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GLOBAL WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS FOR MULTIDIMENTIONAL COMPRESSIBLE FLOW SUBJECT TO LARGE EXTERNAL POTENTIAL FORCES

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1. Introduction

1.1. Background In this paper, we are concerned with compressible, viscous, isentropic flow in three (and two) space dimensions. The fluid motion is described in the following form by the conservation laws of mass and momentum:

$$\rho_t + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) - \mu\Delta u - (\lambda + \mu)\nabla(\operatorname{div} u) = \rho f. \tag{1.2}$$

Here $t \geq 0$ is time, $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ ($n = 2$ or 3) is the spatial coordinate,

$$\rho(x, t), \ \ u(x, t) = (u^1(x, t), \dots, u^n(x, t)), \ \ p(\rho) = a\rho^\gamma \ \ (a > 0, \ \ \gamma \geq 1)$$

represent respectively the fluid density, velocity, and pressure, $f(x) = (f^1(x), \dots, f^n(x))$ is the external force, and μ, λ are viscous coefficients which satisfies $\mu > 0$, $3\lambda+2\mu \geq 0$ by physical requests.

The local (in time) solvability to the various initial boundary value problems for the full Navier-Stokes equations (which include also the conservation law of energy) was obtained by Nash [11], Solonnikov [14], and Tani [16]. The first result about the global theory is that of Matsumura and Nishida [7], who proved the global existence of H^3-solutions around a constant state for the Cauchy problem without external forces. Afterwards, in the case that external potential force field is small enough, and for the interior or exterior problems, almost the same results were derived by Matsumura and Nishida [8]–[9], and Valli [18]. But there have been no remarkable results in the case with *large* external potential forces except for that of Matsumura and Padula [10], who proved the stability of the corresponding stationary state (more precisely, the global existence of H^3-solutions which tend toward the stationary solution) for the interior problems.

On the other hand, discontinuous, namely weak solutions play an important role in the physical as well as in the mathematical theory, and the problem of global existence of those have been attracting the attention of many mathematicians. In the case of 'small data', the most general results are those of Hoff [1]–[3] extending that of [7] (that is to say, for the Cauchy problem without external forces), who proved the global existence of weak solutions in the cases of $\gamma = 1$, $\gamma > 1$, and recently for the full system. In the case of 'large data', however, many problems are open even for the isentropic model. Various researches for these problems have been done by Padula [12]–[13], Lions [5], Vaigant and Kazhikhov [17], Mamontov [6] and so on, but global existence results of weak solutions have been obtained only in fairly restricted forms even under the spatial periodic condition, and there are no satisfactory conclusions in a physical viewpoint.

Under these backgrounds, placing emphasis on *large* external forces and *weak* solutions, we consider the Cauchy problem of (1.1)–(1.2) with the initial data

$$(\rho,\ u)(x, 0) = (\rho_0,\ u_0)(x), \quad \inf \rho_0 > 0 \tag{1.3}$$

and with the external force in the form

$$f(x) = -\nabla\phi(x),$$

where ϕ satisfies suitable decay properties in the far field (see (2.1)–(2.3) below). In particular, we shall derive an asymptotic stability of the corresponding stationary state, more precisely, prove the global existence of weak solutions when the initial perturbation is suitably small in $L^2 \cap L^\infty$ for density and in H^1 for velocity, and the ratio of specific heats is close to 1.

To show the existence of the global weak solution, we shall basically follow the arguments [1]–[2] by Hoff. This weak solution will be constructed as a limit of smooth approximated solutions which satisfy the equations (1.1)–(1.2) with mollified initial data, so that the main argument in this paper is to obtain a priori estimates for these approximated solutions. The required estimates will be obtained by way of the energy methods. We shall start with the energy balance law used in [10], afterwards apply the arguments of [1]–[2], but various difficulties will arise because the stationary solution is not a constant owing to the external force. One of the most essential part of this paper is that we can overcome the above difficulties by making use of Poincaré-type inequalities for unbounded domains and estimating carefully the weighted (in the spatial direction) L^2 norm for density.

1.2. Notations and Brief Overview of the Analysis At this stage we give a brief overview of the analysis for the three-dimensional case, with introducing some notations.

Provided that an arbitrary potential force ϕ with suitable decay properties is given (see (2.1) for more precise), let $(\tilde{\rho},\ 0)$ be the corresponding stationary solution which will be exactly given by (2.4). Then supposing that there exists a sufficiently smooth solution (ρ, u) defined up to a positive time T, we define

$$\Phi_1(T) := \sup_{0\leq t\leq T} \left\{\|u(t)\|^2 + \|(\rho - \tilde{\rho})(t)\|^2\right\} + \int_0^T \|\nabla u(t)\|^2 dt,$$

$$\Phi_2(T) := \sup_{0\leq t\leq T} \|\nabla u(t)\|^2 + \int_0^T \|\dot{u}(t)\|^2 dt,$$

$$\Phi_3(T) := \sup_{0\leq t\leq T} \sigma(t)\|\dot{u}(t)\|^2 + \int_0^T \sigma(t)\|\nabla \dot{u}(t)\|^2 dt$$

and

$$\Psi(T) := \|\rho - \tilde{\rho}\|^2_{L^\infty(\mathbf{R}^n\times[0,T])}.$$

Here,

$$\sigma(t) := \min\{1, t\},$$

and $\dot{u}$ is called as the material derivative of u, generally given by

$$\begin{aligned} \dot{f} = \frac{D}{Dt} f &:= \left[\frac{\partial}{\partial t} + u\cdot\nabla\right] f \\ &= f_t + u^j f_j. \end{aligned}$$

($f_j := f_{x_j}$, and summation over repeated indices is understood.) Moreover, with respect to the notations for norms which we shall use frequently later, we denote the usual L^p norm in the spatial direction by $\|\cdot\|_p$, in particular the L^2 norm by $\|\cdot\|$ for simplicity.

The initial perturbation, on the other hand, will be measured in the norm given by

$$C_0 := \|\rho_0 - \tilde{\rho}\|^2_\infty + \|\rho_0 - \tilde{\rho}\|^2 + \|u_0\|^2_{H^1}.$$

Then, our goal is to obtain the following a priori estimate (see Proposition 4 below):

There exist positive constants γ_0, ε_0, $\underline{\rho}$, $\overline{\rho}$, C and θ independent of T such that, if Φ, $\Psi \leq 1$, $\underline{\rho} \leq \rho(x, t) \leq \overline{\rho}$, $1 \leq \gamma \leq \gamma_0$ and $C_0 \leq \varepsilon_0$, then $\Phi(T) + \Psi(T) \leq CC_0^\theta$.

Here we have denoted $\Phi := \Phi_1 + \Phi_2 + \Phi_3$. Once we obtain this estimate, the remaining arguments to obtain the global weak solution and its asymptotic behavior are almost the same as that of [1]–[2].

This paper is organized as follows:

In Section 2, we shall give a precise formulation of our results after referring to the stationary solution.

In Section 3, we shall start to derive the required a priori estimates, in particular, deal with Φ. Then we can conclude that Φ is estimated by the initial perturbation term C_0, the weighted norm for density

$$R_1(T) := \int_0^T \int_{\mathbf{R}^3} \frac{|\rho - \tilde{\rho}|^2}{(1+|x|)^2} dx dt,$$

which will arise by the decay properties of ϕ, and the higher order terms

$$R_2(T) := \int_0^T \|\nabla u(t)\|_3^3 dt, \quad R_3(T) := \int_0^T \sigma(t) \|\nabla u(t)\|_4^4 dt,$$

owing to the convection term $u \cdot \nabla u$ in (1.2).

Section 4 is devoted to the key estimates for R_1, also for R_2 and R_3. To do that, we shall employ the quantities F (called 'effective viscous flux') and $\Omega = (\Omega^{j,k})$ (which is related to vorticity) given as follows:

$$\begin{aligned} F \quad &:= \frac{2\mu + \lambda}{\tilde{\rho}^\gamma} \operatorname{div} u - a \left\{ \left(\frac{\rho}{\tilde{\rho}} \right)^\gamma - 1 \right\}, \\ \Omega^{j,k} &:= \frac{u_k^j - u_j^k}{\tilde{\rho}^\gamma}. \end{aligned}$$

We note here that Hoff used the similar quantities in [1]–[2], but we have divided them by $\tilde{\rho}^\gamma$ from technical reasons.

In Section 5, we shall estimate Ψ and complete the all estimates. This part of the argument is similar to [1] except for some adjustments.

REMARK. The argument in the two-dimensional case is similar except for a little adjustments about the weight and some exponents, as guessed easily from the following lemmas. More precisely, replacing the weight $1/(1+|x|)^2$ by

$$\frac{1}{(1+|x|)^2\{1+\log(1+|x|)\}^2}$$

and improving exponents about the Sobolev's embedding, we can apply the proof for $n = 3$ also for $n = 2$. Therefore, we shall mainly discuss the case of $n = 3$, and the indication to $n = 2$ will be given at the points where it is required.

1.3. Auxiliary Lemmas As the last part of introduction, we recall some inequalities frequently used below.

First, combining standard Sobolev inequalities (see Ziemer [19]) and Hölder's inequality, we then derive the following elementary estimates:

Lemma 1 (Sobolev's Embedding). *For any $f \in H^1(\mathbf{R}^n)$,*

$$\begin{aligned} &\|f\|_p^p \le C_p \|f\|^{(6-p)/2} \|\nabla f\|^{(3p-6)/2}, && \text{for } p \in [2,6], && \text{when } n = 3, \\ &\|f\|_p^p \le C_p \|f\|^2 \|\nabla f\|^{p-2}, && \text{for } p \ge 2, && \text{when } n = 2. \end{aligned}$$

Moreover we shall prepare the following inequalities which are obtained by easy calculations. Similar inequalities are proved in Ladyzhenskaya [4].

Lemma 2 (Poincaré-Type Inequality). *For any* $f \in H^1(\mathbf{R}^n)$,

$$\int_{\mathbf{R}^3} \frac{|f(x)|^2}{(1+|x|)^2} dx \leq C \int_{\mathbf{R}^3} |\nabla f|^2, \qquad \text{when } n = 3,$$

$$\int_{\mathbf{R}^2} \frac{|f(x)|^2}{(1+|x|)^2 \{1+\log(1+|x|)\}^2} dx \leq C \int_{\mathbf{R}^2} |\nabla f|^2, \qquad \text{when } n = 2.$$

2. Precise Formulation of the Results

In this section, we shall give a precise formulation of our results.

2.1. Stationary Solution Now, we shall discuss the external potential force and the corresponding stationary solution. We assume that the given potential ϕ is sufficiently smooth (it is enough $\phi \in H^4$), and that its first and second derivatives have decays in a suitable sense. As is known from the following argument, in the three-dimensional case, it is sufficient that

$$|D\phi(x)| \leq \frac{C}{1+|x|}, \quad |D^2\phi(x)| \leq \frac{C}{(1+|x|)^2} \tag{2.1}$$

where $D^k f := \{(\partial/\partial x)^\alpha f \mid |\alpha| = k\}$. In what follows, we assume that ϕ satisfies

$$\|\phi\|_{H^4} + \||x||D\phi|\|_\infty + \||x|^2|D^2\phi|\|_\infty \leq M \tag{2.2}$$

for some $M < \infty$. In the two-dimensional case, on the other hand, we assume a little stronger condition

$$\|\phi\|_{H^4} + \||x|\log(1+|x|)|D\phi|\|_\infty + \|\{|x|\log(1+|x|)\}^2|D^2\phi|\|_\infty \leq M. \tag{2.3}$$

Now, we take a constant $\rho_\infty > 0$ and consider a stationary solution $\big(\tilde{\rho}(x), \tilde{u}(x)\big)$ satisfying the condition

$$\big(\tilde{\rho}(x), \tilde{u}(x)\big) \to (\rho_\infty, 0) \text{ as } |x| \to \infty.$$

Since the stationary solution will turn out to be unique and $\tilde{u}$ be zero on Section 3 (see Remark in the proof of (3.1)), it suffices to look for the stationary solution in the form $(\tilde{\rho}, 0)$ from the beginning. Then by (1.2),

$$a\nabla(\tilde{\rho}^\gamma) = -\tilde{\rho}\nabla\phi,$$

and a formal calculation leads us to obtain

$$\tag{2.4} \tilde{\rho}(x) = \begin{cases} \rho_\infty \exp\left[-\dfrac{1}{a}\phi(x)\right], & \text{if} \quad \gamma = 1, \\ \left[{\rho_\infty}^{\gamma-1} - \dfrac{\gamma-1}{a\gamma}\phi(x)\right]^{1/(\gamma-1)}, & \text{if} \quad \gamma > 1. \end{cases}$$

Therefore, in order to avoid the vacuum state, we must expect

$$\tag{2.5} \sup \phi < \frac{a\gamma}{\gamma-1}{\rho_\infty}^{\gamma-1}, \quad \text{if} \quad \gamma > 1.$$

In fact, it is easy to see that if γ is close enough to 1 for given ϕ then (2.5) holds. More precisely, there exist constants $\overline{\gamma}(M) > 1$ and $\underline{\rho}(M)$, $\overline{\rho}(M) > 0$ such that the condition $1 \leq \gamma \leq \overline{\gamma}$ implies (2.5), and in particular

$$\tag{2.6} \underline{\rho} < \inf \tilde{\rho} \leq \sup \tilde{\rho} < \overline{\rho}.$$

2.2. Main Theorem To begin with, we shall give the definition of weak solutions.

DEFINITION. We say that (ρ, u) is a weak solution of Cauchy problem (1.1)–(1.3) provided that $\rho \in L^\infty_{\text{loc}}\big(0, \infty; L^\infty(\mathbf{R}^n)\big)$, $u \in L^\infty_{\text{loc}}\big(0, \infty; H^1(\mathbf{R}^n)\big)$, and for all test functions $\psi \in \mathcal{D}\big(\mathbf{R}^n \times (-\infty, \infty)\big)$,

$$\int_{\mathbf{R}^n} \rho_0 \psi(\,\cdot\,, 0) dx + \int_0^\infty \int_{\mathbf{R}^n} (\rho\psi_t + \rho u \cdot \nabla\psi)\, dx dt = 0,$$

and

$$\begin{aligned} &\int_{\mathbf{R}^n} \rho_0 u_0^j \psi(\,\cdot\,, 0) + \int_0^\infty \int_{\mathbf{R}^n} \left\{\rho u^j \psi_t + \rho u^j u \cdot \nabla\psi + p(\rho)\psi_j\right\} dx dt \\ &- \int_0^\infty \int_{\mathbf{R}^n} \left\{\mu \nabla u^j \cdot \nabla\psi + (\mu+\lambda)(\operatorname{div} u)\psi_j\right\} dx dt = \int_0^\infty \int_{\mathbf{R}^n} \rho\phi_j \psi dx dt, \quad j = 1, \ldots, n. \end{aligned}$$

Then, we can formulate our results as follows:

Theorem 1 (Main Theorem). *In* (1.1)–(1.2), *let* $n = 3$, $p = a\rho^\gamma$ $(a > 0, \gamma \geq 1)$ *and fix a positive constant* ρ_∞ *and the system parameters* a, μ, λ. *We also assume that an arbitrary* $f = -\nabla\phi$ *satisfying* (2.2) *for some* $M < \infty$ *is given. Then, there exist positive constants* γ_0 $(\in (1, \overline{\gamma}))$, ε_0 (*depending only on* M, ρ_∞, a, μ *and* λ) *such that*: *if*

$$\begin{cases} 1 \leq \gamma \leq \gamma_0, \\ C_0 = \|\rho_0 - \tilde{\rho}\|_\infty^2 + \|\rho_0 - \tilde{\rho}\|^2 + \|u_0\|_{H^1}^2 \leq \varepsilon_0, \end{cases}$$

then the Cauchy problem (1.1) – (1.3) *has a global weak solution* (ρ, u) *satisfying*

$$\begin{cases} \rho - \tilde{\rho} \in C\left([0,\infty); H^{-1}(\mathbf{R}^3)\right), \ \rho(\cdot\,,t) - \tilde{\rho} \in (L^2 \cap L^\infty)(\mathbf{R}^3), \quad \textit{a.e. } t > 0, \\ u \in C\left([0,\infty); L^2(\mathbf{R}^3)\right); \end{cases}$$
$$0 < \inf \rho \le \sup \rho < \infty.$$

Moreover, $(\rho, u) \to (\tilde{\rho}, 0)$ *as* $t \to \infty$ *in the sense that, for all* $p \in (2, \infty]$,

$$\lim_{t\to\infty} \left\| \left(\rho(\cdot\,,t) - \tilde{\rho},\ u(\cdot\,,t)\right) \right\|_p = 0.$$

REMARK. In the isothermal case $\gamma = 1$, for an arbitrary ϕ satisfying (2.2), if C_0 is sufficiently small, then the same statement as Theorem 1 holds.

REMARK. We can obtain further regularity of the solution. See [1]–[3].

Theorem 2 (Two-Dimensional Case). *When* $n = 2$*, replacing the assumption* (2.2) *in* Theorem 1 *by* (2.3)*, the similar statement to* Theorem 1 *holds.*

3. L^2 Bounds

In this section, we start to derive a priori estimates for smooth solutions of (1.1)–(1.2). As stated in the previous section, we shall mainly discuss the three-dimensional case.

To begin, let $\overline{\gamma}$, $\underline{\rho}$, $\overline{\rho}$ be as (2.6) and (ρ, u) be a smooth solution of (1.1)–(1.2) which is defined up to a positive time T. (The assumption that (ρ, u) is smooth means that $(\rho - \tilde{\rho}, u) \in (C^{k_1} \cap H^{k_2})(\mathbf{R}^n \times [0, T])$ for sufficiently large k_1, k_2.) And we assume that $\gamma \in [1, \overline{\gamma}]$, $\rho(x,t) \in [\underline{\rho}, \overline{\rho}]$ and Φ, Ψ, $C_0 \le 1$. Moreover, $C > 0$ will denote a generic positive constant which may depend on M, ρ_∞, a, μ and λ, but not on γ (as long as $\gamma \le \overline{\gamma}$) and T. Under these assumptions, we remark that

$$|\nabla \tilde{\rho}(x)| \le \frac{C}{1+|x|}, \quad |D^2 \tilde{\rho}(x)| \le \frac{C}{(1+|x|)^2}.$$

In the following proposition, we derive a bound for the quantity Φ.

Proposition 1. $\Phi(T) \le C\left\{C_0 + (\gamma-1)^2 R_1(T) + R_2(T) + R_3(T)\right\}$.

The proof consists of three separate energy-type estimates:

Lemma 3.

$$\Phi_1(T) \le CC_0, \tag{3.1}$$
$$\Phi_2(T) \le C\{C_0 + (\gamma-1)^2 R_1(T) + R_2(T)\}, \tag{3.2}$$
$$\Phi_3(T) \le C\{C_0 + (\gamma-1)^2 R_1(T) + R_2(T) + R_3(T)\}. \tag{3.3}$$

Proof of (3.1). First, applying the mass equation (1.1) and the fact that $\nabla p(\tilde\rho) = -\tilde\rho\nabla\phi$, we rewrite the momentum equation (1.2) in the form

$$\rho\dot u + \rho\left\{\frac{\nabla p(\rho)}{\rho} - \frac{\nabla p(\tilde\rho)}{\tilde\rho}\right\} - \mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) = 0. \tag{3.4}$$

Multiplying (3.4) by u and integrating the resultant equation, we obtain

$$\begin{aligned}\int_0^t\!\!\int \rho u\cdot\dot u + \int_0^t\!\!\int \rho u\cdot\left\{\frac{\nabla p(\rho)}{\rho} - \frac{\nabla p(\tilde\rho)}{\tilde\rho}\right\}&\\ - \int_0^t\!\!\int u\cdot\{\mu\Delta u + (\lambda+\mu)\nabla(\operatorname{div} u)\} = 0.&\end{aligned} \tag{3.5}$$

Here and in what follows we omit the symbols of integral variables, e.g. '$dxd\tau$', 'dx', and so on, in integral notation unless we are confused. Now noting that

$$\int \rho\dot f\,dx = \frac{d}{dt}\int \rho f\,dx$$

holds in general, the first term on the left hand side of (3.5) is

$$\int_0^t\!\!\int \rho\frac{D}{Dt}\frac{|u|^2}{2} = \frac12\int\rho|u|^2\Big|_0^t.$$

Next, the second term is

$$\begin{aligned}\int_0^t\!\!\int \rho u\cdot\nabla\int_{\tilde\rho}^{\rho}\frac{p'(s)}{s}ds &= -\int_0^t\!\!\int \operatorname{div}(\rho u)\int_{\tilde\rho}^{\rho}\frac{p'(s)}{s}ds\\ &= \int_0^t\!\!\int \rho_t\int_{\tilde\rho}^{\rho}\frac{p'(s)}{s}ds\\ &= \int G(\rho)\Big|_0^t,\end{aligned}$$

where we define

$$G(\rho) := \int_{\tilde\rho}^{\rho}\int_{\tilde\rho}^{r}\frac{p'(s)}{s}dsdr.$$

Integrating by parts also in the third integral on the left side of (3.5), we obtain the energy-balance relation:

$$\frac12\int\left\{\rho|u|^2 + G(\rho)\right\}\Big|_0^t + \int_0^t\!\!\int\left\{\mu|\nabla u|^2 + (\lambda+\mu)(\operatorname{div} u)^2\right\} = 0. \tag{3.6}$$

An easy observation that

$$C^{-1}(\rho-\tilde\rho)^2 \le G(\rho) \le C(\rho-\tilde\rho)^2$$

which follows by use of $G(\tilde\rho) = G'(\tilde\rho) = 0$ and $\rho \in [\underline{\rho}, \overline{\rho}]$, completes the proof of (3.1). $\square$

REMARK. We know here the uniqueness of the stationary solution at least in a $H^3 \times H^4$ ($\supset C^1 \times C^2$)-neighborhood of $(\rho_\infty, 0)$. Indeed, we let $(\tilde\rho, \tilde u)$ be such a solution then, by (3.6), it follows that $|\nabla \tilde u| \equiv 0$, namely $\tilde u \equiv \text{const.} = 0$.

Proof of (3.2). We split the second term on the left side of (3.4) as follows:

$$\begin{aligned}
\rho\left\{\frac{\nabla p(\rho)}{\rho} - \frac{\nabla p(\tilde\rho)}{\tilde\rho}\right\} &= a\frac{\gamma}{\gamma-1}\rho\nabla\left(\rho^{\gamma-1} - \tilde\rho^{\gamma-1}\right)\\
&= a\frac{\gamma}{\gamma-1}\rho\nabla\left[\left\{\left(\frac{\rho}{\tilde\rho}\right)^{\gamma-1} - 1\right\}\tilde\rho^{\gamma-1}\right]\\
&= a\frac{\gamma}{\gamma-1}\left[\tilde\rho^{\gamma}\frac{\rho}{\tilde\rho}\nabla\left\{\left(\frac{\rho}{\tilde\rho}\right)^{\gamma-1} - 1\right\} + \rho\left\{\left(\frac{\rho}{\tilde\rho}\right)^{\gamma-1} - 1\right\}\nabla\left(\tilde\rho^{\gamma-1}\right)\right]\\
&= a\tilde\rho^{\gamma}\nabla\left\{\left(\frac{\rho}{\tilde\rho}\right)^{\gamma} - 1\right\} + a\gamma\frac{\rho}{\tilde\rho}\left(\rho^{\gamma-1} - \tilde\rho^{\gamma-1}\right)\nabla\tilde\rho.
\end{aligned}$$

Then, (3.4) becomes

$$\begin{aligned}
&\rho\dot u + a\tilde\rho^{\gamma}\nabla\left(\frac{\rho^{\gamma}}{\tilde\rho^{\gamma}} - 1\right) + a\gamma\frac{\rho}{\tilde\rho}\left(\rho^{\gamma-1} - \tilde\rho^{\gamma-1}\right)\nabla\tilde\rho\\
&- \mu\Delta u - (\lambda+\mu)\nabla(\operatorname{div} u) = 0.
\end{aligned}\tag{3.7}$$

Multiplying by $\dot u$ and integrating, we thus obtain

$$\begin{aligned}
&\int_0^t\int \rho|\dot u|^2 - \mu\int_0^t\int \dot u\cdot\Delta u - (\lambda+\mu)\int_0^t\int \dot u\cdot\nabla(\operatorname{div} u)\\
&= -a\int_0^t\int \tilde\rho^{\gamma}\dot u\cdot\nabla\left(\frac{\rho^{\gamma}}{\tilde\rho^{\gamma}} - 1\right) - a\gamma\int_0^t\int \frac{\rho}{\tilde\rho}\left(\rho^{\gamma-1} - \tilde\rho^{\gamma-1}\right)\dot u\cdot\nabla\tilde\rho.
\end{aligned}\tag{3.8}$$

- The second term on the left of (3.8) is

$$\begin{aligned}
-\mu\int_0^t\int u_t^j u_{ll}^j - \mu\int_0^t\int u^j u_j^k u_{ll}^k &= \mu\int_0^t\int u_{tl}^j u_l^j + \mu\int_0^t\int\Big(u_l^j u_j^k u_l^k + u^j\underbrace{u_{jl}^k u_l^k}_{=(|u_l^k|^2)_j/2}\Big)\\
&= \frac{\mu}{2}\int|\nabla u|^2\Big|_0^t + \int_0^t\int\mathcal{O}\left(|\nabla u|^3\right).
\end{aligned}$$

- By similar calculations, the third term on the left of (3.8) is

$$\frac{\lambda+\mu}{2}\int|\operatorname{div} u|^2\Big|_0^t + \int_0^t\int\mathcal{O}\left(|\nabla u|^3\right).$$

- On the other hand, the second term on the right of (3.8) is bounded by

$$C(\gamma-1)\int_0^t\int|\rho-\tilde{\rho}||\dot{u}||\nabla\tilde{\rho}|\leq C(\gamma-1)\int_0^t\int\frac{|\rho-\tilde{\rho}|}{1+|x|}|\dot{u}|.$$

- Finally, the first term on the right can be rewritten as follows:

$$\begin{aligned}
&a\int_0^t\int\dot{u}\cdot\nabla(\tilde{\rho}^\gamma)\left(\frac{\rho^\gamma}{\tilde{\rho}^\gamma}-1\right)+a\int_0^t\int\mathrm{div}(\dot{u})(\rho^\gamma-\tilde{\rho}^\gamma)\\
&=a\int_0^t\int u_t\cdot\nabla(\tilde{\rho}^\gamma)\left(\frac{\rho^\gamma}{\tilde{\rho}^\gamma}-1\right)+a\int_0^t\int(u\cdot\nabla u)\cdot\nabla(\tilde{\rho}^\gamma)\left(\frac{\rho^\gamma}{\tilde{\rho}^\gamma}-1\right)\\
&\quad+a\int_0^t\int(\mathrm{div}\,u_t)(\rho^\gamma-\tilde{\rho}^\gamma)+a\int_0^t\int\mathrm{div}(u\cdot\nabla u)(\rho^\gamma-\tilde{\rho}^\gamma)\\
&=:\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.
\end{aligned}$$

First,

$$\mathrm{II}\leq C\int_0^t\int|u||\nabla u||\nabla\tilde{\rho}|\leq C\int_0^t\int|\nabla u|^2.$$

Next, in light of

$$(\rho^\gamma)_t=-\left\{\gamma\rho^\gamma\,\mathrm{div}\,u+u\cdot\nabla(\rho^\gamma)\right\} \tag{3.9}$$

which follows from the mass equation (1.1), we obtain that

$$\begin{aligned}
\mathrm{I}&=a\int u\cdot\nabla(\tilde{\rho}^\gamma)\left(\frac{\rho^\gamma}{\tilde{\rho}^\gamma}-1\right)\Bigg|_0^t+a\int_0^t\int u\cdot\nabla(\tilde{\rho}^\gamma)\frac{1}{\tilde{\rho}^\gamma}\left\{\gamma\rho^\gamma\,\mathrm{div}\,u+u\cdot\nabla(\rho^\gamma)\right\}\\
&=:\mathrm{I}_1+\mathrm{I}_2+\mathrm{I}_3;
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}_1&\leq C\left\{\int|u||\rho-\tilde{\rho}|(t)+\int|u_0||\rho_0-\tilde{\rho}|\right\},\\
\mathrm{I}_2&\leq C\int_0^t\int|u||\nabla u||\nabla\tilde{\rho}|\leq C\int_0^t\int|\nabla u|^2,\\
\mathrm{I}_3&=-a\int_0^t\int\mathrm{div}\left(u\cdot\nabla(\rho^\gamma)\frac{1}{\tilde{\rho}^\gamma}u\right)\rho^\gamma\\
&\leq C\int_0^t\int\left\{|u||\nabla u||\nabla\tilde{\rho}|+|u|^2\left(|\nabla\tilde{\rho}|^2+|D^2\tilde{\rho}|\right)\right\}\\
&\leq C\int_0^t\int|\nabla u|^2.
\end{aligned}$$

Similarly,

$$\mathrm{III}=a\int(\mathrm{div}\,u)(\rho^\gamma-\tilde{\rho}^\gamma)\Big|_0^t+a\int_0^t\int(\mathrm{div}\,u)\left\{\gamma\rho^\gamma\,\mathrm{div}\,u+u\cdot\nabla(\rho^\gamma)\right\}$$

$$=: \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3;$$

$$\mathrm{III}_1 \le C\left\{\int |\nabla u||\rho - \tilde\rho|(t) + \int |\nabla u_0||\rho_0 - \tilde\rho|\right\},$$
$$\mathrm{III}_2 \le C\int_0^t\int |\nabla u|^2.$$

For the terms containing the second derivatives of u,

$$\begin{aligned}
&\mathrm{III}_3 + \mathrm{IV}\\
&= a\int_0^t\int \rho^\gamma \underbrace{\{-\operatorname{div}(u\operatorname{div}u) + \operatorname{div}(u\cdot\nabla u)\}}_{=-\left(u^j u^k_k\right)_j + \left(u^j u^k_j\right)_k \le C|\nabla u|^2} - a\int_0^t\int \tilde\rho^\gamma \operatorname{div}(u\cdot\nabla u) + a\int_0^t\int \gamma\rho^\gamma(\operatorname{div}u)^2\\
&\le C\int_0^t\int \left(|\nabla u|^2 + |u||\nabla u||\nabla\tilde\rho|\right)\\
&\le C\int_0^t\int |\nabla u|^2.
\end{aligned}$$

Substituting these estimates back into (3.8) and applying the previous bound (3.1), we then obtain (3.2). $\square$

Proof of (3.3). Noting that

$$\left[\frac{\partial}{\partial t} + \operatorname{div}(u\,\cdot)\right](\rho f) = \rho\dot f,$$

we operate $\sigma\dot u^j[\partial/\partial t + \operatorname{div}(u\,\cdot)]$ to $(3.7)^j$ and integrate. Then we obtain

$$\begin{aligned}
&\int_0^t\sigma\int \rho\frac{D}{Dt}\frac{|\dot u|^2}{2} - \mu\int_0^t\sigma\int \dot u^j\left\{\Delta u^j_t + \operatorname{div}(\Delta u^j u)\right\}\\
&\quad - (\lambda+\mu)\int_0^t\sigma\int \dot u^j\left\{\operatorname{div}u_{jt} + \operatorname{div}\big((\operatorname{div}u_j)u\big)\right\}\\
&= -a\int_0^t\sigma\int \dot u^j\tilde\rho^\gamma\left\{\left(\frac{\rho^\gamma}{\tilde\rho^\gamma}\right)_{jt} + \operatorname{div}\left(\left(\frac{\rho^\gamma}{\tilde\rho^\gamma}\right)_j u\right)\right\}\\
&\quad - a\gamma\int_0^t\sigma\int \dot u^j\rho\frac{D}{Dt}\left\{\frac{\tilde\rho_j}{\tilde\rho}(\rho^{\gamma-1} - \tilde\rho^{\gamma-1})\right\}.
\end{aligned}\tag{3.10}$$

- The first term on the left side of (3.10) is

$$\frac{\sigma}{2}\int\rho|\dot u|^2\Big|_0^t - \frac{1}{2}\int_0^t\sigma'\int\rho|\dot u|^2 = \frac{\sigma(t)}{2}\int\rho|\dot u|^2(t) + \int_0^t\int\mathcal{O}\left(|\dot u|^2\right).$$

- Next, the second term on the left of (3.10) is

$$
\begin{aligned}
&\mu \int_0^t \sigma \int \left\{ \left(\dot{u}^j\right)_k u^j_{tk} + \left(\dot{u}^j\right)_k u^j_{ll} u^k \right\} \\
&= \mu \int_0^t \sigma \int \left(\dot{u}^j\right)_k \left(u^j_t + u^l u^j_l\right)_k - \mu \int_0^t \sigma \int \left\{ - \left(\dot{u}^j\right)_k \left(u^l u^j_l\right)_k + \left(\dot{u}^j\right)_k u^k u^j_{ll} \right\} \\
&= \mu \int_0^t \sigma \int |\nabla \dot{u}|^2 + \mu \int_0^t \sigma \int \Big\{ \underbrace{\left(\dot{u}^j\right)_{kl} u^l u^j_k - \left(\dot{u}^j\right)_{lk} u^k u^j_l}_{=0} + \mathcal{O}\left(|\nabla \dot{u}||\nabla u|^2\right) \Big\} \\
&= \mu \int_0^t \sigma \int |\nabla \dot{u}|^2 + \int_0^t \sigma \int \mathcal{O}\left(|\nabla \dot{u}||\nabla u|^2\right).
\end{aligned}
$$

- Similarly, the third term on the left of (3.10) is

$$
(\lambda + \mu) \int_0^t \sigma \int |(\dot{\operatorname{div} u})|^2 + \int_0^t \sigma \int \mathcal{O}\left(|(\dot{\operatorname{div} u})||\nabla u|^2\right).
$$

- On the other hand, the second term on the right side of (3.10) is

$$
\begin{aligned}
&- a\gamma \int_0^t \sigma \int \dot{u}^j \rho \Bigg\{ \left(u \cdot \nabla \frac{\tilde{\rho}_j}{\tilde{\rho}} \right) \left(\rho^{\gamma-1} - \tilde{\rho}^{\gamma-1}\right) + \frac{\tilde{\rho}_j}{\tilde{\rho}} \Bigg(\underbrace{\frac{D}{Dt} \rho^{\gamma-1}}_{=(1-\gamma)\rho^{\gamma-1} \operatorname{div} u} - u \cdot \nabla \left(\tilde{\rho}^{\gamma-1}\right) \Bigg) \Bigg\} \\
&\leq C \int_0^t \sigma \int |\dot{u}| \big(|\nabla \tilde{\rho}||u| + |\nabla u| \big).
\end{aligned}
$$

- Finally, due to (3.9), the first term on the right of (3.10) turns out to be

$$
\begin{aligned}
&a \int_0^t \sigma \int \left(\dot{u}^j \tilde{\rho}^\gamma\right)_j \frac{-\gamma \rho^\gamma \operatorname{div} u}{\tilde{\rho}^\gamma} + a \int_0^t \sigma \int \left(\dot{u}^j \tilde{\rho}^\gamma\right)_j \frac{-u \cdot \nabla(\rho^\gamma)}{\tilde{\rho}^\gamma} \\
&+ a \int_0^t \sigma \int \left(\dot{u}^j \tilde{\rho}^\gamma\right)_k \left(\frac{\rho^\gamma}{\tilde{\rho}^\gamma} \right)_j u^k \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}
$$

Here,

$$
\mathrm{I} \leq C \int_0^t \sigma \int |\nabla u| \big(|\nabla \dot{u}| + |\nabla \tilde{\rho}||\dot{u}| \big),
$$

and

$$
\begin{aligned}
\mathrm{II} + \mathrm{III} &= a \int_0^t \sigma \int \left[\left\{ \left(\dot{u}^j \tilde{\rho}^\gamma\right)_j \frac{u^k}{\tilde{\rho}^\gamma} \right\}_k \rho^\gamma - \left\{ \left(\dot{u}^j\right)_k \tilde{\rho}^\gamma u^k \right\}_j \frac{\rho^\gamma}{\tilde{\rho}^\gamma} \right] \\
&\leq C \int_0^t \sigma \int |\nabla \dot{u}| \big(|\nabla \tilde{\rho}||u| + |\nabla u| \big).
\end{aligned}
$$

Substituting these estimates back into (3.10) and applying (3.1)–(3.2), we thus obtain (3.3). $\square$

4. Weighted L^2 Bounds for Density and L^p Bounds for Velocity

In this section, we shall derive bounds for the terms R_1, R_2, and R_3, so that we close the estimates for Φ as in Proposition 2. All the assumptions and notations described in Section 3 will continue in this section.

Proposition 2. *If* $(\gamma - 1)$ *is sufficiently small, then*

$$\Phi(T) \leq C\left\{C_0 + \Phi(T)^{3/2} + \Psi(T)^2\right\}.$$

4.1. Basic L^p Estimates To begin, we shall state important estimates based on singular integral operator theory, with a formal proof. For more details, see Hoff [1] and Stein [15].

Lemma 4. *Let* $p,\ q \in (1, \infty)$. *Then for* $t > 0$,

$$\|\nabla u(t)\|_p \leq C_p\left(\|F(t)\|_p + \|\Omega(t)\|_p + \|(\rho - \tilde{\rho})(t)\|_p\right), \tag{4.1}$$

$$\|\nabla F(t)\|_q,\ \|\operatorname{div}\Omega(t)\|_q \leq C_q\left\{\|\dot{u}(t)\|_q + \|\nabla u(t)\|_q + (\gamma - 1)\left\|\frac{(\rho - \tilde{\rho})(t)}{1 + |x|}\right\|_q\right\}. \tag{4.2}$$

Proof. These inequalities are followed from the Martinkiewicz theorem, namely the fact that the operator '∇^2/Δ' is linear and bounded on L^p.

- (4.1) is easily derived by the following bound: *for each* $f \in W^{1,p}$,

$$\|\nabla f\|_p \leq C_p(\|\operatorname{div} f\|_p + \|\operatorname{curl} f\|_p).$$

This inequality is, for example, obtained by the following formal calculation:

$$\begin{aligned}(\nabla f)^{j,l} &= \frac{\partial_l}{\Delta}\Delta f^j \\ &= \frac{\partial_l}{\Delta}\left\{\partial_j \operatorname{div} f + \partial_k(\operatorname{curl} f)^{j,k}\right\} \\ &= \frac{\partial_j\partial_l}{\Delta}\operatorname{div} f + \frac{\partial_k\partial_l}{\Delta}(\operatorname{curl} f)^{j,k}.\end{aligned}$$

- To prove (4.2), we rewrite the equation (3.7) using the quantities F, Ω as follows:

$$\nabla F + \mu \operatorname{div}\Omega = \frac{\rho}{\tilde{\rho}^\gamma}\dot{u} + \mathcal{O}\big((\gamma - 1)|\nabla\tilde{\rho}||\rho - \tilde{\rho}| + |\nabla\tilde{\rho}||\nabla u|\big).$$

Operating $\nabla \operatorname{div}/\Delta$, we get the required result for ∇F, therefore also for $\operatorname{div}\Omega$. $\square$

4.2. Weighted L^2 Bounds for Density and L^p Bounds for Velocity Using the above estimates we shall derive bounds for R_1, R_2 and R_3, to obtain Proposition 2.

Lemma 5. *If* $(\gamma - 1)$ *is sufficiently small, then*

$$R_1(T) \leq C\left\{C_0 + \Phi(T)\right\}, \tag{4.3}$$

$$R_2(T) \leq C\left\{C_0^{3/2} + \Phi(T)^{3/2} + \Psi(T)^{3/2}\right\}, \tag{4.4}$$

$$R_3(T) \leq C\left\{C_0^2 + \Phi(T)^2 + \Psi(T)^2\right\}. \tag{4.5}$$

To prove, we here write down again the mass equation in the following form:

$$(2\mu + \lambda)\frac{D}{Dt}(\rho - \tilde{\rho}) + a\rho(\rho^\gamma - \tilde{\rho}^\gamma) = -\rho\tilde{\rho}^\gamma F - (2\mu + \lambda)u \cdot \nabla\tilde{\rho}. \tag{4.6}$$

Proof of (4.3). Multiplying (4.6) by $\rho(\rho - \tilde{\rho})/(1 + |x|)^2$, we then get

$$\begin{aligned}
&\frac{2\mu + \lambda}{2}\rho\frac{D}{Dt}\frac{|\rho - \tilde{\rho}|^2}{(1 + |x|)^2} + a\rho^2\frac{(\rho^\gamma - \tilde{\rho}^\gamma)(\rho - \tilde{\rho})}{(1 + |x|)^2} \\
&= -\rho^2\tilde{\rho}^\gamma\frac{F}{1 + |x|} \cdot \frac{\rho - \tilde{\rho}}{1 + |x|} - (2\mu + \lambda)\frac{u \cdot \nabla\tilde{\rho}}{1 + |x|} \cdot \frac{\rho - \tilde{\rho}}{1 + |x|} + \mathcal{O}\left(|\rho - \tilde{\rho}|^2|u|\left|\nabla\frac{1}{(1 + |x|)^2}\right|\right).
\end{aligned}$$

Therefore,

$$\rho\frac{D}{Dt}\frac{|\rho - \tilde{\rho}|^2}{(1 + |x|)^2} + C^{-1}\frac{|\rho - \tilde{\rho}|^2}{(1 + |x|)^2} \leq C\left\{\frac{|F|^2}{(1 + |x|)^2} + \frac{|u|^2}{(1 + |x|)^2}\right\}.$$

Integrating and by use of (4.2),

$$\begin{aligned}
&\int \rho\frac{|\rho - \tilde{\rho}|^2}{(1 + |x|)^2}\bigg|_0^t + C^{-1}\int_0^t\int\frac{|\rho - \tilde{\rho}|^2}{(1 + |x|)^2} \\
&\leq C\int_0^t\left(\|\nabla F\|^2 + \|\nabla u\|^2\right) \\
&\leq C\int_0^t\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) + C(\gamma - 1)^2\int_0^t\int\frac{|\rho - \tilde{\rho}|^2}{(1 + |x|)^2}.
\end{aligned}$$

Thus if $(\gamma - 1)$ is sufficiently small, (4.3) holds. □

Now, it will be convenient to state some bounds as a lemma that follow easily from (4.3) and (4.2).

Lemma 6. *If* $(\gamma - 1)$ *is sufficiently small, then*

$$\left.\begin{array}{l} \sup\limits_{0\le t\le T} \sigma(t)\|\nabla F(t)\|^2 + \int_0^T \|\nabla F(t)\|^2 dt \\ \sup\limits_{0\le t\le T} \sigma(t)\|\operatorname{div}\Omega(t)\|^2 + \int_0^T \|\operatorname{div}\Omega(t)\|^2 dt \end{array}\right\} \le C\{C_0 + \Phi(T)\}. \tag{4.7}$$

Proof of (4.5). In light of (4.1), it suffices to prove the following bounds:

$$\int_0^T \sigma\|F\|_4^4, \int_0^T \sigma\|\Omega\|_4^4 \le C\left(C_0^2 + \Phi^2\right), \tag{4.8}$$

$$\int_0^T \sigma\|\rho - \tilde{\rho}\|_4^4 \le C\left(C_0^2 + \Phi^2 + \Psi^2\right). \tag{4.9}$$

- For (4.8), we estimate as follows:

$$\begin{aligned} \int_0^T \sigma\|F\|_4^4 &\le \int_0^T \sigma\|F\|\|\nabla F\|^3 \\ &\le \sup_{0\le t\le T}\left(\|F\|^2 + \sigma\|\nabla F\|^2\right)\int_0^T \|\nabla F\|^2 \\ &\le C\left(C_0^2 + \Phi^2\right). \end{aligned}$$

Here we applied (4.7) in the last inequality. The bound for Ω is proved in a similar way.

- The proof of (4.9) is similar to that of (4.3). Multiplying (4.6) by $\sigma\rho(\rho - \tilde{\rho})^3$, we then obtain

$$\sigma\rho\frac{D}{Dt}|\rho - \tilde{\rho}|^4 + C^{-1}\sigma|\rho - \tilde{\rho}|^4 \le C\sigma\left(|F|^4 + |u|^4\right).$$

Integrating the first term in t by parts,

$$\sigma(T)\int \rho|\rho - \tilde{\rho}|^4(T) + \int_0^T \sigma\int |\rho - \tilde{\rho}|^4 \le C\int_0^1\int |\rho - \tilde{\rho}|^4 + \int_0^T \sigma\int\left(|F|^4 + |u|^4\right).$$

In light of that the first term on the right side is bounded by $CC_0\Psi(1)$, and that the second integral on the right has already been estimated in (4.8) (where the term for u may be estimated similarly), we then obtain (4.9). □

Proof of (4.4). We divide the proof into the two cases as follows:

- When $T \ge 1$, applying (4.5) to obtain

$$\int_0^T \|\nabla u\|_3^3 \le \left(\int_0^T \|\nabla u\|^2\right)^{1/2}\left(\int_0^T \|\nabla u\|_4^4\right)^{1/2}$$

$$\leq C\left(C_0^{3/2} + \Phi^{3/2} + \Psi^{3/2}\right).$$

• When $T \leq 1$, we shall discuss as same as the proof of (4.5). Specifically, it suffices to estimate $\int_0^1 \|F\|_3^3$, $\int_0^1 \|\Omega\|_3^3$ and $\int_0^1 \|\rho - \tilde{\rho}\|_3^3$. For the term of F, we have that

$$\begin{aligned}
\int_0^1 \|F\|_3^3 &\leq \int_0^1 \|F\|^{3/2}\|\nabla F\|^{3/2} \\
&\leq \left(\int_0^1 \|F\|^6\right)^{1/4}\left(\int_0^1 \|\nabla F\|^2\right)^{3/4} \\
&\leq CC_0^{3/4}(C_0 + \Phi)^{3/4} \\
&\leq C\left(C_0^{3/2} + \Phi^{3/2}\right).
\end{aligned}$$

The required bounds for Ω is obtained similarly. Finally, for the term of ρ, multiplying (4.6) by $\rho\,\mathrm{sgn}(\rho - \tilde{\rho})|\rho - \tilde{\rho}|^2$ and integrating, we then get

$$\int \rho|\rho - \tilde{\rho}|^3(1) + \int_0^1 \int |\rho - \tilde{\rho}|^3 \leq C \int \rho_0|\rho_0 - \tilde{\rho}|^3 + \int_0^1 \left(\|F\|_3^3 + \|u\|_3^3\right).$$

Noting that the first term of the right side is bounded by $CC_0^{3/2}$, we then obtain the required estimates. $\square$

REMARK. When $n = 2$, the proof is similar except for some exponents concerned with Sobolev's embedding. For example, note that

$$\int_0^T \sigma\|F\|_4^4 \leq \int_0^T \sigma\|F\|^2\|\nabla F\|^2,$$

to prove (4.8), and so on.

5. Pointwise Bounds and Closing the Estimates

In this section, we derive pointwise bounds for the density and close the all estimates. Therefore, we will be able to complete the proof of Main Theorem by repeating the arguments of Hoff [1]–[3]. All the assumptions and notations described in Section 3 will continue to hold throughout this section.

Proposition 3. *If $(\gamma - 1)$ is sufficiently small, then for some $\theta > 0$,*

$$\Psi(T) \leq C\left\{C_0^\theta + \Phi(T) + \Psi(T)^2\right\}.$$

Proposition 4. *If $(\gamma - 1)$ and C_0 are sufficiently small, then*

$$\Phi(T) + \Psi(T) \leq CC_0^\theta.$$

Once we have Propositions 2 and 3, we can obtain Proposition 4 by an elementary argument based on the continuity of Φ and Ψ in t and on smallness of C_0. Therefore, it remains to prove Proposition 3, which is easily derived from the following lemma:

Lemma 7. *If* $(\gamma - 1)$ *is sufficiently small, then*

$$\Psi(T) \le C\,\{C_0 + \Phi(T) + \Psi(1)\}, \qquad \textit{for } T \ge 1, \tag{5.1}$$

$$\Psi(T) \le C\left\{C_0^\theta + \Phi(T) + \Psi(T)^2\right\}, \qquad \textit{for } T \le 1. \tag{5.2}$$

Proof of (5.1). Multiplying (4.6) by $\mathrm{sgn}(\rho - \tilde{\rho})|\rho - \tilde{\rho}|^{7/3}$, we then get

$$\frac{D}{Dt}|\rho - \tilde{\rho}|^{10/3} + C^{-1}|\rho - \tilde{\rho}|^{10/3} \le C\left(|F|^{10/3} + |u|^{10/3}\right). \tag{5.3}$$

Integrating along particle trajectories to obtain that, for $t \in [1, T]$,

$$\|(\rho - \tilde{\rho})(t)\|_\infty^{10/3} \le \|(\rho - \tilde{\rho})(1)\|_\infty^{10/3} + C\int_1^t \left(\|F\|_\infty^{10/3} + \|u\|_\infty^{10/3}\right).$$

Applying the embedding $W^{1,10/3}(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$ to the second term on the right side, we then get

$$\Psi(T)^{5/3} \le \Psi(1)^{5/3} + C\int_1^T \left\{\|u\|_{10/3}^{10/3} + \left(\|F\|_{10/3}^{10/3} + \|\nabla u\|_{10/3}^{10/3}\right) + \|\nabla F\|_{10/3}^{10/3}\right\}.$$

- For the first term in the integral on the right side,

$$\int_1^T \|u\|_{10/3}^{10/3} \le \int_1^T \|u\|^{4/3}\|\nabla u\|^2 \le \sup_{1\le t\le T}\|u\|^{4/3}\int_1^T \|\nabla u\|^2 \le CC_0^{5/3}.$$

- Next, by Lemma 4 (4.1),

$$\int_1^T \left(\|F\|_{10/3}^{10/3} + \|\nabla u\|_{10/3}^{10/3}\right) \le C\int_1^T \left(\|F\|_{10/3}^{10/3} + \|\Omega\|_{10/3}^{10/3} + \|\rho - \tilde{\rho}\|_{10/3}^{10/3}\right).$$

For the term of F,

$$\int_1^T \|F\|_{10/3}^{10/3} \le \int_1^T \|F\|^{4/3}\|\nabla F\|^2 \le \sup_{1\le t\le T}\|F\|^{4/3}\int_1^T \|\nabla F\|^2 \le CC_0^{2/3}\{C_0 + \Phi(T)\}.$$

We apply a similar argument to Ω. And the term for ρ may be estimated by (4.6) as

$$\int_1^T \|\rho - \tilde{\rho}\|_{10/3}^{10/3} \le C\left\{C_0^{5/3} + \int_1^T \left(\|F\|_{10/3}^{10/3} + \|u\|_{10/3}^{10/3}\right)\right\}.$$

- Finally, applying (4.2), we then get

$$\int_1^T \|\nabla F\|_{10/3}^{10/3} \le C \int_1^T \left(\|\dot{u}\|_{10/3}^{10/3} + \|\nabla u\|_{10/3}^{10/3} + \|\rho - \tilde{\rho}\|_{10/3}^{10/3} \right),$$

and in light of $T \ge 1$, using

$$\int_1^T \|\dot{u}\|_{10/3}^{10/3} \le \int_1^T \|\dot{u}\|^{4/3} \|\nabla \dot{u}\|^2 \le \sup_{1 \le t \le T} \|\dot{u}\|^{4/3} \int_1^T \|\nabla \dot{u}\|^2 \le \Phi(T)^{5/3},$$

then the required bounds are easily derived. $\square$

REMARK. When $n = 2$, using that $W^{1,4}(\mathbf{R}^2) \hookrightarrow L^\infty(\mathbf{R}^2)$, we can derive the required estimates in a similar way.

Proof of (5.2). Integrating (4.6) over a fixed particle path $x(t)$, we obtain that, for $t \in (0, T]$,

$$|\rho - \tilde{\rho}|(x(t), t) \le C \left\{ C_0^{1/2} + \int_0^t \left(\|F\|_\infty + \|u\|_\infty \right) + \int_0^t |\rho - \tilde{\rho}|(x(s), s) ds \right\}.$$

Applying Gronwall's inequality in light of $T \le 1$, and taking appropriate supremums, we then get

$$\Psi(T)^2 \le C \left(C_0 + A^2 \right).$$

Here, by the embedding $W^{1,4}(\mathbf{R}^3) \hookrightarrow L^\infty(\mathbf{R}^3)$,

$$A := \int_0^T \left(\|F\|_\infty + \|u\|_\infty \right) \le \int_0^T \left\{ \|u\|_4 + \left(\|F\|_4 + \|\nabla u\|_4 \right) + \|\nabla F\|_4 \right\}.$$

- First,

$$\int_0^T \|u\|_4 \le \int_0^T \|u\|^{1/4} \|\nabla u\|^{3/4} \le \sup_{0 \le t \le T} (\|u\| + \|\nabla u\|) \le \Phi(T)^{1/2}.$$

- Next, again by Lemma 4 (4.1),

$$\int_0^T \left(\|F\|_4 + \|\nabla u\|_4 \right) \le C \int_0^T \left(\|F\|_4 + \|\Omega\|_4 + \|\rho - \tilde{\rho}\|_4 \right).$$

Here,

$$\int_0^T \|F\|_4 \le \int_0^T \|F\|^{1/4} \|\nabla F\|^{3/4} \le \left(\int_0^T \|F\|^{2/5} \right)^{5/8} \left(\int_0^T \|\nabla F\|^2 \right)^{3/8}$$

$$\leq CC_0^{1/8}(C_0 + \Phi(T))^{3/8} \leq C\left(C_0^{1/2} + \Phi(T)^{1/2}\right).$$

The bounds for Ω is similar. And the term for ρ can be estimated as

$$\int_0^T \|\rho - \tilde{\rho}\|_4 \leq CC_0^{1/4}\Psi(T)^{1/4} \leq C\left\{C_0^{1/3} + \Psi(T)\right\}.$$

- Finally, again by Lemma 4 (4.2),

$$\int_0^T \|\nabla F\|_4 \leq C\int_0^T \left(\|\dot{u}\|_4 + \|\nabla u\|_4 + \|\rho - \tilde{\rho}\|_4\right),$$

and note here that

$$\begin{aligned}\int_0^T \|\dot{u}\|_4 &\leq \int_0^T \|\dot{u}\|^{1/4}\|\nabla\dot{u}\|^{3/4} \\ &\leq \left(\int_0^T \sigma^{-3/4}\right)^{1/2}\left(\int_0^T \|\dot{u}\|^2\right)^{1/8}\left(\int_0^T \sigma\|\nabla\dot{u}\|^2\right)^{3/8} \\ &\leq C\Phi(T)^{1/2}.\end{aligned}$$

All these estimates complete the proof. □

REMARK. The proof in the case of $n = 2$ is similar.

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