Mizuma, Y.
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AN ESTIMATE OF THE RIBBON NUMBER BY THE JONES POLYNOMIAL

Dedicated to Professor Yukio Matsumoto on the occasion of his 60th birthday

YOKO MIZUMA

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Abstract

For a ribbon knot we define the notion of its ribbon number. In this paper we estimate the ribbon number for a ribbon knot by using the Jones polynomial. As a corollary we determine the ribbon number of the Kinoshita-Terasaka knot.

1. Introduction

To investigate the complexity of a ribbon knot, we define the notion of the ribbon number. This obvious measure of a ribbon knot's complexity is often hard to determine. In fact, even in a simple case of the Kinoshita-Terasaka knot, its ribbon number is hard to determine. In this paper, we estimate the ribbon number by using a formula for the first derivative at -1 of the Jones polynomial of a ribbon knot of 1-fusion in [6]. As a corollary we determine the ribbon number of the Kinoshita-Terasaka knot.

1.1. Definitions and theorems.

DEFINITION 1.1. A *ribbon disk* is an immersed 2-disk of D^2 into S^3 with only transverse double points such that the singular set consists of ribbon singularities, that is, the preimage of each ribbon singularity consists of a properly embedded arc in D^2 and an embedded arc interior to D^2 (see Fig. 1). A knot is a *ribbon knot* if it bounds a ribbon disk in S^3 (cf. [3], [4]).

For a ribbon knot we define its ribbon number as follows.

DEFINITION 1.2. The *ribbon number* of a ribbon knot is defined as the minimal number of ribbon singularities needed for a ribbon disk bounded by the ribbon knot.

Here we have some remarks of Definition 1.2.

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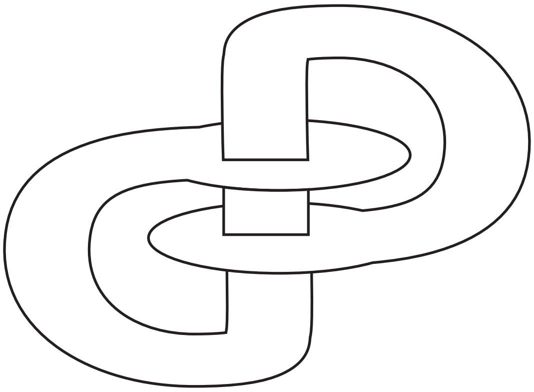

Fig. 1. A ribbon disk with two ribbon sigularities

REMARK 1.3. A ribbon knot whose ribbon number is zero is a trivial knot and there does not exist a ribbon knot whose ribbon number is one.

REMARK 1.4. The ribbon number of a ribbon knot K is greater than or equal to the genus of K ([1]).

Now we can state the following theorem.

Theorem 1.5. *Let K be a ribbon knot satisfying $\Delta_K(t) = 1$ and $J'_K(-1) \neq 0$, where $\Delta_K(t)$ is the Alexander polynomial of K and $J'_K(-1)$ is the first derivative at -1 of the Jones polynomial of K. Then the ribbon number of K is greater than or equal to three.*

To state the next theorem we review a ribbon knot of 1-fusion as follows.

DEFINITION 1.6. We call a knot K in S^3 a ribbon knot of 1-*fusion*, if it has a knot diagram described in Fig. 2, where n is even and each small rectangle named C_i is determined by $c_i \in \{-1, 0, 1\}$ $(i = 1, 2, \dots, n)$ and there are disjointly embedded $(n+1)$ bands in the big rectangle, being knotted, twisted and mutually linked (cf. [6], [5]). The diagram is called 1-*fusion diagram* of K and gives a ribbon disk bounded by K.

REMARK 1.7. A ribbon knot of 1-fusion is a band sum of 2-component trivial link and vice versa.

Here we can state the following theorem.

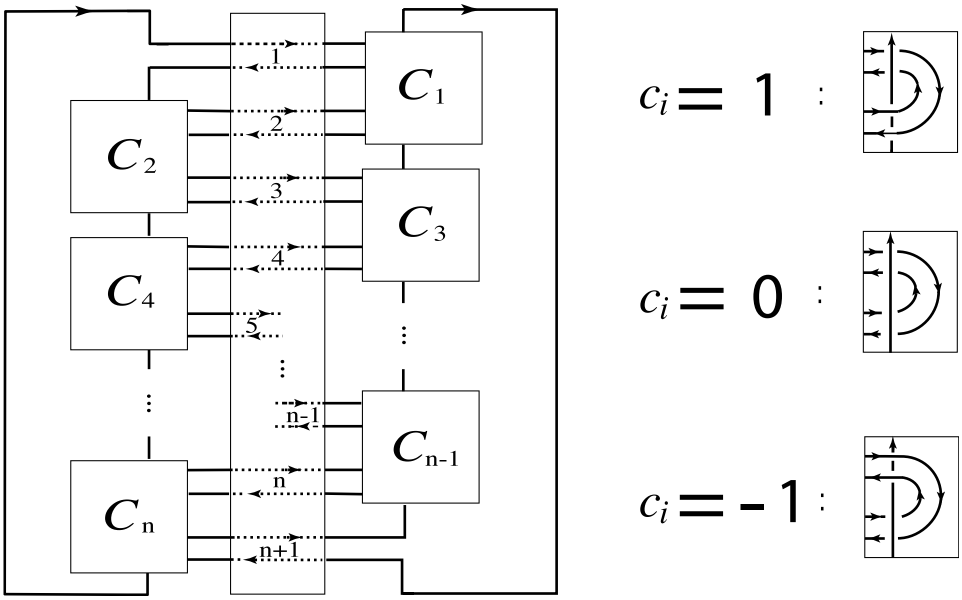

Fig. 2. A 1-fusion diagram

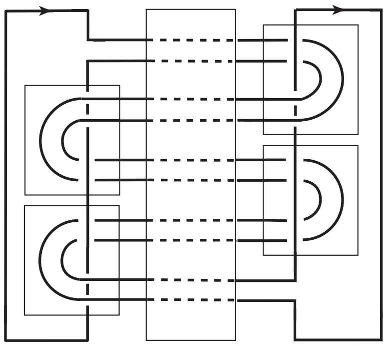

Fig. 3. (1, 1, 0, −1)

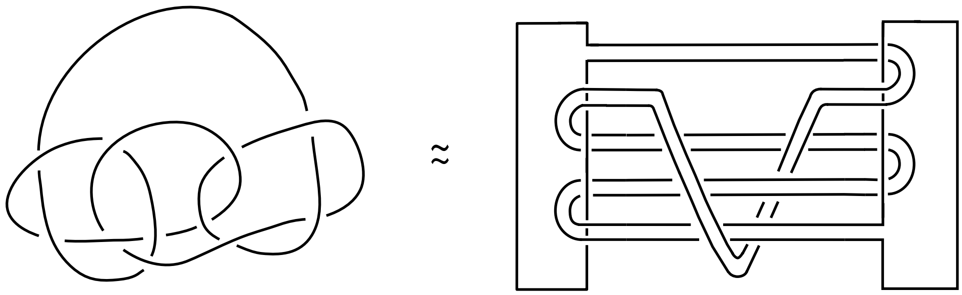

Fig. 4. The Kinoshita-Terasaka knot

Theorem 1.8. *If K has a 1-fusion diagram with* $(c_1, c_2, c_3, c_4) = (1, 1, 0, -1)$ *as shown in* Fig. 3*, where* $J'_K(-1) \neq 0$*, then the ribbon number of K is three.*

1.2. An application of theorems. Now we consider the ribbon number of the Kinoshita-Terasaka knot, which has a 1-fusion diagram as in Fig. 4 and $\Delta(t) = 1$ and $J'(-1) = 48$. Hence we obtain the following theorem from Theorem 1.8.

Theorem 1.9. *The ribbon number of the Kinoshita-Terasaka knot is three.*

Note that the genus of the Kinoshita-Terasaka knot is two ([2]).

2. Proof

Now we start to prove theorems.

Proof of Theorem 1.5. Let K be a ribbon knot satisfying $\Delta_K(t) = 1$ and $J'_K(-1) \neq 0$. K is not a trivial knot, so the ribbon number of K is greater than or equal to two. Note that a ribbon disk with two ribbon singularities bounded by a non-trivial knot is bounded by a ribbon knot which has one of eight 1-fusion diagrams as shown in Fig. 5, where C_i is determined by $c_i \in \{-1, 1\}$ $(i = 1, 2, 3)$. By using a formula for the Alexander polynomial of a ribbon knot of 1-fusion in [5], $\Delta(t) \neq 1$ for each knot in the left side of Fig. 5 and $\Delta(t) = 1$ for each knot in the right side of Fig. 5. By using a formula for the first derivative at -1 of the Jones polynomial of a ribbon knot of 1-fusion, $J'(-1) = 0$ for each knot in the right side of Fig. 5 (Example 1.12 in [6]). Hence the ribbon number of K is not two. This completes the proof. □

Proof of Theorem 1.8. By using a formula for the Alexander polynomial of a ribbon knot of 1-fusion in [5], a knot K which has a 1-fusion diagram as shown in

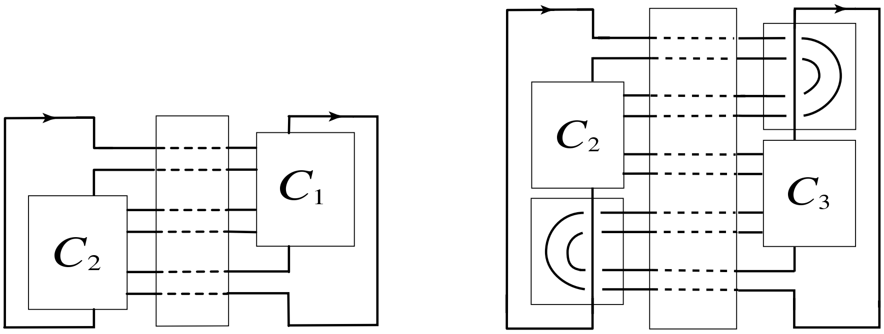

Fig. 5. (c_1, c_2) and $(0, c_2, c_3, 0)$

Fig. 3 satisfies $\Delta_K(t) = 1$. By Theorem 1.5, the ribbon number of K is greater than or equal to three. The diagram in Fig. 3 gives a ribbon disk bounded by K which has three ribbon singularities, hence the ribbon number of K is three. □

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Research Institute for Mathematical Sciences
Kyoto University
Sakyo-ku, Kyoto 606-8502, Japan
e-mail: mizuma@kurims.kyoto-u.ac.jp

The Fields Institute for Research in Mathematical Sciences
222 College Street, Toronto, Ontario, M5T 3J1, Canada
e-mail: ymizuma@fields.utoronto.ca