$.

Proof. We have $\overline{l_1}^* \subset \overline{E_2}^* \setminus \{P\}$ clearly. Since $x \cap l_1 = \{x_1\}$ and $\overline{x} \cap \Delta \subset \overline{E_1}$, we know by Proposition 2.2(v) that $\overline{x} \cap \overline{l_1} = \emptyset$ and $\overline{l_1}$ is a (-1)-curve in $\overline{X} \setminus \overline{x}$. Since $(\overline{\Gamma} \cdot \overline{l_1})_{\overline{X}} = (\overline{E_2} \cdot \overline{l_1})_{\overline{X}} = 1$, we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$. Thus we have (i). By (i) and Lemma 3.7(iv), we obtain (ii). By (ii) and Proposition 2.2(vi), we obtain (iii) and (iv). □

Proposition 3.9. *Assume that* $\overline{H}^* = \overline{E_1}^*$. *Then one obtains the following three cases*:
(i) $\mathcal{Y} = 3Y_1 + Y_2$ (Y_i : *line*). *In this case,* $\overline{Y_2}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_2}^*} = 1$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_2}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$.
(ii) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : *line*). *In this case,* $\overline{Y_3}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 2$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_2}^*} = (\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_3}^*} = 1$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_3}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$.
(iii) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ (Y_i : *line*). *In this case,* $\overline{Y_4}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_i}^*} = 1$ $(i = 1, 2, 3, 4)$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_4}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$.
Moreover, for the cases (i), (ii) *and* (iii), *one obtains the weighted dual graphs of* $\widehat{Y} \cup E \cup \widehat{l}$ *of type* (VII), (VIII) *and* (IX) *in* Theorem 1 *respectively.*

Proof. By Lemma 3.8 and by using separation, we obtain the assertions. □

Lemma 3.10. *Assume that* $\overline{H}^* = \overline{E_2}^*$. *Then one obtains the following*:
(i) $\overline{l_1}^* \subset \overline{E_1}^* \setminus \{P\}$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$.
(ii) *There exists a unique irreducible component* Y_{i_1} *of* Y *such that* $\overline{Y_{i_1}}^* = \{P\}$ *and* $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_{i_1}}^*} = 3$.
(iii) $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_{i_1}}^*} = 1$.
(iv) $\mathcal{Y} = Y_{i_1} + \sum_{i \neq i_1} k_i Y_i$.

Proof. By using (i), we obtain (ii),(iii) and (iv) easily. Hence it suffices to show (i). First we have $\overline{l_1}^* \subset \overline{E_1}^* \setminus \{P\}$ clearly. Now we note that $1 \leq (\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} \leq 4$. We assume that $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 2$ (*resp*. $3, 4$). Similarly to the proof of Proposition 3.6, we obtain the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1}$ and we get the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ in Fig. 6(a) (*resp*. (b), (c)), where we denote by E_1' and E_2' the proper transforms of E_1 and E_2 respectively. However, these graphs are not linear trees. This is a contradiction. Thus we obtain $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$. □

Proposition 3.11. *Assume that* $\overline{H}^* = \overline{E_2}^*$. *Then one obtains the following three cases*:

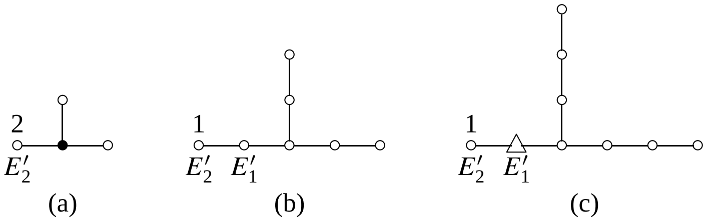

Fig. 6.

(i) $\mathcal{Y} = 3Y_1 + Y_2$ *(Y_i : line). In this case,* $\overline{Y_2}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_2}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_2}^*} = 1$.
(ii) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ *(Y_i : line). In this case,* $\overline{Y_3}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_3}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 2$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_2}^*} = (\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_3}^*} = 1$.
(iii) $\mathcal{Y} = Y_1 + Y_2 + Y_3 + Y_4$ *(Y_i : line). In this case,* $\overline{Y_4}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_4}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_i}^*} = 1$ $(i = 1, 2, 3, 4)$.
Moreover, for the cases (i), (ii) *and* (iii), *one obtains the weighted dual graphs of* $\widehat{Y} \cup E \cup \widehat{l}$ *of type* (X), (XI) *and* (XII) *in* Theorem 1 *respectively.*

Proof. By Lemma 3.10 and by using separation, we have the assertions. □

3.2.2. The case $\overline{Y}^*$ is a line. By Proposition 2.2(vii), we note that $Y = Y_1 \cup Y_2$ where Y_1 is a line through x_1 and where Y_2 is not a line through x_1 and that $\overline{H}^* = \overline{Y}^* = \overline{Y_2}^*$. Since $\overline{\psi}|_{\overline{X}} : \overline{X} \to \mathbb{P}^2$ is a birational morphism, we also note that $\overline{Y_2}^* \neq \overline{E_1}^*, \overline{E_2}^*$. Then we obtain the following:

Lemma 3.12. (i) $\overline{Y_2}^*$ *is a line through* P.
(ii) $\overline{Y_1}^* = \{P\}$.
(iii) *There exist exactly two lines* l_1 *and* l_2 *in* X *through* x_1 *such that* $l_1, l_2 \not\subset Y$.
(iv) $\overline{l_i}^* \subset \overline{E}^* \setminus \{P\}$ $(i = 1, 2)$.
(v) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y_1}^* \cup \overline{l_1}^* \cup \overline{l_2}^*$ *(exactly three points).*

Proof. By Proposition 2.2(vii), $\overline{Y}^* \cup \overline{E}^* = \overline{Y_2}^* \cup \overline{E_1}^* \cup \overline{E_2}^*$ does not have any cycles. Hence $\overline{Y_2}^*$ passes through the intersection point P of $\overline{E_1}^*$ and $\overline{E_2}^*$. This shows (i). Similarly to Lemma 3.5, we obtain (ii),(iii),(iv) and (v) easily. □

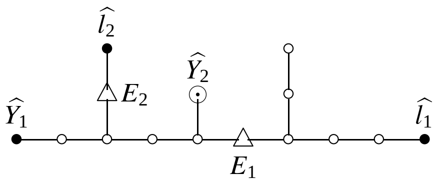

Fig. 7.

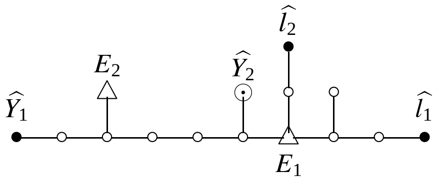

Fig. 8.

Proposition 3.13. *The case $\overline{Y}^*$ is a line cannot occur.*

Proof. We obtain the following three cases:
(1) $\overline{l_1}^*, \overline{l_2}^* \subset \overline{E_2}^* \setminus \{P\}$.
(2) $\overline{l_1}^* \subset \overline{E_1}^* \setminus \{P\}$, $\overline{l_2}^* \subset \overline{E_2}^* \setminus \{P\}$.
(3) $\overline{l_1}^*, \overline{l_2}^* \subset \overline{E_1}^* \setminus \{P\}$.
(1) In this case, we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_i}^*} = 1$ for $i = 1, 2$ since $\overline{x} \cap \overline{l_i} = \emptyset$. Hence we have $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 2$ and $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 1$. By Lemma 3.12(v), we obtain $(\overline{\Gamma}^* \cdot \overline{E_1}^*) = (\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 1$. This is a contradiction.
(2) In this case, we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_2}^*} = 1$ since $\overline{x} \cap \overline{l_2} = \emptyset$. We also obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 1$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 3$ and $(\overline{\Gamma}^* \cdot \overline{Y_2}^*)_{\overline{Y_1}^*} = 1$. Similarly to the proof of Proposition 3.6, we obtain the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1} \cup \widehat{l_2}$ in Fig. 7, and we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction.
(3) In this case, we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*) = (\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 4$ by Lemma 3.12(v). Hence we obtain $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 1$ and $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} + (\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_2}^*} = 3$. We may assume that $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 2$ and $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_2}^*} = 1$. Then we note that $(\overline{\Gamma}^* \cdot \overline{Y_2}^*)_{\overline{Y_1}^*} = 1$. Similarly to the case (2), we obtain the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1} \cup \widehat{l_2}$ in Fig. 8, and we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction. $\square$

3.3. The case $\overline{\mathcal{E}}$ = line + conic. Let $\overline{\mathcal{E}} = \overline{E_1} + \overline{E_2}$ ($\overline{E_1}$: line, $\overline{E_2}$: conic) be the restriction of Δ to $\overline{X}$. Then $\overline{E_1}^*$ and $\overline{E_2}^*$ meet tangentially to the second order at one point, which is denoted by P. By Proposition 2.2(vi) and (vii), we know that $\overline{Y}^*$ consists of finite points and $\overline{H}^* = \overline{E_1}^*$.

Lemma 3.14. (i) *There exists only one line l_1 in X through x_1 such that $l_1 \not\subset Y$.*
(ii) $\overline{l_1}^* \subset \overline{E_2}^* \setminus \{P\}$.
(iii) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y}^* \cup \overline{l_1}^*$ (*at most five points*).
(iv) $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_P = 7$.
(v) *There exists a unique irreducible component Y_{i_1} of Y such that $\overline{Y_{i_1}}^* = \{P\}$.*
(vi) $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_P = 2$.
(vii) $\mathcal{Y} = 2Y_{i_1} + \sum_{i \neq i_1} k_i Y_i$.

Proof. Similarly to Lemma 3.3, we obtain (i),(ii) and (iii) easily. Since we can obtain (v) and (vii) by using (iv) and (vi), it suffices to show (iv) and (vi).
(iv) Since $\overline{x} \cap \overline{l_1} = \emptyset$, similarly to the proof of Lemma 3.8(i), we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$. By using (iii), we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_P = (\overline{\Gamma}^* \cdot \overline{E_2}^*) - (\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 8 - 1 = 7$.
(vi) Let $\tau : (\mathbb{P}^2)' \to \mathbb{P}^2$ be the blowing-up at P with exceptional curve e. Let $(\overline{\Gamma}^*)'$ and $(\overline{E_i}^*)'$ be the proper transforms of $\overline{\Gamma}^*$ and $\overline{E_i}^*$ by τ respectively. Here we note that $\overline{E_1}^*$, $\overline{E_2}^*$ and e meet only at one point, which is denoted by P', and that each pair of them meets transversally at P'. By using (iv), we obtain $((\overline{\Gamma}^*)' \cdot (\overline{E_2}^*)')_{(\mathbb{P}^2)', P'} = 6$, that is, $(\overline{\Gamma}^*)'$ and $(\overline{E_2}^*)'$ meet tangentially to the sixth order at P'. Hence $(\overline{\Gamma}^*)'$ and $(\overline{E_1}^*)'$ meet transversally at P'. Thus we obtain $((\overline{\Gamma}^*)' \cdot (\overline{E_1}^*)')_{P'} = 1$ and $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_P = 2$. □

Proposition 3.15. *One obtains the following two cases*:
(i) $\mathcal{Y} = 2Y_1 + 2Y_2$ (Y_i : *line*). *In this case,* $\overline{Y_1}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 2$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_2}^*} = 2$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 7$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$.
(ii) $\mathcal{Y} = 2Y_1 + Y_2 + Y_3$ (Y_i : *line*). *In this case,* $\overline{Y_1}^* = \{P\}$,
$(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 2$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_2}^*} = (\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_3}^*} = 1$,
$(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{Y_1}^*} = 7$, $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_{\overline{l_1}^*} = 1$.
Moreover, for the cases (i) *and* (ii), *one obtains the weighted dual graphs of* $\widehat{Y} \cup E \cup \widehat{l}$ *of type* (XIII) *and* (XIV) *in* Theorem 1 *respectively.*

Proof. By Lemma 3.14 and by using separation, we obtain the assertions. □

3.4. Non-existence of the case $\overline{\mathcal{E}}$ = line + line + line. Let $\overline{\mathcal{E}} = \overline{E_1} + \overline{E_2} + \overline{E_3}$ ($\overline{E_i}$: line) be the restriction of Δ to $\overline{X}$. Here we note that $\overline{E_1}$, $\overline{E_2}$ and $\overline{E_3}$ meet only at one point. We set $\{P\} := \operatorname{Sing} \overline{E}^*$. Then we have the following two cases:

(1) $\overline{Y}^*$ consists of finite points;
(2) $\overline{Y}^*$ is a line.

3.4.1. The case $\overline{Y}^*$ consists of finite points.

Lemma 3.16. (i) *One may assume that* $\overline{H}^* = \overline{E_1}^*$.
(ii) *There exist two lines* l_1 *and* l_2 *in* X *through* x_1 *such that* $l_1, l_2 \not\subset Y$.
(iii) $\overline{l_i}^* \neq \{P\}$ $(i = 1, 2)$.
(iv) *One may assume that* $\overline{l_1}^* \subset \overline{E_2}^* \setminus \{P\}$ *and* $\overline{l_2}^* \subset \overline{E_j}^* \setminus \{P\}$ $(j = 2$ *or* $3)$.
(v) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y}^* \cup \overline{l_1}^* \cup \overline{l_2}^*$ (*at most six points*).
(vi) $(\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{l_i}^*} = 1$ $(i = 1, 2)$.

Proof. Similarly to Lemma 3.14, we obtain the assertions. □

Proposition 3.17. *The case* $\overline{Y}^*$ *consists of finite points cannot occur.*

Proof. Since $1 \leq \sum_{Q \in \overline{E_2}^* \setminus \{P\}} (\overline{\Gamma}^* \cdot \overline{E_2}^*)_Q \leq 2$ by Lemma 3.16(iv) and (vi), we obtain $(\overline{\Gamma}^* \cdot \overline{E_2}^*)_P \geq 2$. Hence we know by using Lemma 3.16(iii) and (v) that $P \in \overline{\Gamma}^*$ and that there exists a unique irreducible component Y_{i_1} of Y such that $\overline{Y_{i_1}}^* = \{P\}$. We note that $\sum_{i=1}^{2} (\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{l_i}^*} = 2$ and

$$\sum_{i \neq i_1} (\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{Y_i}^*} = (\overline{\Gamma}^* \cdot \overline{E_1}^*) - (\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_{i_1}}^*} \leq 4 - 1 = 3.$$

Then we obtain

$$\begin{aligned} 12 = (\overline{\Gamma}^* \cdot \overline{E}^*) &= (\overline{\Gamma}^* \cdot \overline{E}^*)_P + \sum_{i=1}^{2} (\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{l_i}^*} + \sum_{i \neq i_1} (\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{Y_i}^*} \\ &\leq (\overline{\Gamma}^* \cdot \overline{E}^*)_P + 5. \end{aligned}$$

Hence we obtain $(\overline{\Gamma}^* \cdot \overline{E}^*)_P \geq 7$. On the other hand, at most one of $\overline{E_1}^*$, $\overline{E_2}^*$ and $\overline{E_3}^*$ meets $\overline{\Gamma}^*$ tangentially at P. Hence we obtain $(\overline{\Gamma}^* \cdot \overline{E}^*)_P = \sum_i (\overline{\Gamma}^* \cdot \overline{E_i}^*)_P \leq 4+1+1 = 6$. This is a contradiction. □

3.4.2. The case $\overline{Y}^*$ is a line. By Proposition 2.2(vii), we note that $Y = Y_1 \cup Y_2$ where Y_1 is a line through x_1 and where Y_2 is not a line through x_1 and that $\overline{H}^* = \overline{Y}^* = \overline{Y_2}^*$. Then, similarly to Lemma 3.16, we obtain the following:

Lemma 3.18. (i) $\overline{Y_2}^*$ *is a line through* P.
(ii) $\overline{Y_1}^* = \{P\}$.
(iii) *There exist exactly three lines* l_1, l_2 *and* l_3 *in* X *through* x_1 *such that* $l_1, l_2, l_3 \not\subset Y$.
(iv) $\overline{l_i}^* \subset \overline{E}^* \setminus \{P\}$ $(i = 1, 2, 3)$. *One may assume that* $\overline{l_1}^* \subset \overline{E_1}^* \setminus \{P\}$.
(v) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y_1}^* \cup \overline{l_1}^* \cup \overline{l_2}^* \cup \overline{l_3}^*$ (*exactly four points*).
(vi) $(\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{l_i}^*} = 1$ $(i = 1, 2, 3)$.

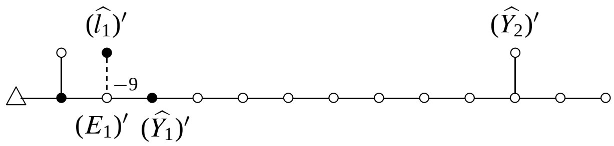

Fig. 9.

Proposition 3.19. *The case $\overline{Y}^*$ is a line cannot occur.*

Proof. By Lemma 3.18(vi), we obtain $(\overline{\Gamma}^* \cdot \overline{E}^*)_P = (\overline{\Gamma}^* \cdot \overline{E}^*) - \sum_{i=1}^{3} (\overline{\Gamma}^* \cdot \overline{E}^*)_{\overline{l_i}^*} = 9$. On the other hand, we obtain $(\overline{\Gamma}^* \cdot \overline{E}^*)_P = \sum_{i=1}^{3} (\overline{\Gamma}^* \cdot \overline{E_i}^*)_P \leq 4 + 1 + 1 = 6$. This is a contradiction. $\square$

3.5. Non-existence of the case $\overline{\mathcal{E}}$ is a cuspidal cubic. Let $\overline{\mathcal{E}} = \overline{E_1}$ ($\overline{E_1}$: cuspidal cubic) be the restriction of Δ to $\overline{X}$. We put $\{P\} := \operatorname{Sing} \overline{E_1}^*$. By Proposition 2.2(vi), we see that $\overline{Y}^*$ is a line. By Proposition 2.2(vii), we note that $Y = Y_1 \cup Y_2$ where Y_1 is a line through x_1 and where Y_2 is not a line through x_1 and that $\overline{H}^* = \overline{Y}^* = \overline{Y_2}^*$. Then we obtain the following:

Lemma 3.20. (i) *There exists only one line l_1 in X through x_1 such that $l_1 \not\subset Y$.*
(ii) $\overline{\Gamma}^* \cap \overline{E}^* = \overline{Y_1}^* \cup \overline{l_1}^*$ (*exactly two points*).
(iii) $P \in \overline{Y_1}^* \cup \overline{l_1}^*$.

Proof. Similarly to Lemma 3.18, we obtain (i) and (ii). Hence it suffices to show (iii). Now we have the following two cases:
(1) $P \notin \overline{Y_1}^* \cup \overline{l_1}^*$;
(2) $P \in \overline{Y_1}^* \cup \overline{l_1}^*$.
We assume that $P \notin \overline{Y_1}^* \cup \overline{l_1}^*$. Then we easily obtain $(\overline{Y_2}^* \cdot \overline{E_1}^*) = (\overline{Y_2}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 3$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{l_1}^*} = 1$ and $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_{\overline{Y_1}^*} = 11$. Since we know the intersection of $\overline{\Gamma}^*$, $\overline{E_1}^*$ and $\overline{Y_2}^*$, we obtain by using separation the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1}$. We note that $\widehat{Y} \cup E$ is the boundary curve of a smooth compactification of $\mathbb{C}^2$ which is not simple normal crossing. By applying the blowing-ups three times on $\operatorname{Sing} E_1$, we obtain the weighted dual graph of the simple normal crossing boundary of a smooth compactification of $\mathbb{C}^2$ in Fig. 9, where we denote by $(\widehat{Y_1})'$, $(\widehat{Y_2})'$, $(\widehat{l_1})'$ and $(E_1)'$ the proper transforms of $\widehat{Y_1}$, $\widehat{Y_2}$, $\widehat{l_1}$ and E_1 respectively. By contracting suitable (-1)-curves in this boundary successively, we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction. Hence we obtain $P \in \overline{Y_1}^* \cup \overline{l_1}^*$. $\square$

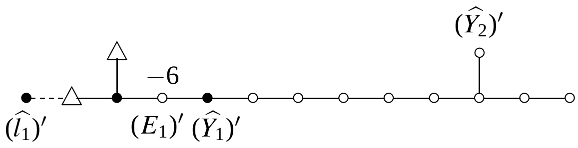

Fig. 10.

We define a point Q by $\{P, Q\} := \overline{Y_1}^* \cup \overline{l_1}^* = \overline{\Gamma}^* \cap \overline{E_1}^*$. Then we have the following two cases:
(1) The intersection of $\overline{\Gamma}^*$ and $\overline{E_1}^*$ at P is tangential.
(2) The intersection of $\overline{\Gamma}^*$ and $\overline{E_1}^*$ at P is not tangential.

Proposition 3.21. *The case $\overline{\mathcal{E}}$ is a cuspidal cubic cannot occur.*

Proof. (1) We assume that the intersection of $\overline{\Gamma}^*$ and $\overline{E_1}^*$ at P is tangential. Then we obtain $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_P = 3$ and $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_Q = 9$. Now we assume that $\overline{l_1}^* = \{Q\}$. Since $\overline{x} \cap \overline{l_1} = \emptyset$, by Proposition 2.2(v), $\overline{l_1}$ is a (-1)-curve in $\overline{X} \setminus \overline{x}$. Since $(\overline{\Gamma} \cdot \overline{l_1})_{\overline{X}} = (\overline{E_1} \cdot \overline{l_1})_{\overline{X}} = 1$, we obtain $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_Q = 1$. This is a contradiction. Hence we obtain $\overline{Y_1}^* = \{Q\}$. From this and by Proposition 2.2(vii), we obtain $(\overline{Y_2}^* \cdot \overline{E_1}^*) = (\overline{Y_2}^* \cdot \overline{E_1}^*)_Q = 3$, that is, $\overline{Y_2}^*$ and $\overline{E_1}^*$ meet only at Q tangentially to the third order. Similarly to the proof of Lemma 3.20, we obtain the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1}$ and, by applying the blowing-up at a certain point of E_1, we obtain the weighted dual graph of the simple normal crossing boundary of a smooth compactification of $\mathbb{C}^2$ in Fig. 10, where we denote by $(\widehat{Y_1})'$, $(\widehat{Y_2})'$, $(\widehat{l_1})'$ and $(E_1)'$ the proper transforms of $\widehat{Y_1}$, $\widehat{Y_2}$, $\widehat{l_1}$ and E_1 respectively. By contracting suitable (-1)-curves in this boundary successively, we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction.
(2) We assume that the intersection of $\overline{\Gamma}^*$ and $\overline{E_1}^*$ at P is not tangential. Then we obtain $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_P = 2$, $(\overline{\Gamma}^* \cdot \overline{E_1}^*)_Q = 10$ and, similarly to (1), $\overline{Y_1}^* = \{Q\}$. By Proposition 2.2(vii), we obtain $(\overline{Y_2}^* \cdot \overline{E_1}^*) = (\overline{Y_2}^* \cdot \overline{E_1}^*)_Q = 3$, that is, $\overline{Y_2}^*$ and $\overline{E_1}^*$ meet only at Q tangentially to the third order. Similarly to the proof of (1), we obtain the weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l_1}$ and, by applying the blowing-ups twice on a certain point of E_1, we obtain the weighted dual graph of the simple normal crossing boundary of a smooth compactification of $\mathbb{C}^2$ in Fig. 11, where we denote by $(\widehat{Y_1})'$, $(\widehat{Y_2})'$, $(\widehat{l_1})'$ and $(E_1)'$ the proper transforms of $\widehat{Y_1}$, $\widehat{Y_2}$, $\widehat{l_1}$ and E_1 respectively. By contracting suitable (-1)-curves in this boundary successively, we obtain the weighted dual graph of the boundary of a minimal normal compactification of $\mathbb{C}^2$ which is not a linear tree. This is a contradiction. $\square$

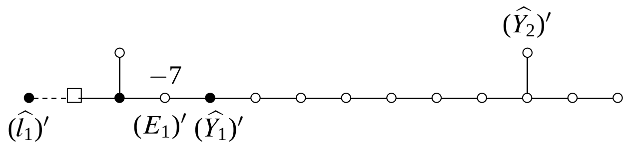

Fig. 11.

4. Construction of Linearizing Automorphisms

In this section, we shall prove Theorems 2 and 3. For each weighted dual graph of $\widehat{Y} \cup E \cup \widehat{l}$ of type (I) through (XIV) in Theorem 1, we know by its proof the shape of the divisor $(\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$ and the intersection of the divisors $\overline{\Gamma}^*$ and $(\overline{\psi}|_{\overline{X}})_*(\overline{\mathcal{E}})$. Hence, by Proposition 2.5, we can write down the defining equation of (X, Y) of the same type as in Theorem 2. Next we construct an automorphism of $\mathbb{C}^3$ which transforms the hypersurface $X \setminus Y$ of $\mathbb{C}^3$ onto a hyperplane of $\mathbb{C}^3$ which shows Theorem 3. It suffices to consider the defining equations of (X, Y) of type (VI), (X), (XI) and (XII). Indeed, for the other types, we can easily construct such automorphisms, which are elements of the subgroup $J(3, \mathbb{C}) \vee A(3, \mathbb{C})$ of $\mathrm{Aut}(\mathbb{C}^3)$. Here we denote by (x, y, z) a coordinate system of $\mathbb{C}^3$ and by $\mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z]$ the group of $\mathbb{C}$-algebra isomorphisms of the polynomial ring of three variables x, y and z over $\mathbb{C}$. Then we obtain the natural group isomorphism $\mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z] \xrightarrow{\sim} \mathrm{Aut}(\mathbb{C}^3)$, $\sigma \mapsto \Phi_\sigma$, where Φ_σ is defined by

$$\Phi_\sigma : \begin{cases} x' = \sigma(x) \\ y' = \sigma(y) \\ z' = \sigma(z). \end{cases}$$

In the following, we shall mainly describe elements of $\mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z]$.

4.1. The type (VI). For this type, it suffices to consider the following hypersurface of $\mathbb{C}^3$:

$$(1 + a_1 x + a_2 x^2)y + (a_3 x + a_4 x^2)y^2 + xy^3 + x^3 z + a_5 x^2 = 0,$$

where a_i are complex parameters. After performing the two coordinate transformations $x' := x$, $y' := y + a_5 x^2$, $z' := z$ and $x'' := x'$, $y'' := y'$, $z'' := z' + p(x', y')$ where $p(x', y')$ is a suitable polynomial of x' and y', we obtain the following hypersurface S_1 of $\mathbb{C}^3$:

$$S_1 : (1 + a_1 x + a_2 x^2)y + (a_3 x + a_4 x^2)y^2 + xy^3 + x^3 z = 0,$$

where a_i are complex parameters. Hence it suffices to construct an automorphism of $\mathbb{C}^3$ which transforms S_1 onto a hyperplane of $\mathbb{C}^3$. According to Proposition 2.2 in Russell [14], we define $\mathbb{C}$-algebra homomorphisms $\sigma_1, \tau_1 : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, z]$ as follows:

$$\sigma_1 : \begin{cases} \sigma_1(x) := x \\ \sigma_1(y) := f_1(x, y) + x^3 z \\ \sigma_1(z) := \{g_1(x, \sigma_1(y)) - y\}/x^3, \end{cases}$$

$$\tau_1 : \begin{cases} \tau_1(x) := x \\ \tau_1(y) := g_1(x, y) - x^3 z \\ \tau_1(z) := \{y - f_1(x, \tau_1(y))\}/x^3, \end{cases}$$

where $f_1, g_1 \in \mathbb{C}[x, y]$ are defined by

$$\begin{aligned} f_1(x, y) &:= (1 + a_1 x + a_2 x^2)y + (a_3 x + a_4 x^2)y^2 + (x)y^3, \\ g_1(x, y) &:= \{1 - a_1 x + (-a_2 + a_1^2)x^2\}y + \{-a_3 x + (3a_1 a_3 - a_4)x^2\}y^2 \\ &\quad +\{-x + (2a_3^2 + 4a_1)x^2\}y^3 + (5a_3 x^2)y^4 + (3x^2)y^5. \end{aligned}$$

Proposition 4.1. $\sigma_1, \tau_1 \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z]$ *and* $\sigma_1^{-1} = \tau_1$. *In particular, the automorphism* Φ_{σ_1} *transforms* S_1 *onto a hyperplane of* $\mathbb{C}^3$ *and* $\Phi_{\sigma_1}^{-1} = \Phi_{\tau_1}$.

Proof. First we check that σ_1 and τ_1 can be defined as $\mathbb{C}$-algebra endomorphisms of $\mathbb{C}[x, y, z]$. Now we get the following equalities by computing directly:

$$f_1(x, g_1(x, y)) \equiv g_1(x, f_1(x, y)) \equiv y \mod (x^3). \tag{1}$$

By using (1), we obtain

$$\begin{aligned} f_1(x, y) &\equiv f_1(x, y) + x^3 z \equiv \sigma_1(y) \\ &\equiv f_1(x, g_1(x, \sigma_1(y))) \mod (x^3). \end{aligned} \tag{2}$$

By using (2) and (1) again, we obtain

$$\begin{aligned} y &\equiv g_1(x, f_1(x, y)) \equiv g_1(x, f_1(x, g_1(x, \sigma_1(y)))) \\ &\equiv g_1(x, \sigma_1(y)) \mod (x^3). \end{aligned}$$

Similarly, we obtain $y \equiv f_1(x, \tau_1(y)) \mod (x^3)$. Thus we see that both of $\sigma_1(z)$ and $\tau_1(z)$ are polynomials of x, y, z and hence we can define σ_1 and τ_1 as $\mathbb{C}$-algebra endomorphisms of $\mathbb{C}[x, y, z]$. Since we can easily check $\sigma_1\tau_1(x) = \tau_1\sigma_1(x) = x$, $\sigma_1\tau_1(y) = \tau_1\sigma_1(y) = y$ and $\sigma_1\tau_1(z) = \tau_1\sigma_1(z) = z$, we obtain $\sigma_1, \tau_1 \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z]$ and $\sigma_1^{-1} = \tau_1$. $\square$

4.2. The types (X), (XI) and (XII). For these types, it suffices to consider the following hypersurface S_2 of $\mathbb{C}^3$:

$$S_2 : y + x\left(xz + \sum_{i,j\geq 0} a_{ij}x^i y^j\right) = 0,$$

where a_{ij} are complex parameters. Now we construct an automorphism of $\mathbb{C}^3$ which transforms S_2 onto a hyperplane of $\mathbb{C}^3$. We define $\mathbb{C}$-algebra homomorphisms $\sigma_2, \tau_2 : \mathbb{C}[x, y, z] \to \mathbb{C}[x, y, z]$ as follows:

$$\sigma_2 : \begin{cases} \sigma_2(x) := x \\ \sigma_2(y) := y + xf_2(x, y, z) \\ \sigma_2(z) := z - g_2(x, y, z), \end{cases}$$

$$\tau_2 : \begin{cases} \tau_2(x) := x \\ \tau_2(y) := y - xf_2(x, y, z) \\ \tau_2(z) := z + h_2(x, y, z), \end{cases}$$

where $f_2, g_2, h_2 \in \mathbb{C}[x, y, z]$ are defined by

$$\begin{aligned} f_2(x, y, z) &:= xz + \sum_{i,j\geq 0} a_{ij}x^i y^j, \\ g_2(x, y, z) &:= \sum_{i\geq 0, j\geq 1} a_{ij}\left\{\sum_{k=1}^{j}\binom{j}{k} y^{j-k}x^k f_2^k\right\} x^i \Big/ x, \\ h_2(x, y, z) &:= \sum_{i\geq 0, j\geq 1} a_{ij}\left\{\sum_{k=1}^{j}\binom{j}{k} (y - xf_2)^{j-k}x^k f_2^k\right\} x^i \Big/ x. \end{aligned}$$

Proposition 4.2. $\sigma_2, \tau_2 \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z]$ *and* $\sigma_2^{-1} = \tau_2$. *In particular, the automorphism* Φ_{σ_2} *transforms* S_2 *onto a hyperplane of* $\mathbb{C}^3$ *and* $\Phi_{\sigma_2}^{-1} = \Phi_{\tau_2}$.

Proof. For any $j \geq 1$, we note that x divides $\sum_{k=1}^{j}\binom{j}{k} y^{j-k}x^k f_2^k$ and $\sum_{k=1}^{j}\binom{j}{k}(y-xf_2)^{j-k}x^k f_2^k$. Therefore g_2 and h_2 are polynomials of x, y, z and, in particular, σ_2 and τ_2 can be defined as $\mathbb{C}$-algebra endomorphisms of $\mathbb{C}[x, y, z]$. Here we can check the equalities $\sigma_2(f_2) = \tau_2(f_2) = f_2$ by direct computation. By using these equalities, we can easily get $\sigma_2\tau_2(x) = \tau_2\sigma_2(x) = x$, $\sigma_2\tau_2(y) = \tau_2\sigma_2(y) = y$ and $\sigma_2\tau_2(z) = \tau_2\sigma_2(z) = z$. Hence we obtain $\sigma_2, \tau_2 \in \mathrm{Aut}_{\mathbb{C}}\, \mathbb{C}[x, y, z]$ and $\sigma_2^{-1} = \tau_2$. □

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