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Osaka J. Math.
43 (2006), 371–399

ACTIVE SUMS II

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(Received March 24, 2005)

Abstract

We exhibit several finite groups that are not active sums of cyclic subgroups. We show that this is the case for groups with H_1G of odd order and H_2G of even order. As particular examples of this we have the alternating groups A_n for $n \geq 4$, some special and some projective linear groups. Our next set of examples consists of p-groups where the normalizer and the centralizer of every element coincide. We also have an example of a 2-group where the above conditions are not satisfied; thus we had to devise an ad hoc argument. We observe that the examples of p-groups given also provide groups that are not molecular.

Introduction

Given a generating family $\mathcal{F}$ of subgroups of a group G, closed under conjugation and with a partial order compatible with inclusion, a new group, called the active sum of $\mathcal{F}$, can be constructed, taking into account the multiplication in the subgroups and their mutual actions given by conjugation.

In Active Sums I many examples of groups that are active sums of cyclic subgroups are given. In the first section of the present paper we show that, for a finite group G, if H_1G has odd order and H_2G has even order, then G is not the active sum of cyclic subgroups. As a consequence the alternating groups A_n for $n \geq 4$, some special and some projective linear groups are not active sums of cyclic subgroups.

The remaining sections deal with p-groups.

Section 2 is devoted to groups such that the normalizer of every element coincides with its centralizer. Among these we present Pizaña's group, Pizaña-like groups, el chamuco, a family with trivial center and a family with cyclic commutator subgroup.

In Section 3 we study the group of Belana-Tomàs. In this 2-group normalizers of some elements do not coincide with their centralizers. Not being able to apply the techniques of Sections 1 and 2, we have to give an ad hoc argument to prove that it is not an active sum of cyclic subgroups.

Atomic groups are defined as groups normally generated by one element, and molecular groups as active sums of atomic groups (see Ribenboim [8]). Non-molecular groups are not exhibited in that paper. Since atomic p-groups are cyclic, every finite

2000 Mathematics Subject Classification. Primary 20J05, 20D99; Secondary 20D30.

p-group that is not the active sum of cyclic subgroups is not molecular. Thus, the examples in Sections 2 and 3 are not molecular. We make this explicit in Section 4.

We would like to thank J.A. Belana, M.A. Pizaña and F. Tomàs for the groups that bear their names in this paper.

1. Homology conditions

Fix a finite group G and a generating active family $\mathcal{F}$ of cyclic subgroups of G. We start with a technical lemma:

Lemma 1.1. *Let $F, H \in \mathcal{F}$, $g \in G$ and $x \in H$. If $H \leq F \geq H^g$ and the order of F is even, then $x^{-1}x^g \in F^2$ (where $F^2 = \{y^2 \mid y \in F\}$).*

Proof. Let $|F| = m$ and assume $F = \langle a \rangle$. Then $H = \langle a^k \rangle$ for some $k \in \mathbb{Z}_m$. Since $H^g \leq F$, we have that $(a^k)^g = a^r$ for some $r \in \mathbb{Z}_m$. This means that

$$a^{-k}\left(a^k\right)^g = a^{r-k}.$$

Notice that, since a^r and a^k are conjugate, they have the same order. Using the fact that m is even, it is not hard to see that $2|r-k$. This proves the lemma for a generator a^k of H. Taking now an arbitrary $x \in H$, we have that $x = a^{ks}$ for some $s \in \mathbb{N}$. Since F is Abelian we have

$$a^{-ks}\left(a^{ks}\right)^g = a^{-ks}\left(\left(a^k\right)^g\right)^s = \left(a^{-k}\left(a^k\right)^g\right)^s.$$

We conclude that $a^{-ks}(a^{ks})^g = a^{(r-k)s}$. Since $r-k$ is even, we are done. □

We use the lemma to prove:

Proposition 1.2. *If the order of $H_1(G)$ is odd and $\mathcal{F}$ is regular, then every element of $\mathcal{F}$ has odd order.*

Proof. Let $r = \max\{n \in \mathbb{N} \mid 2^n$ divides the order of some element of $\mathcal{F}\}$. Let's assume that $r \geq 1$. For every $F \in \mathcal{F}$, let R_F be the subgroup of F generated by elements of the form $x^{-1}x^g$ where $g \in G$ and $x \in H \leq F \geq H^g$ in $\mathcal{F}$. Define $\beta_F\colon F \to \mathbb{Z}_2$ as zero if 2^r does not divide the order of F, and as the only epimorphism there is if it does. We claim that $\beta_F(R_F) = \{0\}$. This is clear if β_F is trivial, and an easy consequence of Lemma 1.1 otherwise. The homomorphism β_F induces then a homomorphism $\widehat{\beta}_F\colon F/R_F \to \mathbb{Z}_2$. If $H \in \mathcal{F}$ is such that $H \leq F$, the inclusion of H in F induces a homomorphism $H/R_H \to F/R_F$. It is not hard to see that the diagram

$$\begin{tikzcd} H/R_H \arrow[r, "\widehat{\beta}_H"] \arrow[d] & \mathbb{Z}_2 \\ F/R_F \arrow[ur, "\widehat{\beta}_F"'] & \end{tikzcd}$$

commutes. We can then induce a homomorphism $\varphi\colon \varinjlim_{F\in\mathcal{F}} F/R_F \to \mathbb{Z}_2$ such that the diagram

$$\begin{tikzcd} F/R_F \arrow[r] \arrow[dr, "\widehat{\beta}_F"'] & \varinjlim_{F\in\mathcal{F}} F/R_F \arrow[d, "\varphi"] \\ & \mathbb{Z}_2 \end{tikzcd}$$

commutes.

Let F be such that $|F| = 2^r k$, with k odd. If $F = \langle a\rangle$ then a^k has order 2^r. Since the order of $a^kG' \in G/G' = H_1(G)$ is even and $H_1(G)$ is odd, we must have $a^k \in G'$. Thus $a^k \in F \cap G'$. Since k is odd we have that $\widehat{\beta}_F(a^k) = 1$. Therefore $\widehat{\beta}_F(F \cap G'/R_F)$ is not trivial. Since $\widehat{\beta}_F$ factors through $\varinjlim_{F\in\mathcal{F}} F/R_F$, we conclude that $\mathcal{F}$ is not regular. $\square$

Theorem 1.3. *If $|H_1(G)|$ is odd and $|H_2G|$ is even, then G is not the active sum of any active family of cyclic subgroups of G.*

Proof. Let $\mathcal{F}$ be a regular and independent generating active family of subgroups of G. By the previous proposition, every element of $\mathcal{F}$ must have odd order. Since 2 divides the order of H_2G, we can find a subgroup N of $H_2(G)$ such that $H_2G/N \simeq \mathbb{Z}_2$. If $\widehat{G}$ is a covering group for G, we have a stem extension (the short exact sequence $A \rightarrowtail B \twoheadrightarrow C$ is a stem extension if the image of A in B is contained in $Z(B) \cap B'$)

$$H_2G \rightarrowtail \widehat{G} \twoheadrightarrow G.$$

Dividing by N the first two factors, we obtain a stem extension of the form

$$\mathbb{Z}_2 \rightarrowtail H \twoheadrightarrow G.$$

According to Proposition 3.13 in [4], the active sum of the family projects onto H. $\square$

EXAMPLE (Special linear groups). Groups that satisfy the conditions of Theorem 1.3 are $SL_2(4)$, $SL_3(2)$, $SL_3(3)$, $SL_4(2)$, and $SL_3(4)$, according to [6]. Thus these groups are not active sums of cyclic subgroups.

EXAMPLE (Alternating groups). The alternating groups A_n, $n \geq 4$, are examples of groups that satisfy the conditions of Theorem 1.3. Indeed, according to [6], p.284, the Schur multiplier of A_n is $\mathbb{Z}_2$ if $n = 5$ or $n \geq 8$, and $\mathbb{Z}_6$ if $n = 6, 7$. Furthermore, the Schur multiplier of A_4 is $\mathbb{Z}_2$ [6], p.278. It is well know that the Schur multiplier is isomorphic to the second homology group. Observe that $A_4/A_4' \simeq \mathbb{Z}_3$ and $A_n/A_n' = 1$ for $n \geq 5$. Thus we have:

Proposition 1.4. *For $n \geq 4$, the group A_n is not the active sum of cyclic subgroups.*

EXAMPLE (Projective special linear groups). Many of the projective special linear groups satisfy the conditions of Theorem 1.3. According to [6], pp.244–246, the groups $PSL_n(q)$ are simple with the exception of $PSL_2(2)$ and $PSL_2(3)$. Furthermore, the Schur multiplier is given by

- $\mathbb{Z}_2$ for $PSL_2(4)$, $PSL_3(2)$, $PSL_4(2)$, and $PSL_3(3)$;
- $\mathbb{Z}_6$ for $PSL_2(9)$;
- $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ for $PSL_3(4)$;
- $\mathbb{Z}_{(n,q-1)}$ in all other cases.

Thus we have:

Proposition 1.5. *The group $PSL_n(q)$ is not the active sum of cyclic subgroups if it is one of the groups $PSL_2(4), PSL_3(2), PSL_4(2), PSL_3(3), PSL_3(4)$, or if $2|(n, q-1)$.*

Proof. According to [6], p.245, $PSL_2(3) \simeq A_4$, and we know that A_4 is not the active sum of cyclic subgroups. The other possibilities are covered by Proposition 1.3. □

2. When the centralizer and the normalizer of any element coincide

2.1. Conditions on the family. For groups G, such that $N_G(x) = C_G(x)$ for every $x \in G$, there is a strong condition of which cyclic groups can belong to a regular generating active family of cyclic subgroups of G. Namely:

Proposition 2.1. *Let G be a finite group such that $N_G(x) = C_G(x)$ for every $x \in G$. If $\mathcal{F}$ is a regular generating active family of cyclic subgroups of G, then for every $F \in \mathcal{F}$, $F \cap G'$ is trivial.*

Proof. Assume $\mathcal{F} = \langle F_i \rangle_I$, with F_i cyclic for every i. As in Section 1.2 in [4], define R_i to be the normal subgroup of F_i generated by elements of the form $x^{-1}x^g$, where either $x, g \in F_i$ or $x \in F_j$ and $F_j \leq F_i \geq F_j^g$ in $\mathcal{F}$. Since F_i is Abelian, $x, g \in F_i$ implies $x^{-1}x^g$ is trivial. Assume $x \in F_j$ and $F_j \leq F_i \geq F_j^g$ in $\mathcal{F}$. Since F_i is cyclic we must have $F_j = F_j^g$. If $F_j = \langle h \rangle$, then $g \in N_G(h) = C_G(h)$. Now,

x is some power of h, therefore $x^{-1}x^g$ is trivial. We have shown that R_i is the trivial group.

Let p be any prime number, and let α be the largest number such that p^α divides the order of G. We construct a family of morphisms $F_i \to \mathbb{Z}_{p^\alpha}$ in two steps. First define $\rho_i\colon F_i \to S_p(F_i)$ as the projection, where $S_p(F_i)$ denotes the Sylow p-subgroup of F_i. We will follow ρ_i by $\mu_i\colon S_p(F_i) \to \mathbb{Z}_{p^\alpha}$. However, we will define μ for all the cyclic subgroups of G.

Let $\mathcal{H}$ be the active family of all the cyclic subgroups of G. We define $\mu_H\colon S_p(H) \to \mathbb{Z}_{p^\alpha}$ recursively on the order of H, such that every μ_H is a monomorphism, and that they are compatible with inclusions and conjugations. If H is trivial define $\mu_H\colon S_p(H) \to \mathbb{Z}_{p^\alpha}$ as the trivial homomorphism. Assume now that for every $H \in \mathcal{H}$ with $o(H) < n$ we have defined $\mu_H\colon S_p(H) \to \mathbb{Z}_{p^\alpha}$ mono, and compatible with inclusions and conjugations. Assume $H \in \mathcal{H}$ has order n. If n is not a power of p, then $o(S_p(H)) < n$. In this case define $\mu_H = \mu_{S_p(H)}$. Assume now that $n = p^{s+1}$. Choose a transversal T for $\mathcal{H}$, and for every $t \in T$ choose H_t a representative. Let $t \in T$ be such that $o(H_t) = p^{s+1}$. Since H_t is cyclic, it has a unique subgroup K of order p^s. Then choose $\mu_{H_t}\colon S_p(H_t) = H_t \to \mathbb{Z}_{p^\alpha}$ mono such that the diagram

$$\begin{tikzcd} K \arrow[rr, hook] \arrow[dr, "\mu_K"'] & & H_t \arrow[dl, "\mu_{H_t}"] \\ & \mathbb{Z}_{p^\alpha} & \end{tikzcd}$$

commutes. Now, if L is in the same class that H_t, choose $g \in G$ such that $L^g = H_t$, and define μ_L as the composition

$$L \xrightarrow{(\ )^g} H_t \xrightarrow{\mu_{H_t}} \mathbb{Z}_{p^\alpha}.$$

If $h \in G$ is such that $L^h = H_t$, then we have that $L^{gh^{-1}} = L$. Since L is cyclic, we must have that gh^{-1} acts trivially on L, otherwise we would contradict that the normalizer and the centralizer of a generator of L coincide.

Let $f_i = \mu_{F_i} \circ \rho_i\colon F_i \to \mathbb{Z}_{p^\alpha}$. It is not hard to see that these morphisms induce a morphism $f\colon \varinjlim F_i/R_i = \varinjlim F_i \to \mathbb{Z}_{p^\alpha}$.

Assume that there is an i such that $F_i \cap G'$ is not trivial. Let p be a prime such that $p|o(F_i \cap G')$. We must then have a subgroup $H \subseteq F_i \cap G'$ of order p. Then the monomorphism $\mu_H\colon H \to \mathbb{Z}_{p^\alpha}$ factors as

$$H \hookrightarrow F_i \cap G' \hookrightarrow F_i \longrightarrow \varinjlim F_i \xrightarrow{f} \mathbb{Z}_{p^\alpha}.$$

We conclude that $\mathcal{F}$ is not regular. $\square$

2.2. Pizaña's group. We use Proposition 2.1 in the group defined below to reduce the number of active families of cyclic subgroups to be considered. Then we show that it is not the active sum of cyclic subgroups.

DEFINITION 2.2 (Pizaña's group). Let P be the following group. As a set $P = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Given (a, b, c) and (x, y, z) in P, we define their product as

$$(a, b, c)(x, y, z) = (a + x + 2cy, b + y + 2cx + 2cy, c + z + 2bx + 2cx),$$

where the sum is modulo 4 in all the coordinates.

We want to show that the group P is not the active sum of cyclic subgroups. We start with some basic properties of P. We calculate explicitly the inverse of an element in P.

Lemma 2.3. *Let $(a, b, c) \in P$. The inverse of (a, b, c) in P is given by*

$$(a, b, c)^{-1} = (3a + 2bc, 3b + 2ac + 2bc, 3c + 2ac + 2ab).$$

A routine calculation then shows that

$$[(a, b, c), (x, y, z)] = \left(2 \begin{vmatrix} b & c \\ y & z \end{vmatrix}, 2 \begin{vmatrix} a+b & c \\ x+y & z \end{vmatrix}, 2 \begin{vmatrix} a & b+c \\ x & y+z \end{vmatrix}\right).$$

Conversely, there are eight elements in P of the form $(2h, 2k, 2l)$. It is easy to see that each one of them can be expressed as the commutator of a couple of elements. For example $(0, 0, 2) = [(1, 0, 2), (1, 1, 0)]$ and $(2, 2, 2) = [(1, 3, 1), (2, 1, 0)]$. We have thus shown:

Lemma 2.4. *The commutator P' of P consists of those elements of the form $(2h, 2k, 2l)$.*

The next lemma describes the center of P:

Lemma 2.5. *The center $Z(P)$ of P is equal to the commutator P'.*

Proof. For the not so easy part, assume that $(a, b, c) \in Z(P)$. Then $(a, b, c + 1) = (a, b, c)(0, 0, 1) = (0, 0, 1)(a, b, c) = (a + 2b, 3b + 2a, 1 + c + 2a)$ means that a and b are even. That is to say $(a, b, c) = (2h, 2k, c)$. Using that this element commutes with $(1, 0, 0)$, we can see that c is also even. □

In particular, P is a class 2 group. Another formula for P:

Lemma 2.6. *If* $(a, b, c), (x, y, z) \in P$

$$(a, b, c)^{(x,y,z)} = \left(a + 2\begin{vmatrix} b & c \\ y & z \end{vmatrix}, b + 2\begin{vmatrix} a+b & c \\ x+y & z \end{vmatrix}, c + 2\begin{vmatrix} a & b+c \\ x & y+z \end{vmatrix}\right).$$

Proposition 2.7. *If* $(a, b, c) \in P$ *has order* 2, *then* $(a, b, c) \in P'$.

Proof. Assume $(a, b, c)^2 = (0, 0, 0)$. Since $(a, b, c)^2 = (2a+2bc, 2b+2ac+2bc, 2c+2ab+2ac)$ we conclude that

$$2|a + bc,$$
$$2|b + ac + bc,$$
$$2|c + ab + ac.$$

If a is odd, then since $2|a+bc$ we have that b, c are odd. This contradicts $2|b+ac+bc$. Therefore a is even. Since $2|c + ab + ac$, we conclude that c is also even. Therefore b is even as well. □

We now show:

Proposition 2.8. *For every* $w \in P$, $N_P(w) = C_P(w)$.

Proof. Observe that the statement of the proposition is equivalent to showing that, if $[(a, b, c), (x, y, z)]$ is a power of (a, b, c) then $[(a, b, c), (x, y, z)] = 0$.

If (a, b, c) has order 2, we have that $(a, b, c) \in Z(P)$. Therefore the commutator of (a, b, c) with any other element is trivial, and we are done. Assume then that (a, b, c) has order 4. This means on the one hand that at least one of a, b, c is odd, and on the other that the only non-trivial power of (a, b, c) that $[(a, b, c), (x, y, z)]$ can be is 2. The seven possible cases are listed in the following table. The first element in every row is the element (a, b, c), the second element is the commutator $[(a, b, c), (x, y, z)]$, and the third element is $(a, b, c)^2$. It is very easy to see that it is not possible in any case to solve the equation $[(a, b, c), (x, y, z)] = (a, b, c)^2$.

$(2h + 1, 2k + 1, 2l + 1)$	$(2(z + y), 2(x + y), 2(z + y))$	$(0, 2, 2)$
$(2h + 1, 2k + 1, 2l)$	$(2z, 0, 2(x + y + z))$	$(2, 2, 2)$
$(2h + 1, 2k, 2l + 1)$	$(2y, 2(x + y + z), 2(x + y + z))$	$(2, 2, 0)$
$(2h, 2k + 1, 2l + 1)$	$(2(z + y), 2(x + y + z), 0)$	$(2, 0, 2)$
$(2h, 2k, 2l + 1)$	$(2y, 2(x + y), 2x)$	$(0, 0, 2)$
$(2h, 2k + 1, 2l)$	$(2z, 2z, 2x)$	$(0, 2, 0)$
$(2h + 1, 2k, 2l)$	$(0, 2z, 2(y + z))$	$(2, 0, 0)$

□

Theorem 2.9. *The group P is not the active sum of cyclic subgroups.*

Proof. Let $\mathcal{F} = \langle F_i \rangle_{i \in I}$ be a generating active family of cyclic subgroups of P. Since $N_P(w) = C_P(w)$ for every $w \in P$, we must have $F_i \cap P' = \{e\}$ if we want $\mathcal{F}$ to be regular. However, every non-trivial cyclic subgroup of P intersects P' non-trivially. Therefore $\mathcal{F}$ is not regular. □

2.3. Pizaña-like groups. Let p be a prime number, $p \neq 2$, and $r \geq 2$. Consider the following group $P = P(p,r)$. As a set $P(p,r) = \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^r} \times \mathbb{Z}_{p^{r+1}}$. The multiplication is given by

$$(a,b,c)(x,y,z) = \left(a + x + p^{r-1}cy,\, b + y + p^{r-1}c(x+y),\, c + z + p^r x(b+c)\right),$$

where the first and second entries are modulo p^r, and the third is modulo p^{r+1}. The inverse of (a,b,c) is $(-a+p^{r-1}bc,\, -b+p^{r-1}c(a+b),\, -c+p^r a(b+c))$. This group is not an active sum of cyclic subgroups if 5 is not a square in $\mathbb{Z}_p$. We do not know what happens when 5 is a square in $\mathbb{Z}_p$. Observe that by the reciprocity law we have that

$$\left(\frac{p}{5}\right)\left(\frac{5}{p}\right) = (-1)^{(p-1)(5-1)/4} = (-1)^{p-1} = 1$$

for every prime $p > 2$, where $\left(\frac{p}{q}\right)$ is Legendre's symbol. This means that 5 is a square in $\mathbb{Z}_p$ if and only if p is a square in $\mathbb{Z}_5$. Since the only squares in $\mathbb{Z}_5$ are 0, 1 and 4, we have that 5 is a square in $\mathbb{Z}_p$ if and only if p is congruent to 0, 1 or 4 modulo 5. This is equivalent to p being ± 1 modulo 10 or $p = 5$. What this means is that $P(p,r)$ is not the active sum of cyclic subgroups if the last digit of p is 3 or 7.

We will show on the one hand that $N_P(x) = C_P(x)$, for every $x \in P$, and on the other, that any generating family of cyclic subgroups of P contains an element that intersects P'. By Proposition 2.1, this suffices to show that P is not an active sum of cyclic subgroups. We need then some facts about P.

A routine calculation shows that, for $(a,b,c), (x,y,z) \in P$, we have

$$[(a,b,c),(x,y,z)] = \left(p^{r-1}\begin{vmatrix} c & b \\ z & y \end{vmatrix},\, p^{r-1}\begin{vmatrix} c & a+b \\ z & x+y \end{vmatrix},\, p^r\begin{vmatrix} b+c & a \\ y+z & x \end{vmatrix}\right).$$

Next we determine the center and the commutator of P.

Lemma 2.10. *$P' = \{(p^{r-1}k, p^{r-1}l, p^r m) \mid 0 \leq k,l,m \leq p-1\}$ and $Z(P) = \{(pk, pl, pm) \mid 0 \leq k,l \leq p^{r-1}-1$ and $0 \leq m \leq p^r - 1\}$.*

Proof. By the above formula of the commutator, it is clear that every element of P' is of the form $(p^{r-1}k, p^{r-1}l, p^r m)$. It is also very easy to see that every element of the form (pk, pl, pm) is in $Z(P)$.

Assume $(a, b, c) \in Z(P)$. Then $(a, b, c)(0, 1, 0) = (0, 1, 0)(a, b, c)$. That is to say $(a + p^{r-1}c, b + 1 + p^{r-1}c, c) = (a, b + 1, c + p^r a)$. Therefore $p|a$ and $p|c$. Similarly, using that $(a, b, c)(1, 0, 0) = (1, 0, 0)(a, b, c)$ we conclude that $p|b$. Therefore (a, b, c) is of the required form.

Now, to generate all the elements of P' all we need is

$$\begin{aligned}
[(1,0,0),(0,1,0)] &= \left(0, 0, -p^r\right),\\
[(1,0,0),(0,0,1)] &= \left(0, -p^{r-1}, -p^r\right),\\
[(0,1,0),(0,0,1)] &= \left(-p^{r-1}, -p^{r-1}, 0\right).
\end{aligned}$$ □

It is clear then, that P is a class 2 group. We want to show that every element of order p is in P'. An inductive argument proves:

Lemma 2.11. *For every $n \in \mathbb{N}$ and every $(a, b, c) \in P$ we have that $(a, b, c)^n$ is equal to*

$$\left(na + \frac{n(n-1)}{2}p^{r-1}bc, nb + \frac{n(n-1)}{2}p^{r-1}c(a+b), nc + \frac{n(n-1)}{2}p^r a(b+c)\right).$$

In particular, since $p \neq 2$, we have that $(a, b, c)^p = (pa, pb, pc)$.

Lemma 2.12. *If an element of P has order p, then it belongs to P'.*

We have shown in particular that every non-trivial cyclic subgroup of P intersects P' non-trivially.

It remains to show that for every $u \in P$ we have that $N_P(u) = C_P(u)$ if 5 is not a square in $\mathbb{Z}_p$. Observe that if $(a, b, c) \neq (0, 0, 0)$ is an element of P, then the only powers of it that belong to P' are those of the form $(a, b, c)^{ko(a,b,c)/p}$, with $1 \leq k \leq p - 1$. What we have to show then, is that the equation

$$[(a, b, c), (x, y, z)] = (a, b, c)^{ko(a,b,c)/p} \tag{1}$$

has a solution for some $(a, b, c) \neq (0, 0, 0)$ if and only if 5 is a square in $\mathbb{Z}_p$.

Now, if $p|a$ and $p|b$ and $p|c$ then (a, b, c) is in the center of P and the commutator of (a, b, c) with any element is trivial. Therefore we must have that p does not divide all of a, b and c.

Assume first that p does not divide c. In this case $o(a, b, c) = p^{r+1}$. We have that $(a, b, c)^{ko(a,b,c)/p} = (0, 0, kp^r c)$. Using the formula for the commutator we have that equation (1) is

$$(0,0,kp^r c) = \left(p^{r-1} \begin{vmatrix} c & b \\ z & y \end{vmatrix}, p^{r-1} \begin{vmatrix} c & a+b \\ z & x+y \end{vmatrix}, p^r \begin{vmatrix} b+c & a \\ y+z & x \end{vmatrix} \right).$$

That is to say, we must solve the system

$$\begin{aligned} cy &\equiv bz \mod (p), \\ cx + cy &\equiv az + bz \mod (p), \\ bx + cx - ay - az &\equiv kc \mod (p). \end{aligned}$$

It is not hard to transform this system into

$$\begin{aligned} cy &\equiv bz \mod (p), \\ cx &\equiv az \mod (p), \\ bx - ay &\equiv kc \mod (p). \end{aligned}$$

If we multiply the third one by c and use the first and second equations we obtain

$$0 \equiv abz - abz \equiv bcx - acy \equiv kc^2 \mod (p).$$

since p and c are relatively prime, and $1 \leq k \leq p-1$, we see that there is no solution in this case.

Assume then that $c = pc'$. Since we can not have $p|a$ and $p|b$, we have that the order of (a,b,c) is p^r. In this case equation (1) is

$$\left(kp^{r-1}a, kp^{r-1}b, kp^r c'\right) = \left(p^{r-1} \begin{vmatrix} pc' & b \\ z & y \end{vmatrix}, p^{r-1} \begin{vmatrix} pc' & a+b \\ z & x+y \end{vmatrix}, p^r \begin{vmatrix} b+pc' & a \\ y+z & x \end{vmatrix} \right).$$

The system we obtain now is

$$\begin{aligned} ka &\equiv -bz \mod (p), \\ kb &\equiv -az - bz \mod (p), \\ kc' &\equiv bx - ay - az \mod (p). \end{aligned}$$

It is not hard to see that if the equations are satisfied, then $p|a$ implies $p|b$, and that $p|b$ implies $p|a$. We conclude that p does not divide a nor b. Then, from the first equation we obtain $z \equiv -b^{-1}ka \mod (p)$. If we substitute this value on the second equation we obtain $kb \equiv b^{-1}ka^2 + ka$. Since $1 \leq k \leq p-1$ and p does not divide b, we can cancel k and multiply by b to obtain

$$a^2 + ab - b^2 \equiv 0 \mod (p).$$

Solving for a we obtain

$$a = \frac{-b \pm b\sqrt{5}}{2}.$$

We conclude that if the system has a solution then 5 is a square in $\mathbb{Z}_p$. Conversely, if 5 is a square in $\mathbb{Z}_p$ then it is easy to see that the system has a solution taking $b = 1$, $a = (-1 + \sqrt{5})/2$. Thus we have shown:

Theorem 2.13. *Let p be a prime number, $p \neq 2$. If 5 is not a square in $\mathbb{Z}_p$, then the group $P(p, r)$ in not an active sum of cyclic subgroups.*

Though the equations are a little bit different, it can be shown that $P(2, r)$ with $r \geq 2$ is not an active sum of cyclic subgroups.

2.4. El chamuco. We follow the same strategy with the following group. The group in question is group (xv) in [2], p.146. The presentation given there is

$$\left\langle a, b, c \;\middle|\; a^9 = b^3 = c^3 = [a, b] = 1, a^c = ab, b^c = a^6 b \right\rangle.$$

Alternatively, it can be defined as $G = \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ as a set, with multiplication

$$(k, l, m)(x, y, z) = \left(k + x + 6kz + 6lz + 3kz^2, kz + l + y, m + z\right),$$

where the sum is modulo 9 in the first entry, and modulo 3 on the second and third. Then a corresponds to $(1, 0, 0)$, b to $(0, 1, 0)$ and c to $(0, 0, 1)$. Notice however that $(1, 1, 1)$ corresponds to cba, not to abc.

In the second presentation, the inverse of the element (k, l, m) is $(6km + 6km^2 + 8k + 6lm, km + 2l, 2m)$. We also have

$$[(k, l, m), (x, y, z)] = \left(3\left(2lz + kz^2 + 2kz + 2m^2 x + mx + ym\right), 2mx + kz, 0\right).$$

It is not hard to see that G' is generated by $(3, 0, 0)$ and $(0, 1, 0)$.

Lemma 2.14. *For every $(k, l, m) \in G$ we have $(k, l, m)^3 \in G'$.*

Proof. Observe that $(k, l, m)^3 = (3k + 6km^2, 0, 0)$. □

Proposition 2.15. *For every $w \in G$ we have that $N_G(w) = C_G(w)$.*

Proof. Let $(k, l, m) \in G$. We want to show that, if $(k, l, m)^{(x,y,z)} \in \langle (k, l, m) \rangle$, then $(k, l, m)^{(x,y,z)} = (k, l, m)$.

Assume first that (k, l, m) is in G'. Then (k, l, m) is of the form $(3k', l, 0)$. Observe that $(3k', l, 0)^{(x,y,z)} = (3k' + 6lz, l, 0)$, and that $(3k', l, 0)^r = (3k'r, lr, 0)$. For these last to be equal, we must have that $l(r-1) \equiv 0 \mod 3$. If $3|l$ then $(3k', l, 0)^{(x,y,z)} = (3k', l, 0)$. Assume then that $l \not\equiv 0 \mod 3$. We must then have $r \equiv 1 \mod 3$. Since we also must have $3k' + 6lz \equiv 3k'r \mod 9$, we conclude that $2lz \equiv k'(r-1) \mod 3$. Since $3|r-1$, we conclude that $3|lz$. Therefore $3|z$. We have then $(3k', l, 0)^{(x,y,z)} = (3k', l, 0)$. We conclude that $N_G(w) = C_G(w)$ if $w \in G'$.

Assume now that $(k, l, m) \notin G'$. If (k, l, m) has order 3, and $(k, l, m)^{(x,y,z)} = (k, l, m)^r$, then $[(k, l, m), (x, y, z)] = (k, l, m)^{r-1}$. Since $(k, l, m)^2 \notin G'$, we conclude that $r = 1$. Suppose that the order of (k, l, m) is 9. Then $r - 1$ can only be 3 or 6. We do the case $r = 3$ and leave the other to the reader. With $r - 1 = 3$ we have that $[(k, l, m), (x, y, z)] = (k, l, m)^{r-1}$ is

$$\left(3\left(2lz + kz^2 + 2kz + 2m^2x + mx + ym\right), 2mx + kz, 0\right) = \left(3k + 6km^2, 0, 0\right).$$

Since the order of (k, l, m) is 9, we have that $3k + 6km^2 = 3k(1 + 2m^2) \not\equiv 0 \mod 9$. We conclude that $k \not\equiv 0 \mod 3$, and $m = 0$. The above equation is then

$$\left(3\left(2lz + kz^2 + 2kz\right), kz, 0\right) = (3k, 0, 0).$$

Then $3|z$, and the equation becomes

$$(0, 0, 0) = (3k, 0, 0),$$

that has no solution, since $k \not\equiv 0 \mod 3$. □

We concluded in the above proof that (k, l, m) has order 9 if and only if $k \not\equiv 0 \mod 3$ and $m = 0$. Then the elements of order 9 are of the form $(3k' + 1, l, 0)$ or $(3k' + 2, l, 0)$. There are 18 elements of order 9 in G. G' has 8 elements of order 3. That leaves us with 54 elements of order 3 that are not elements of G', or 27 subgroups of order 3 whose intersection with G' is trivial. All these 27 groups are represented by a generator in the following table:

(0,0,1)	(1,0,1)	(2,0,1)
(3,0,1)	(4,0,1)	(5,0,1)
(6,0,1)	(7,0,1)	(8,0,1)
(0,1,1)	(1,1,1)	(2,1,1)
(3,1,1)	(4,1,1)	(5,1,1)
(6,1,1)	(7,1,1)	(8,1,1)
(0,2,1)	(1,2,1)	(2,2,1)
(3,2,1)	(4,2,1)	(5,2,1)
(6,2,1)	(7,2,1)	(8,2,1)

each column of it is a conjugacy class in G. It is not hard to see that the elements of a single column do not generate G. However, the elements of two columns do generate. If we consider all the groups above as an active family of subgroups of G and form the active sum S, it is not hard to see that S/S' is isomorphic to $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. Since $G/G' \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3$ we can not have $S \cong G$.

Therefore, if we still have some hope of getting $S \cong G$ then we can only include two of the columns in our active family. To see that it does not matter which two columns we include, we define an automorphism of G that sends the first column to the second, the second to the third, and the third to the first. The automorphism fixes a and b and sends c to ca. Since a, b and ca generate G, to see that this is indeed an automorphism it suffices to see that a, b and c satisfy the same relations as a, b and ca, but this is easy.

Assume then that we take as our active family of cyclic subgroups of G the first two columns. We want to show that the active sum S of the family is not isomorphic to G. We show this by exhibiting an epimorphism from S to the group

$$H := \left\langle w, x, y, z \mid w^3 = x^3 = y^3 = z^3 = [z, \_] = [x, y] = 1, x^w = xy, y^w = yz \right\rangle .$$

This suffices since H has the same order as G, but is not isomorphic to it. Then S can not be isomorphic to G, since there can not be an epimorphism from G to H.

To define the epimorphism $S \to H$, we use the universal property of the active sum S, defining the images in H of the elements in the first two columns of the table above. This is what we have in the following table, where the top element in each column is sent to the bottom element of the column.

c	ca^3	ca^6	cb	cba^3	cba^6	cb^2	cb^2a^3	cb^2a^6
w	wz^2	wz	wyz	wy	wyz^2	wy^2	wy^2z^2	wy^2z

ca	ca^4	ca^7	cba	cba^4	cba^7	cb^2a	cb^2a^4	cb^2a^7
x	x	x	xy	xy	xy	xy^2z	xy^2z	xy^2z

To show that we get a homomorphism, we must check that the assignment behaves well with respect to conjugation. This we leave to the reader. Since w and x generate H, we do get an epimorphism $S \to H$.

H is isomorphic to group (xi) of [2], p.145, and indeed not isomorphic to G.

2.5. A family with trivial center. Let p be a prime number. Multiplying by the matrix

$$\varphi = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

produces an automorphism $\varphi \colon \mathbb{Z}_p^2 \to \mathbb{Z}_p^2$ of order 3. We are interested in the family of groups $G = G(p) := \mathbb{Z}_3 \propto^{\varphi} \mathbb{Z}_p^2$. We show

Proposition 2.16. *If p is a prime number, and -3 is not a square in $\mathbb{Z}_p$, then the group $G(p)$ is not an active sum of cyclic subgroups.*

We work with the following presentation of G:

$$G := \left\langle a, b, c \;\middle|\; a^p = b^p = c^3 = [a, b] = 1, a^c = b^{-1}, b^c = ab^{-1} \right\rangle.$$

Lemma 2.17. *If p is a prime number, and -3 is not a square in $\mathbb{Z}_p$, then for every $w \in G$ we have that $C_G(w) = N_G(w)$.*

Proof. Notice that every element of G can be written as $a^i b^j c^k$ with $0 \leq i, j \leq p-1$ and $0 \leq k \leq 2$. Assume first that $w = a^i b^j$. Then we have that

$$(a^i b^j)^{a^x b^y c^z} = (a^i b^j)^{c^z}.$$

We want to show that, if $c^z \in N_G(w)$ then $c^z \in C_G(w)$. Clearly, it suffices to show this for $z = 1$. Assume then that $(a^i b^j)^c = (a^i b^j)^s$ for some $s \in \mathbb{Z}_p$. With a little calculation this equation becomes

$$a^j b^{-(i+j)} = a^{si} b^{sj}.$$

This means that

$$\begin{aligned} si &\equiv j \mod (p), \\ sj + i + j &\equiv 0 \mod (p). \end{aligned}$$

Substituting j on the second equation, we obtain

$$i(s^2 + s + 1) \equiv 0 \mod (p).$$

Since -3 is not a square in $\mathbb{Z}_p$, we have that $s^2 + s + 1 \equiv 0 \mod (p)$ has no solution in $\mathbb{Z}_p$. Therefore $i = 0$. Then $j = 0$ as well, and $w = e$.

Next consider the case where $w = a^i b^j c$. It is not hard to show that this element has order 3, and that the exponent of c, when w^2 is written of the form $a^x b^y c^z$, is 2. Furthermore, given any $v \in G$, the exponent of c when w^v is written in the form $a^x b^y c^z$ is 1. Therefore, the equation $w^v = w^2$ can not be solved for this w. Similar considerations apply if w is of the form $w = a^i b^j c^2$. □

It is easy to see that $G' = \langle a, b \rangle$. According to Proposition 2.1, any regular generating family of cyclic subgroups of G can only contain groups of the form $\langle a^i b^j c \rangle$. As we said before, all these groups have order 3. Since all Sylow 3-groups of G are conjugate, we have that all these groups are conjugate to $\langle c \rangle$. Observe that c and ac generate G. Thus, the only regular generating active family of cyclic subgroups of G is $\mathcal{F} = \{\langle a^i b^j c \rangle | i, j \in \mathbb{Z}_p\}$. Let S be the active sum of this family.

We will show that there is an epimorphism from S to the group H, constructed in the following way. First consider the group

$$M(p) = \left\langle x, y, z \;\middle|\; x^p = y^p = z^p = [z, \_] = 1, y^x = yz \right\rangle.$$

We have an automorphism ψ of $M(p)$ defined on the generators as $\psi(x) = y^{-1}$, $\psi(y) = xy^{-1}$ and $\psi(z) = z$. It is not hard to see that ψ has order 3. Let $H = \mathbb{Z}_3 \propto^{\psi} M(p)$.

Using the universal property of the active sum S, to define a homomorphism $S \to H$, we have to find, for every element $a^i b^j c$, an element in H, in such a way that the correspondence is compatible with conjugation. We will work with the following presentation of H:

$$H = \left\langle x, y, z, w \;\middle|\; x^p = y^p = z^3 = w^p = [w, \_] = 1, [x, y] = w, x^z = y^{-1}, y^z = xy^{-1} \right\rangle.$$

We will need the following lemma

Lemma 2.18. *For every $i, j \in \mathbb{Z}_p$ there exists a unique $\gamma(i, j) \in \mathbb{Z}_p$ such that $x^i y^j w z^{\gamma(i,j)}$ has order* 3.

Proof. First we list some useful formulas satisfied in H for every $i, j \in \mathbb{Z}_p$:

$$\begin{gathered}
y^j x^i = x^i y^j w^{-ij}, \\
z y^j = x^{-j} z, \\
z x^i = x^{-i} y^i z w^{i(i+1)/2}, \\
z^2 y^j = x^j y^{-j} z^2 w^{-j(-j+1)/2}, \\
z^2 x^i = y^{-i} z^2.
\end{gathered}$$

Then

$$(x^i y^j z)^3 = w^{(1/2)((i+j)^2+i+2ij-j)}.$$

Define $\gamma(i, j) = -(1/6)((i + j)^2 + i + 2ij - j)$. □

Then the element in H corresponding to $a^i b^j c$ is $x^i y^j z w^{\gamma(i,j)}$. Observe that the conjugation relations are preserved since the exponents of x and y behave with respect to w, in H, in the same way that the exponents of a and b behave with respect to c in G. We have thus shown Proposition 2.16.

2.6. A family with cyclic commutator subgroup. Let $p \neq 2$ be a prime number. We consider the following group

$$G = \left\langle a_1, a_2, a_3 \;\middle|\; a_1^{p^2} = a_2^{p^2} = a_3^{p^2} = [a_3, \_] = 1, [a_2, a_1] = a_3^p \right\rangle.$$

The following construction produces the same group. $G = (\mathbb{Z}_{p^2})^3$ as a set, with multiplication

$$(a, b, c) \cdot (x, y, z) = (a + x, b + y, c + z + pbx),$$

where the operations are modulo p^2 in each coordinate.

It is easy to see that, for any $(a, b, c) \in G$

$$(a, b, c)^{-1} = (-a, -b, -c + pab)$$

and for any $k \in \mathbb{Z}$

$$(a, b, c)^k = \left(ka, kb, kc + \frac{k(k-1)}{2} pab\right).$$

In particular $(a, b, c)^p = (pa, pb, pc)$. Therefore $G^p = \{(pa, pb, pc) \mid a, b, c \in \{0, 1, \ldots, p-1\}\}$. Furthermore, if (x, y, z) is another element of G, we have

$$(a, b, c)^{(x,y,z)} = (a, b, c + p(bx - ay)).$$

We also have

$$[(a, b, c), (x, y, z)] = (0, 0, p(bx - ay)).$$

From this equation, it is easy to see that $G' = \{(0, 0, pn) \mid n = 1, \ldots, p\} \cong \mathbb{Z}_p$. We have that $G/G' \cong \mathbb{Z}_{p^2} \oplus \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p$. It is not hard to see that $Z(G) = \{(pa, pb, c) \mid a, b, c \in \{0, \ldots, p-1\}\}$.

Lemma 2.19. *The normalizer of any element in G is equal to its centralizer.*

Lemma 2.20. $\Phi(G) = G^p$ *with* $\Phi(G)$ *Frattini subgroup of* G.

Proof. It is well known that for finite p-groups $\Phi(G) = G'G^p$. But in this case $G' \subset G^p$. $\square$

REMARK 2.21. We have that $G/\Phi(G) \cong (\mathbb{Z}_p)^3$. According to Theorem 1.1, p.173, [5], a subset $X \subset G$ generates G if and only if the image of X generates $G/\Phi(G)$.

It is clear that every element of order p is of the form (pa, pb, pc) with $a, b, c \in \{0, \ldots, p-1\}$ not all zero. There are $p^3 - 1$ elements of order p. Therefore, there are $(p^3 - 1)/(p - 1) = p^2 + p + 1$ subgroups of order p. It is easy to see that for every element of order p, (pa, pb, pc), there are p^3 elements (x, y, z) such that $(x, y, z)^p = (pa, pb, pc)$. Indeed, (x, y, z) must be of the form $(a + pl, b + pm, c + pn)$. Therefore (pa, pb, pc) belongs to p^2 subgroups of G of order p^2.

A picture of the cyclic subgroups of G looks as follows, where the labels on the arrows represent the number of arrows coming out of the node.

$$\begin{tikzcd}
 & & & & (1, pb, c+pn) \\
 & & & (p, 0, pc) \arrow[ur, "p^2"] & \\
(pa, pb, 1) & (0, 0, p) \arrow[l, "p^2"'] & (0, 0, 0) \arrow[l, "1"'] \arrow[ur, "p"] \arrow[r, "p"] \arrow[dr, "p(p-1)"'] & (0, p, pc) \arrow[r, "p^2"] & (pa, 1, c+pn) \\
 & & & \underset{p \nmid b}{(p, pb, pc)} \arrow[dr, "p^2"] & \\
 & & & & (1, b+pm, c+pn)
\end{tikzcd}$$

Observe that for any $g, h \in G$ we have that $(g^h)^p = g^p$.

We want to show:

Theorem 2.22. *The group G is not the active sum of cyclic subgroups.*

Proof. We begin by showing that if an active generating family $\mathcal{F}$ of cyclic subgroups of G is regular, then every member of $\mathcal{F}$ intersects G' trivially.

Suppose $\mathcal{F} = \langle F_i \rangle_i$. Recall that the condition for regularity involves the colimit

$$\varinjlim_i F_i / R_i \tag{2}$$

in Abelian groups, where R_i is the normal subgroup of F_i generated by elements of the form $x^{-1}x^g$ with $x, y \in F_i$ or $x \in F_j$, $g \in G$, and $F_j \leq F_i \geq F_j^g$. Assume now that $F_i \cap G'$ is non-trivial. We have two cases, either $F_i = G'$ or F_i is of the form $\langle (pa, pb, 1) \rangle$. In both cases, $F_i \subset Z(G)$. Therefore, in both cases, $R_i = \{1\}$. Observe furthermore that the diagram over which the colimit (2) is taken, has arrows given by either contention or conjugation, and the colimit is the direct sum of the colimits taken over the connected components of the diagram. Since F_i is central in G, the only possible arrows in the diagram can come from conjugation. Therefore, the connected component to which this F_i belongs can only be one of:

$$F_i,\ G' \longrightarrow F_i \quad \text{or} \quad \begin{tikzcd}[row sep=tiny]
 & F_i \\
 & F_{j_1} \\
G' \arrow[uur] \arrow[ur] \arrow[r] \arrow[ddr] & F_{j_2} \\
 & \vdots \\
 & F_{j_l}
\end{tikzcd}.$$

In any of these cases, it is clear that the arrow $\beta_i \colon F_i/R_i \to \varinjlim_j F_j/R_j$ does not satisfy $\beta_i(F_i \cap G'/R_i) = 0$. Thus $\mathcal{F}$ can not be regular.

In the picture of the cyclic subgroups of G above, this means that any regular $\mathcal{F}$ can not contain groups from the left branch.

Assume now that $\mathcal{F}$ is an active regular generating family of cyclic subgroups of G. We want to show that $\mathcal{F}$ can not be independent. To do this, recall that independence involves a colimit of the form

$$\varinjlim_{t \in T} A_t \tag{3}$$

where T is a transversal and $A_t = F_t G'/G'$. This colimit is required to be isomorphic to G/G'.

Since $\mathcal{F}$ is generating, by Observation 2.21, we need at least three cyclic subgroups A, B, C in $\mathcal{F}$ that generate $G/\Phi(G)$. On the one hand, this means that these three subgroups have order p^2, and on the other, that $A \cap B = A \cap C = B \cap C = \{1\}$. Assuming A, B, C are part of the transversal T, what we said above means that they belong to three different connected components of the diagram defining colimit (3). Since $\left|AG'/G'\right| = p^2$ and similarly for B and C, examining the possible shapes of the connected components of the corresponding diagram, we conclude that the size of colimit (3) is at least p^6. Since $|G/G'| = p^5$, $\mathcal{F}$ can not be independent. $\square$

3. Belana-Tomàs' group

Here is another example of a group that is not the active sum of cyclic subgroups.

DEFINITION 3.1 (Belana-Tomàs' group). Let B be the following group. As a set $B = \mathbb{Z}_{16} \times \mathbb{Z}_8 \times \mathbb{Z}_4$. Given (x, y, z) and (u, v, w) in B, we define their product as

$$(x, y, z)(u, v, w) = (x + u + 8zv,\ y + v + 4zu,\ z + w + 2yu),$$

where the sum is modulo 16 in the first coordinate, modulo 8 in the second and modulo 4 in the third.

Belana and Tomàs [1] showed that the group B is not the active sum of any discrete family of proper normal subgroups. The whole section is devoted to the proof of:

Theorem 3.2. *The group B is not the active sum of cyclic subgroups.*

We start with some basic properties of B. We calculate explicitly the inverse of an element in B.

Lemma 3.3. *Let* $(x, y, z) \in B$. *The inverse of* (x, y, z) *in* B *is given by*

$$(x, y, z)^{-1} = (-x + 8yz, -y + 4xz, -z + 2xy).$$

A routine calculation then shows that

$$[(x, y, z), (u, v, w)] = (8(yw + zv), 4(xw + zu), 2(xv + yu)).$$

On the other hand, there are eight elements in B of the form $(8l, 4m, 2n)$, and it is easy to see that each one of them can be expressed as the commutator of a couple of elements. For example $(8, 4, 2) = [(1, 1, 0), (0, 1, 1)]$ and $(0, 4, 2) = [(1, 0, 0), (1, 1, 1)]$. We have thus shown:

Lemma 3.4. *The commutator* B' *of* B *consists of those elements of the form* $(8l, 4m, 2n)$.

The next lemma describes the center of B:

Lemma 3.5. *The center* $Z(B)$ *of* B *consists of those elements of the form* $(2x, 2y, 2z)$.

Proof. For the not so easy part, assume that $(x, y, z) \in Z(B)$. Then $(x, y, z+1) = (x, y, z)(0, 0, 1) = (0, 0, 1)(x, y, z) = (x+8y, y+4x, z+1)$ means that x and y are even. Similarly, using $(1, 0, 0)$ we can see that z is also even. □

This means that B is a class 2 group. Some more formulas for B:

Lemma 3.6. $(x, y, z)^{(u,v,w)} = \left(x + 8 \begin{vmatrix} y & z \\ v & w \end{vmatrix}, y + 4 \begin{vmatrix} x & z \\ u & w \end{vmatrix}, z + 2 \begin{vmatrix} x & y \\ u & v \end{vmatrix} \right).$

Lemma 3.7. *For every* $r \geq 0$, *we have*

$$(x, y, z)^r = \left(rx + 8 \left[\frac{r}{2} \right] yz, ry + 4 \left[\frac{r}{2} \right] xz, rz + 2 \left[\frac{r}{2} \right] xy \right).$$

Proof. Assume first that $r = 2n$ with $n \geq 0$. Then we have that $(x, y, z)^{2n} = (2x+8yz, 2y+4xz, 2z+2xy)^n$. This last one is easy to calculate since all the components are even. Thus we have $(x, y, z)^{2n} = (2nx + 8nyz, 2ny + 4nxz, 2nz + 2nxy)$. Now, for $r = 2n+1$ we have

$$\begin{aligned} (x, y, z)^{2n+1} &= (x, y, z)^{2n}(x, y, z) \\ &= (2nx + 8nyz, 2ny + 4nxz, 2nz + 2nxy)(x, y, z) \\ &= ((2n+1)x + 8nyz, (2n+1)y + 4nxz, (2n+1)z + 2xy). \end{aligned}$$ □

Proposition 3.8. *If* $(x, y, z) \in B$ *has order* 2, *then* $(x, y, z) \in B'$.

Proof. Assume $(x, y, z)^2 = (0, 0, 0)$. Since $(x, y, z)^2 = (2x+8yz, 2y+4xz, 2z+2xy)$ we conclude that

$$8|x + 4yz,$$
$$4|y + 2xz,$$
$$2|z + xy.$$

From the first two we conclude that x and y are even. Using this and the third we conclude that z is also even. Using again the first two and the fact that z is even, we have that $8|x$ and $4|y$. □

The elements we define next will be very important in what follows

Definition 3.9. An element x in a group G is called *twisted* if the normalizer of x in G is different from the centralizer of x in G, otherwise it is called *not-twisted*. A cyclic subgroup H of G generated by one element h is called *twisted* or *not-twisted* according to whether h is or not-twisted. (This is independent of the choice of the generator h.)

We will need a characterization of the twisted element in B:

Lemma 3.10. *An element* $h = (x, y, z) \in B$ *is twisted if and only if* $x \equiv 2 \mod 4$, y *is even and* z *is odd, and this happens if and only if* $h \neq (0, 0, 0)$ *and there is a* $g \in B$ *such that* $h^{-1}h^g = h^{(1/2)o(h)}$.

Proof. Assume that h is of the form $(4u + 2, 2v, 2w + 1)$. Then the order of h is 8 and $h^{(0,1,1)} = h^4$. Thus h is twisted.

Assume now that $h = (x, y, z)$ is twisted. Clearly $h \neq (0, 0, 0)$. Let $g \in N_G(h)\backslash C_G(h)$. Then $[h, g] = h^{-1}h^g$ is an element in $B' \cap \langle(x, y, z)\rangle$. Since every non-trivial element of B' has order 2, we conclude that $h^{-1}h^g = h^{(1/2)o(h)}$. It follows that $o(h) \neq 2$ since $o(h^g) = o(h) \neq o(h^2)$.

Now let $r = o(h)$ and assume $r = 4$. Then $(0, 0, 0) = (x, y, z)^4 = (4x, 4y, 0)$. This means that $4|x$ and $2|y$. Therefore $(x, y, z)^2 = (2x, 2y, 2z)$. On the other hand, if $g = (u, v, w)$, then $h^{-1}h^g = (8(yw + zv), 4(xw + zu), 2(xv + yu)) = (8zv, 4zu, 0)$. Therefore $2|z$ and $h^{-1}h^g = (0, 0, 0)$, a contradiction.

Assume next that $r = 16$. Then x must be odd and $h^8 = (8, 0, 0)$. On the other hand, $h^{-1}h^g = (8(yw + zv), 4(xw + zu), 2(xv + yu)) = (8(yw + zv), 4(w + zu), 2(v + yu))$. Therefore

$$2 \nmid yw + zv,$$

$$2|w + zu,$$
$$2|v + yu.$$

If u were even, then w and v would be even, this in turn implies that $2|yw + zv$, contradicting the first condition above. Thus u must be odd. Assume v is even. Then y must be even by the third condition above, this in turn means that $2|yw+zv$, which is not the case. We conclude that both u and v are odd. If we assume w even, then the second condition above tells us that z is even, concluding $2|yw+zv$ again. Thus we conclude that u, v, and w are odd. Therefore z and y are also odd. We again conclude $2|yw + zv$. Therefore $r \neq 16$.

Assume now $r = 8$. Therefore x must be even. If $4|x$, then $h^4 = (0, 4y, 0)$, concluding that y is odd. On the other hand $h^{-1}h^g = (8(yw+zv), 4(xw+zu), 2(xv+yu)) = (8(yw + zv), 4zu, 2u)$. We conclude that u is simultaneously odd and even. Therefore $x \cong 2 \mod 4$. Assume now that y is odd. Under these conditions $h^4 = (4x, 4y, 0)$ and $h^{-1}h^g = (8(yw + zv), 4zu, 2u)$. We conclude again that u must be simultaneously odd and even. Therefore y is even. Since x and y are both even, z cannot be even, since then h would be in the center of B and then $h^{-1}h^g = (0, 0, 0)$. Therefore z is odd. □

In the group B a conjugate of a twisted element is twisted and an odd power of a twisted element is twisted.

Fig. 1 represents all the cyclic subgroups of B. The group $\langle(l, m, n)\rangle$ is represented in the figure by lmn. Numbers with a bar on top mean the negative of the number, so for example $\bar{4}01$ represents the group $\langle(-4, 0, 1)\rangle$. The arrows represent group inclusions. Notice that there are eight twisted subgroups of B, they are at the bottom center of the figure.

Let $\mathcal{F} = \langle F_i\rangle_{i\in I}$ be a family of pairwise different cyclic subgroups of B, close under conjugation and such that $\mathcal{F}$ generates B. Assume furthermore, that $\mathcal{F}$ has an order $\leq$ compatible with group inclusion and conjugation. Let $S = \divideontimes\mathcal{F}/R$ be the active sum of the family $\mathcal{F}$. We will show that S is not isomorphic to B. It suffices to show that S/S' is not isomorphic to B/B'. Notice that $B/B' \simeq \mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$ as a group.

For every $i \in I$, let R_i be the normal subgroup of F_i generated by elements of the form $h^{-1}h^g$ where there is a $j \in I$ such that $h \in F_j \leq F_i \geq F_j^g$.

Lemma 3.11. *R_i is non-trivial only if F_i is twisted.*

Proof. Assume $F_i = \langle(x, y, z)\rangle$ and take $h^{-1}h^g \in R_i$. Assume that $|F_i| = 2^r$ with $r > 0$. If $o(h) < 2^r$ then h is a square. It is not hard to see that in such a case we have $h^{-1}h^g = 0$ for any $g \in B$. If, on the other hand, $o(h) = 2^r$ then $h^{-1}h^g \in F_i = \langle h\rangle$. If $h^{-1}h^g$ is not trivial, then Lemma 3.10 tells us that h is twisted. □

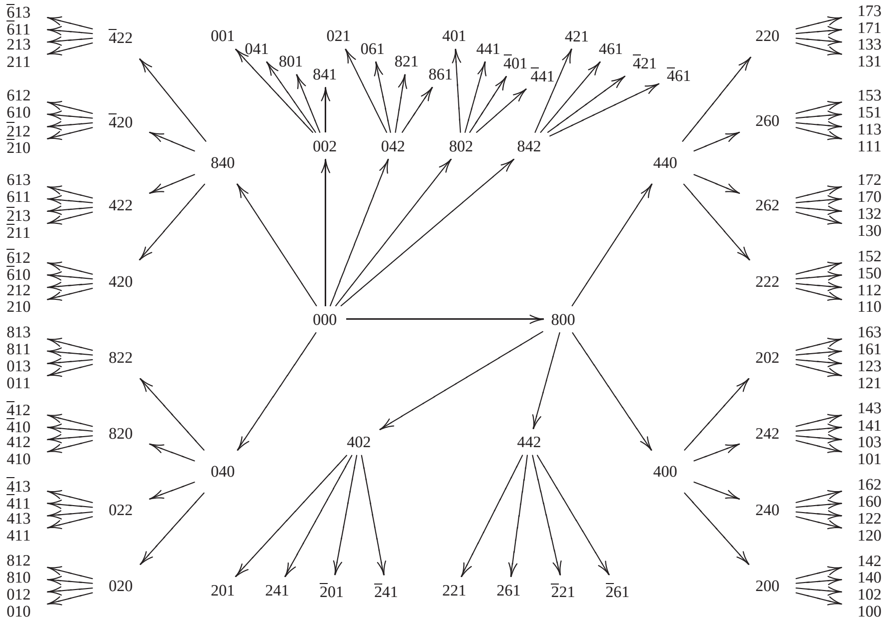

Fig. 1. Belana-Tomàs' group

We consider first the case where $\mathcal{F}$ does not contain a twisted group. Consider the biggest possible such family. Let $\mathcal{G} = \{H \mid H$ is a non-twisted cyclic subgroup of $B\}$, ordered by inclusion.

Lemma 3.12. *There is a family $\langle \varphi_H : H \to \mathbb{Z}_{16} \rangle_{H \in \mathcal{G}}$ that satisfies the following conditions*:

1. *For every $H \in \mathcal{G}$ the homomorphism φ_H is mono.*
2. *For every pair $H, K \in \mathcal{G}$ with $H \subseteq K$, we have $\varphi_K|_H = \varphi_H$.*
3. *If $H, K \in \mathcal{G}$ and $g \in B$ is such that $H^g = K$, then the diagram*

$$\begin{tikzcd} H \arrow[rr, "(\_)^g"] \arrow[dr, "\varphi_H"'] & & K \arrow[dl, "\varphi_K"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes.

Proof. We define the φ_H's by induction on the order of $H \in \mathcal{G}$. If $|H| = 1$ there is nothing to do.

If $|H| = 2$, then define $\varphi_H : H \to \mathbb{Z}_{16}$ as the only non-trivial homomorphism there is. It is clear that conditions 1, 2 and 3 of the lemma are satisfied for the elements of $\mathcal{G}$ of order at most 2.

Assume now that we have defined φ_H for all the elements $H \in \mathcal{G}$ of order at most 2^r with $1 \leq r \leq 3$, satisfying the conditions 1, 2 and 3 above. We want to define φ_H on the elements of order 2^{r+1}. Consider a transversal T for $\mathcal{G}$, and for every $t \in T$ choose a representative $H_t \in t$. Let t be such that $|H_t| = 2^{r+1}$. Now, H_t has a unique subgroup K of order 2^r. Define φ_{H_t} in such a way that the diagram

$$\begin{tikzcd} K \arrow[rr] \arrow[dr, "\varphi_K"'] & & H_t \arrow[dl, "\varphi_{H_t}"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes, where the top arrow is the inclusion. It is not hard to see that such a φ_{H_t} always exists and it is mono. For every $L \in t$ choose $g \in B$ such that $L^g = H_t$. Define $\varphi_L : L \to \mathbb{Z}_{16}$ such that the diagram

$$\begin{tikzcd} L \arrow[rr, "(\_)^g"] \arrow[dr, "\varphi_L"'] & & H_t \arrow[dl, "\varphi_{H_t}"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes.

We show now that φ_L does not depend on the choice of g. If $h \in B$ also satisfies $L^h = H_t$, then we show that $(\_)^g = (\_)^h : L \to H_t$. Indeed, if $L = \langle k \rangle$ and $k^g \neq k^h$, then $k^{-1}k^{gh^{-1}} \in \langle k \rangle$ and it is non-trivial. This means that k is twisted, a contradiction since L is not-twisted.

Conditions 1 and 3 above are clear now for the elements of $\mathcal{G}$ of order at most 2^{r+1}.

As for condition 2, notice that it is enough to prove it in the case where $M \subset L$ with $|M| = 2^r$ and $|L| = 2^{r+1}$. Assume $L \in t$ and that $L^g = H_t$. Then M^g is the only subgroup of H_t of order 2^r. We have the following commutative diagram:

$$\begin{tikzcd} L \arrow[r, "(\_)^g"] & H_t \arrow[dr, "\varphi_{H_t}"] & \\ M \arrow[u] \arrow[r, "(\_)^g"] \arrow[rr, bend right, "\varphi_M"'] & M^g \arrow[u] \arrow[r, "\varphi_{M^g}"] & \mathbb{Z}_{16} \end{tikzcd}$$

where the vertical arrows are inclusions. $\square$

An application of the previous lemma is:

Proposition 3.13. *If the family $\mathcal{F}$ does not have a twisted group, then S/S' is not isomorphic to B/B'.*

Proof. Notice that if we can construct an epimorphism $S \to \mathbb{Z}_{16}$ then we are done. By the universal property of the active sum this is equivalent to finding a family of homomorphisms $\langle \psi_i \colon F_i \to \mathbb{Z}_{16} \rangle_{i \in I}$ satisfying the following properties:

1. There is an $i \in I$ such that ψ_i is epi.
2. If $F_i \leq F_j$ then $\psi_j|_{F_i} = \psi_i$.
3. If $F_j = F_i^g$ for some $g \in B$, then the diagram

$$\begin{tikzcd} F_i \arrow[rr, "(\_)^g"] \arrow[dr, "\psi_i"'] & & F_j \arrow[dl, "\psi_j"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes.

Consider a family $\langle \varphi_H \to \mathbb{Z}_{16} \rangle_{H \in \mathcal{G}}$ built as in Lemma 3.12. Define $\psi_i = \varphi_{F_i}$. Then the family $\langle \psi_i \colon F_i \to \mathbb{Z}_{16} \rangle_{i \in I}$ satisfies conditions 2 and 3 of the proposition. Since every ψ_i is a monomorphism, to prove condition 1 it suffices to show that there is an $i \in I$ with $|F_i| = 16$. Since the family $\mathcal{F}$ generates, we can find elements $g_1, g_2, \ldots, g_n \in \bigcup_{i \in I} F_i$ such that $g_1 g_2 \cdots g_n$ has order 16 in B. It is easy to see that, if the product of two elements in B has order 16, then one of them has order 16. It follows that one of the $g_1, g_2, \ldots, g_n$ has order 16. $\square$

Our next step is to show that, if there is a group in the family $\mathcal{F}$ that does not contain the element $(8, 0, 0)$ then S/S' is not isomorphic to B/B'. For this we need the following lemma:

Lemma 3.14. *Let $\mathcal{H}$ be the family of all the cyclic subgroups of B. There exists a family of homomorphisms $\langle \varphi_H \colon H \to \mathbb{Z}_{16} \rangle_{H \in \mathcal{H}}$ satisfying the following conditions*:

1. *φ_H is mono if $(8, 0, 0) \notin H$.*
2. *φ_H is zero if $(8, 0, 0) \in H$.*
3. *If $H \subseteq K$ in $\mathcal{H}$, then $\varphi_K|_H = \varphi_H$.*
4. *If $H^g = K$ with $H, K \in \mathcal{H}$ and $g \in B$, then the diagram*

$$\begin{tikzcd} H \arrow[rr, "(\_)^g"] \arrow[dr, "\varphi_H"'] & & K \arrow[dl, "\varphi_K"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes.

Proof. We proceed along the same lines as the proof of Lemma 3.12. If H is trivial there is only one homomorphism $\varphi_H : H \to \mathbb{Z}_{16}$. If $|H| = 2$ we define $\varphi_H : H \to \mathbb{Z}_{16}$ as the only non-trivial homomorphism there is if $(8, 0, 0) \notin H$. We define φ_H as zero if $H = \langle(8, 0, 0)\rangle$. It is clear that the conditions of the lemma are satisfied for all the elements of $\mathcal{H}$ of order at most 2.

Assume that we have defined φ_H for all the elements of $\mathcal{H}$ of order at most 2^r with $1 \le r \le 3$, in such a way that the conditions of the lemma are satisfied. Let T be a transversal and choose $H_t \in t$ for every $t \in T$.

Let $t \in T$ be such that $|H_t| = 2^{r+1}$. Define $\varphi_{H_t} : H_t \to \mathbb{Z}_{16}$ to be zero if $(8, 0, 0) \in H_t$. Otherwise, let K be the only subgroup of H_t of order 2^r. Define $\varphi_{H_t} : H_t \to \mathbb{Z}_{16}$ in such a way that the diagram

$$\begin{tikzcd} K \arrow[rr] \arrow[dr, "\varphi_K"'] & & H_t \arrow[dl, "\varphi_{H_t}"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes, where the top arrow is inclusion. Notice that such an arrow does exist and it is a monomorphism.

Given an arbitrary $L \in \mathcal{H}$ of order 2^{r+1}, there are $g \in B$ and $t \in T$ such that $L^g = H_t$. Define φ_L such that the diagram

$$\begin{tikzcd} L \arrow[rr, "(\_)^g"] \arrow[dr, "\varphi_L"'] & & H_t \arrow[dl, "\varphi_{H_t}"] \\ & \mathbb{Z}_{16} & \end{tikzcd}$$

commutes.

If $(8, 0, 0) \notin L$ then $(8, 0, 0) = (8, 0, 0)^{g^{-1}} \notin L^{g^{-1}} = H_t$. Therefore φ_L is mono. So we have condition 1.

If $(8, 0, 0) \in L$ then $(8, 0, 0) = (8, 0, 0)^g \in L^g = H_t$. Therefore φ_L is zero. So we have condition 2.

Assume now that $H \subseteq K$ in $\mathcal{H}$. If $(8, 0, 0) \in H$ then φ_H and φ_K are both zero. If $(8, 0, 0) \notin K$ then we proceed as in Lemma 3.12. Finally, it is not hard to see that the situation $(8, 0, 0) \in K$ and $(8, 0, 0) \notin H$ happens only if H is trivial. This proves 3.

Condition 4 is shown along the same lines. □

As an application of this lemma we have:

Proposition 3.15. *If there is a non-trivial element of the family $\mathcal{F}$ that does not contain* $(8, 0, 0)$*, then $\mathcal{F}$ is not regular. In particular $S/S' \not\cong B/B'$.*

Proof. Let $\langle \varphi_H \colon H \to \mathbb{Z}_{16} \rangle_{H \in \mathcal{H}}$ be a family of morphisms as in Lemma 3.14. For every $i \in I$ define $\psi_i = \varphi_{F_i} \colon F_i \to \mathbb{Z}_{16}$. By the universal property of the colimit, we can induce a homomorphism $\psi \colon \varinjlim_I F_i/R_i \to \mathbb{Z}_{16}$ such that for every $i \in I$ the diagram

$$\begin{tikzcd} F_i/R_i \arrow[r] \arrow[dr, "\psi_i"'] & \varinjlim_I F_i/R_i \arrow[d, "\psi"] \\ & \mathbb{Z}_{16} \end{tikzcd}$$

commutes. Let $j \in I$ and assume that F_j is not trivial and that $(8, 0, 0) \notin F_j$. Since $(8, 0, 0)$ is a member of every twisted group, we have that F_j is not-twisted. Therefore R_j is trivial according to 3.11. Since F_j must have an element of order 2, and every element of order 2 is in the commutator of B, we have that $(F_j \cap B')/R_j = F_j \cap B'$ is non-trivial. Since ψ_j is a monomorphism, we have that $F_j \cap B'$ has non-trivial image under ψ_j. Therefore, the injection $F_j/R_j \to \varinjlim_I F_i/R_i$ cannot map $(F_j \cap B')/R_j$ to zero. That is, $\mathcal{F}$ is not regular. $\square$

Assume from now on that $\mathcal{F}$ contains at least one twisted group and that every non-trivial element of $\mathcal{F}$ contains $(8, 0, 0)$. We will show that in such a case there is an epimorphism $S \to \mathbb{Z}_8 \oplus \mathbb{Z}_8$. Let $\mathcal{K}$ be the family of all those cyclic subgroups of B that contain $(8, 0, 0)$.

Lemma 3.16. *There is a family of homomorphisms $\langle \varphi_K \colon K \to \mathbb{Z}_8 \oplus \mathbb{Z}_8 \rangle_{K \in \mathcal{K}}$ satisfying the following properties*:

1. *For all $K \in \mathcal{K}$,* $\ker \varphi_K = \langle (8, 0, 0) \rangle$.
2. *If $K \subseteq H$ in $\mathcal{K}$, then $\varphi_H|_K = \varphi_K$.*
3. *If $H, K \in \mathcal{K}$ and $g \in B$ are such that $H^g = K$ then the diagram*

$$\begin{tikzcd} H \arrow[rr, "(\_)^g"] \arrow[dr, "\varphi_H"'] & & K \arrow[dl, "\varphi_K"] \\ & \mathbb{Z}_8 \oplus \mathbb{Z}_8 & \end{tikzcd}$$

commutes.

4. $\left\langle \bigcup_{K \in \mathcal{K}} \mathrm{Im}(\varphi_K) \right\rangle = \mathbb{Z}_8 \oplus \mathbb{Z}_8$.

Proof. In Fig. 1, the family $\mathcal{K}$ consists of those groups on the right and the twisted ones at the bottom. We can divide the groups in $\mathcal{K}$ of order bigger than 2 in four classes, according to which of the following elements they contain: $(4,4,0)$, $(4,0,0)$, $(4,4,2)$ and $(4,0,2)$. It is not hard to construct the homomorphisms for the first two classes, sending the first class to the first coordinate of $\mathbb{Z}_8 \oplus \mathbb{Z}_8$ and the second class to the second coordinate of $\mathbb{Z}_8 \oplus \mathbb{Z}_8$. For the twisted groups notice that $(8,0,0) = h^{-1}h^g$ should be mapped to zero. $\square$

Lemma 3.17. *Given a family of elements $\langle (i_r, j_r, k_r) \rangle$ in $\mathbb{Z}_8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2$, the family generates the group if and only if there exist indices r_1, r_2 and r_3 such that the determinant*

$$\begin{vmatrix} i_{r_1} & j_{r_1} & k_{r_1} \\ i_{r_2} & j_{r_2} & k_{r_2} \\ i_{r_3} & j_{r_3} & k_{r_3} \end{vmatrix}$$

is odd.

Proof. The family generates the group if and only if the Abelian group presented by the matrix

$$M = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \\ i_1 & j_1 & k_1 \\ & \vdots & \\ i_r & j_r & k_r \end{pmatrix}$$

with integer coefficients, presents the trivial group. This in turn holds if and only if the 0-th elementary ideal of M is all of $\mathbb{Z}$ (see [3] and [7]), that is, if and only if the 3×3 subdeterminants of the matrix M generate $\mathbb{Z}$. This is the case if and only if there exist indices r_1, r_2 and r_3 such that the determinant

$$\begin{vmatrix} i_{r_1} & j_{r_1} & k_{r_1} \\ i_{r_2} & j_{r_2} & k_{r_2} \\ i_{r_3} & j_{r_3} & k_{r_3} \end{vmatrix}$$

is odd. $\square$

Lemma 3.18. *If every group in $\mathcal{F}$ contains $(8,0,0)$ and there is at least one twisted group in $\mathcal{F}$, then there is an epimorphism $S \to \mathbb{Z}_8 \oplus \mathbb{Z}_8$.*

Proof. Let $\langle \varphi_K : K \to \mathbb{Z}_8 \oplus \mathbb{Z}_8 \rangle_{K \in \mathcal{K}}$ be a family of homomorphisms as constructed in Lemma 3.16. For every $i \in I$ define $\psi_i = \varphi_{F_i}$. Notice that this induces a homomorphism $\psi\colon S \to \mathbb{Z}_8 \oplus \mathbb{Z}_8$. Now, to show that ψ is epi, it suffices to show that there are groups of the form $\langle(2l+1, 2m, n)\rangle$ and $\langle(2l'+1, 2m'+1, n')\rangle$ in $\mathcal{F}$. It is not hard to see that this is the case if the non-twisted groups in $\mathcal{F}$ generate B. Suppose then that we need the twisted group $\langle(4l_1+2, 2m_1, 2n_1+1)\rangle$ of $\mathcal{F}$ to generate all of B. Since the second coordinate of $(4l_1+2, 2m_1, 2n_1+1)$ is even, we need, to generate B, an element with an odd entry in the middle. It is not hard to see that this element should be of the form $(2l_2+1, 2m_2+1, n_2)$. We need a third element to be able to generate B/B'. If this element had an odd entry in the middle then it would be of the form $(2l_3+1, 2m_3+1, n_3)$. Notice however that the determinant

$$\begin{vmatrix} 4l_1+2 & 2m_1 & 2n_1+1 \\ 2l_2+1 & 2m_2+1 & n_2 \\ 2l_3+1 & 2m_3+1 & n_3 \end{vmatrix}$$

is even. According to Lemma 3.17, these three elements do not generate B/B'. Therefore, we must have an element with an even entry in the middle. Now, for the determinant

$$\begin{vmatrix} 4l_1+2 & 2m_1 & 2n_1+1 \\ 2l_2+1 & 2m_2+1 & n_2 \\ l_3 & 2m_3 & n_3 \end{vmatrix}$$

to be odd, we need l_3 to be odd. Therefore, we have groups in $\mathcal{F}$ of the desired form. We conclude that $\psi\colon S \to \mathbb{Z}_8 \oplus \mathbb{Z}_8$ is an epimorphism. □

Since we have covered all the possible cases, we conclude Theorem 3.2.

4. Non-molecular *p*-groups

Recall that an atomic group is one that is normally generated by a cyclic subgroup and that a molecular group is one that is the active sum of atomic subgroups.

Not every group is molecular. Observe that several of the examples of groups that are shown not to be active sum of cyclic subgroups are p-groups. Proposition 2.10 in [4] states that for p-groups, it is equivalent to be molecular to be active sum of cyclic subgroups. Thus we have:

Proposition 4.1. *The following groups are not molecular*:

1. *Pizaña's group* (Section 2.2).
2. *Pizaña like groups* (Section 2.3).
3. *El chamuco* (Section 2.4).
4. *Belana-Tomàs* (Section 3).

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