Murray, J.
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STRONGLY REAL 2-BLOCKS AND THE FROBENIUS-SCHUR INDICATOR

JOHN MURRAY

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Abstract

Let G be a finite group, let k be an algebraically closed field of characteristic 2 and let $\Omega := \{g \in G \mid g^2 = 1_G\}$. It is shown that for a block B of kG, the permutation module $k\Omega$ has a B-composition factor if and only if the Frobenius-Schur indicator of the regular character of B is non-zero or equivalently if and only if B is real with a strongly real defect class.

1. Introduction

Let G be a finite group. In this paper we investigate the permutation module of G acting by conjugation on its involutions, over a field of characteristic 2. This develops the main theme of [10] and [8]. In the former paper G.R. Robinson considered the projective components of this module. In the latter paper the author showed that each such component is irreducible and self-dual and belongs to a 2-block of defect zero. Here we investigate which 2-blocks have a composition factor in the involution module. There are two apparently different ways of characterising such blocks. One method is local and uses the *defect classes* of the block. This gives rise to the definition of a *strongly real* 2-block. The other method is global and uses the Frobenius-Schur indicators of the irreducible characters in the block. Our main result is Theorem 2. The proof of this theorem requires Corollaries 4, 15, 18 and 20.

J.A. Green proved a number of results about p-blocks, using the observation that the group algebra of G is a module for the group $G \times G$. Here we shall exploit the additional fact that the group algebra is a module for the wreath product of G with a cyclic group of order 2. This was also an essential tool in [8].

Throughout this paper k will be an algebraically closed field of characteristic 2. There are various reasons why we limit ourselves to characteristic 2. Our wreath product group is an extension of $G \times G$ by a group of order 2. It is thus fairly uninteresting, from the point of view of blocks over a field of characteristic not equal to 2. In addition, the prime 2 is useful for studying the contragradient operator and real blocks, as pairing arguments of various kinds can be employed.

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Recall also the following classical result. The Frobenius-Schur indicator $\nu(\chi)$ of a generalized character χ of G is the integer $(1/|G|)\sum_{g\in G}\chi(g^2)$. If χ is absolutely irreducible then $\nu(\chi) = 0$ or $1, -1$, depending on whether χ is not real-valued, or χ is real-valued and a χ-module affords a symmetric, respectively anti-symmetric non-degenerate G-invariant bilinear form. Then Frobenius and Schur proved that $|\Omega| = \sum \nu(\chi)\chi(1_G)$, where $\Omega := \{g \in G \mid g^2 = 1_G\}$.

The reader may be interested to know that in odd characteristic, the geometric type (quadratic or sympletic) of an irreducible self dual module is determined by the Frobenius-Schur indicator of a real valued character which contains the Brauer character of the module with odd multiplicity. It is an open problem as to whether there is an analogous Frobenius-Schur indicator in characteristic 2. See [11] for details.

A *component* of a module is a direct summand of the module that is indecomposable. Following Green, a 2-block of G is a component of kG, considered as a $G \times G$-module in the usual way. For the rest of the paper we use B to denote a 2-block of G.

A *defect class* of B is a conjugacy class of G whose sum appears with non-zero multiplicity in the block idempotent 1_B, and on which the central character ω_B of B does not vanish. Defect classes are known to exist and to consist of elements of odd order.

The irreducible complex characters, Brauer characters and indecomposable modules of G are partitioned among its 2-blocks. We use $\mathrm{Irr}(B)$, $\mathrm{IBr}(B)$ and $\mathrm{Pic}(B)$ to denote, respectively, the set of irreducible characters, the Brauer characters and the principal indecomposable characters of G that belong to B. We use ψ to indicate the irreducible Brauer character associated to $\Psi \in \mathrm{Pic}(B)$. If M is a G-module, $M{\downarrow_H}$ denotes the restriction of M to $H \leq G$ and $M{\uparrow^K}$ denotes the induction of M to $K \geq G$. Identical notation applies to the restriction and induction of characters. See [9] for any additional unexplained notation.

The contragradient map ^{o} is defined by $\left(\sum \alpha_g g\right)^o = \sum \alpha_g g^{-1}$. It is a k-algebra involutary anti-automorphism of kG. A block B is said to be *real* if $B^o = B$. A con jugacy class C of G is said to be *real* if it coincides with the class C^o of the inverses of its elements. It is one of the main results of [5] that each 2-block has at least one defect class that is real.

A real conjugacy class of G is said to be *strongly real* if it is the trivial class or if its elements are inverted by involutions. This leads to the following key definition:

DEFINITION 1. A *strongly real* 2-block is a real 2-block that has a strongly real defect class.

It turns out that if B is strongly real then each of its real defect classes is strongly real. This was proved by Gow in [4]. Notice that the principal 2-block is strongly real; the identity class is a strongly real defect class.

We use 1_G both for the identity element of G and its trivial character. The set Ω consisting of the involutions in G, together with 1_G, forms a G-set under conjugation.

We denote the kG-module with permutation basis Ω by $k\Omega$. Our main result in this paper is:

Theorem 2. *Let B be a 2-block of G. Then the following are equivalent*:

(i) *$k\Omega$ has a B-composition factor;*

(ii) $\sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(1_G) > 0$;

(iii) *B is strongly real.*

Let g be an element of G. There is a unique decomposition of $\langle g\rangle$ into a direct product of a cyclic 2-group E and a cyclic $2'$-group O. So $g = g_2 g_{2'} = g_{2'} g_2$, for some $g_2 \in E$ and $g_{2'} \in O$. We call g_2 the 2-part, and $g_{2'}$ the $2'$-part, of g. Both are uniquely determined by g.

In our first lemma we compute the multiplicity of an irreducible kG-module as a composition factor of $k\Omega$.

Lemma 3. *Let $\overline{P}$ be an irreducible kG-module, let P be the projective cover of $\overline{P}$, and let $\Phi \in \mathrm{Pic}(G)$ be principal indecomposable character of P. Then $\overline{P}$ occurs with multiplicity $\nu(\Phi)$ as a composition factor of $k\Omega$. In particular*, $\nu(\Phi) \geq 0$.

Proof. The number of solutions in G to the equation $x^2 = g$, for fixed $g \in G$, is given by $\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi(g)$. Let $g \in G$ be 2-regular and let $x \in G$ satisfy $x^2 = g$. As x_2 and $x_{2'}$ commute, we have $g = x_2^2 x_{2'}^2 = x_{2'}^2 x_2^2$. So $x_2^2 = 1_G$ and $x_{2'}^2 = g_{2'}$. It follows that $x_2 \in \Omega(C_G(g))$, while $x_{2'} = g_{2'}^{1/2}$ is uniquely determined. Conversely, given any involution $t \in C_G(g)$, then $t g_{2'}^{1/2}$ is a solution to $x^2 = g$ in G. We conclude that $\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi(g) = |\Omega \cap C_G(g)|$ coincides with the Brauer character of $k\Omega$. The lemma follows once we note that the virtual Brauer character of the restriction of the generalized character $\sum_{\chi\in\mathrm{Irr}(G)} \nu(\chi)\chi$ to 2-regular elements is given by $\sum_{\Psi\in\mathrm{Pic}(G)} \nu(\Psi)\psi$. □

Our Corollary shows that (i) $\Longleftrightarrow$ (ii) in Theorem 2.

Corollary 4. *The dimension of the sum of all submodules of $k\Omega$ that belong to B is given by $\sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(1_G)$. In particular $\sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(1_G) \geq 0$.*

Proof. Both statements follow from Lemma 3 and the fact that

$$\sum_{\chi\in\mathrm{Irr}(B)} \chi(1_G)\chi = \sum_{\Psi\in\mathrm{Pic}(B)} \psi(1_G)\Psi. \qquad \square$$

Let $\overline{P}$, P and Φ be as in Lemma 3. Suppose that $t \in \Omega$. The Frobenius-Nakayama reciprocity formula [9, 3.1.27] shows that $\overline{P}$ occurs with multiplicity $\langle \Phi_{C_G(t)}, 1_{C_G(t)}\rangle$ as a composition factor of $k_{C_G(t)}\uparrow^G$. Then, using the previous lemma, we get (c.f. [10, Lemma 1])

$$\nu(\Phi) = \langle \Phi, 1_G \rangle + \sum_t \left\langle \Phi_{C_G(t)}, 1_{C_G(t)} \right\rangle,$$

where t ranges over a set of representatives for the conjugacy classes of involutions in G.

We now proceed to the main construction needed for the proof of Theorem 2.

The wreath product group $G \wr \Sigma$ is the semi-direct product of $G \times G$ with the symmetric group Σ on two symbols. Here the conjugation action of the involution $\sigma \in \Sigma$ on $G \times G$ is given by $(g_1, g_2)^\sigma = (g_2, g_1)$, for all $g_1, g_2 \in G$. We shall use the notations

$$\underline{g} := (g, g) \in G \wr \Sigma, \quad \text{for each} \quad g \in G, \quad \text{and}$$
$$\underline{X} := \{\underline{x} \mid x \in X\} \subseteq G \wr \Sigma, \quad \text{for each} \quad X \subseteq G.$$

We highlight the following crucial fact:

Lemma 5. *The centralizer of σ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$.*

Let R be a commutative ring. Then the group algebra RG is a right $RG \wr \Sigma$-module. For, RG is a $RG \times G$-module via $x \cdot (g_1, g_2) = g_1^{-1} x g_2$, for each $x \in RG$ and $g_1, g_2 \in G$. The action of σ on RG is given by the contragradient involution $x \cdot \sigma = x^o$. In more detail we have:

Lemma 6. *The $RG \wr \Sigma$-module RG is isomorphic to the permutation module $\left(R_{\underline{G} \times \Sigma}\right)\uparrow^{G \wr \Sigma}$.*

Proof. The elements of G form an $RG \wr \Sigma$-invariant basis of RG. Moreover if $g_1, g_2 \in G$, then $g_2 = g_1 \cdot (g_1, g_2)$. So G is a transitive $G \wr \Sigma$-set. The stabilizer of $1_G \in RG$ in $G \wr \Sigma$ is $\underline{G} \times \Sigma$. The lemma follows from these facts. □

Suppose that E is a block algebra of RG. Set $E^* := E + E^o$. Then E^* is an $RG \wr \Sigma$-submodule of RG. If $E \neq E^o$, we have $E^* = E\uparrow^{G \wr \Sigma}$. If $E = E^o$, it is still useful to distinguish between the $RG \times G$-module E and the $RG \wr \Sigma$-module E^*, even though the underlying R-modules are the same.

Lemma 7. *Let $E_1, \ldots, E_r$ be the real blocks and $E_{r+1}, E_{r+1}^o, \ldots, E_{r+s}, E_{r+s}^o$ be the nonreal blocks of RG. Then there is a unique indecomposable decomposition of RG as $RG \wr \Sigma$-module*:

$$RG = E_1^* \oplus \cdots \oplus E_{r+s}^*.$$

Proof. This follows from the indecomposable decomposition of RG into a direct sum of its blocks algebras, as $R(G \times G)$-module. □

As a particular case, consider when $R = \mathbb{C}$. Let $\chi \in \mathrm{Irr}(G)$ and let M be a $\mathbb{C}G$-module that affords χ. We use $E(\chi)$ to denote the corresponding Wedderburn component $\mathrm{End}_{\mathbb{C}}(M)$ of $\mathbb{C}G$. Clearly $E(\chi)$ has $G \times G$-character $\overline{\chi} \otimes \chi : (g_1, g_2) \to \chi\left(g_1^{-1}\right)\chi(g_2)$, for $g_1, g_2 \in G$. Suppose now that $\chi = \overline{\chi}$ is real valued. Then $\overline{\chi} \otimes \chi = \chi \otimes \chi$ has two (irreducible) extensions to $G \wr \Sigma$. These will be denoted by χ^{+1} and χ^{-1}. Here if $\varepsilon \in \{\pm 1\}$ then $\chi^{\varepsilon}(g_1, g_2)\sigma = \varepsilon\chi(g_1 g_2)$, for all $g_1, g_2 \in G$. When $\overline{\chi} \neq \chi$, the next lemma shows why it is useful to denote the induced $G \wr \Sigma$-character $(\overline{\chi} \otimes \chi)\uparrow^{G \wr \Sigma}$ by χ^0.

Lemma 8. *Let χ be an irreducible character of G and let $E(\chi)^*$ be the corresponding $G \wr \Sigma$-component of $\mathbb{C}G$. Then $E(\chi)^*$ has character $\chi^{\nu(\chi)}$.*

Proof. This is obvious when $\nu(\chi) = 0$. So we may assume that $\chi = \overline{\chi}$. Then

$$\left\langle \chi^{\pm 1}\downarrow_{\underline{G} \times \Sigma}, 1_{\underline{G} \times \Sigma} \right\rangle = \frac{1}{2|G|} \sum_{g \in G} \left(\chi\left(g^{-1}\right)\chi(g) \pm \chi\left(g^2\right)\right) = \frac{1 \pm \nu(\chi)}{2}.$$

The result now follows from Lemma 6 and Frobenius reciprocity. □

Recall the following result [7, Theorem 1] of Green. A modern proof is [9, 5.10.8].

Lemma 9. *Let D be a defect group of B. Then B has vertex $\underline{D}$, as indecomposable $k(G \times G)$-module.*

We use this to make a preliminary observation about the vertices of the component B^* of the $G \wr \Sigma$-module kG. This will be refined in Proposition 14.

Lemma 10. *Let D be a defect group of B. If B is not real then $\underline{D}$ is a vertex of B^*; if B is real then there exists $e \in N_G(D)$, with $e^2 \in D$, such that $\underline{D}\langle \underline{e}\sigma \rangle$ is a vertex of B^*.*

Proof. Suppose first that B is not real. So $B^* = B\uparrow^{G \wr \Sigma}$. It then follows from Lemma 9 that B^* has vertex $\underline{D}$.

Suppose then that B is real. Lemma 6 shows that B^* is $\underline{G} \times \Sigma$-projective. We choose a vertex V of B^* so that $V \leq \underline{G} \times \Sigma$. Now B^* is a quasi-permutation module, $G \times G$ is a normal subgroup of $G \wr \Sigma$, and $B^*\downarrow_{G \times G} = B$ is indecomposable. A variant of Lemma 9.7 of [2] then implies that $V \cap (G \times G) = V \cap \underline{G}$ is a vertex of B. Using Lemma 9, we may choose D so that $V \cap \underline{G} = \underline{D}$. As $G \wr \Sigma / G \times G$ is a 2-group, and as $B^*\downarrow_{G\times G}$ is indecomposable, Green's indecomposability theorem [6, Theorem 8], implies that $V \not\leq (G \times G)$. The last statement of the lemma follows from this. □

It is easy to compute the stabilizer subgroup of an element g of the $G \wr \Sigma$-set G in the group $\underline{G} \times \Sigma$. This subgroup will be denoted $C_{\underline{G}\times\Sigma}(g)$. We hope that the reader will not confuse this group with $C_{\underline{G}\times\Sigma}(\underline{g}) = \underline{C_G(g)} \times \Sigma$.

Lemma 11. *Let $g \in G$. If g is not G-conjugate to g^{-1}, then $C_{\underline{G}\times\Sigma}(g) = \underline{C_G(g)}$. If $g^t = g^{-1}$, for $t \in G$, then $C_{\underline{G}\times\Sigma}(g) = \underline{C_G(g)}\langle \underline{t}\sigma\rangle$.*

Proof. These statements follow from the fact that $g \cdot \sigma = g^{-1}$. □

For $H \leq G$ and M a kG-module, let M^H denote the sum of all the trivial H-submodules of $M\downarrow_H$. The relative trace map $\mathrm{Tr}_H^G \colon M^H \to M^G$ is defined by $\mathrm{Tr}_H^G(m) = \sum m\cdot g$, for $m \in M^H$. Here g ranges over any set of representatives for the right cosets of H in G. We write $\mathrm{Tr}_H^G(M)$, instead of $\mathrm{Tr}_H^G(M^H)$, for the image of the trace map on M^H.

The reader is warned that B^* is generally not a $\underline{G} \times \Sigma$-algebra, in the sense of Green [7]. In particular, there is no Mackey-type decomposition of a product of the form $\mathrm{Tr}_X^{\underline{G}\times\Sigma}(a)\,\mathrm{Tr}_Y^{\underline{G}\times\Sigma}(b)$, for $X, Y \leq \underline{G} \times \Sigma$. However, we do have the following useful result.

Lemma 12. *Let A be a k-algebra and a kG-module such that each element of G acts on A as a k-algebra automorphism or as a k-algebra anti-automorphism. Suppose also that A^G is contained in the centre $Z(A)$ of A. Then A^G a subalgebra of $Z(A)$. Also $\mathrm{Tr}_H^G(A)$ is an ideal of A^G, for each $H \leq G$.*

Proof. Write the G-action on A in exponential form. It is obvious that $\mathrm{Tr}_H^G(A)$ is a k-subspace of A. Let $a \in A$, $z \in A^G$ and $g \in G$. Suppose that g acts as a k-algebra anti-automorphism. Then $a^g z = \left(z^{g^{-1}}a\right)^g = (za)^g = (az)^g$. Similarly $a^g z = (az)^g$, if g acts as a k-algebra automorphism. It follows that the map $a \to az$ is a kG-endomorphism of A. In particular, if $a \in A^H$, then $\mathrm{Tr}_H^G(a)z = \mathrm{Tr}_H^G(az)$. Taking $H = G$, we get that A^G is a subalgebra of $Z(A)$. More generally, we can conclude that $\mathrm{Tr}_H^G(A)$ is an ideal of A^G. □

We will apply this Lemma to the algebra B^* and the group $\underline{G} \times \Sigma$. Denote by $Z^*(kG)$ the σ-fixed point subalgebra of $Z(kG)$. It has k-basis $\{(C \cup C^o)^+\}$, where C ranges over the conjugacy classes of G. Note that $Z(kG) = kG^{\underline{G}}$ and $Z^*(kG) = kG^{\underline{G}\times\Sigma}$.

Corollary 13. *Let P be a 2-subgroup of G and let $q \in N_G(P)$ with $q^2 \in P$. Then*
(i) $\mathrm{Tr}_{\underline{P}}^{\underline{G}\times\Sigma}(kG)$ *is an ideal of* $Z^*(kG)$ *with k-basis* $\{(X \cup X^o)^+\}$. *Here X ranges over the set of non-real conjugacy classes of G such that P contains a Sylow 2-subgroup of $C_G(x)$, for some $x \in X$.*
(ii) $\mathrm{Tr}_{\underline{P}\langle \underline{q}\sigma\rangle}^{\underline{G}\times\Sigma}(kG)$ *is an ideal of* $Z^*(kG)$ *with k-basis* $\{(X \cup X^o)^+\} \cup \{Y^+\}$. *Here X has the same meaning as in* (i), *while Y ranges over the the set of real conjugacy classes of G such that P contains a Sylow 2-subgroup of $C_G(y)$, and $y^{pq} = y^{-1}$, for some $y \in Y$ and $p \in P$.*

Proof. Lemma 12 implies that both $\mathrm{Tr}_{\underline{P}}^{\underline{G}\times\Sigma}(kG)$ and $\mathrm{Tr}_{\underline{P}\langle \underline{q}\sigma\rangle}^{\underline{G}\times\Sigma}(kG)$ are ideals of $Z^*(kG)$.

In general, suppose that G is a finite group, H is a subgroup of G, and M is a permutation kG-module. Then it is well know that the k-space $\mathrm{Tr}_H^G(M)$ has basis of the form $\{O^+\}$. Here O ranges over the G-orbits on the permutation basis such that H contains a Sylow 2-subgroup of the stabilizer subgroup of some element of O in G. The Corollary follows by applying this, and Lemma 11, to the group $\underline{G} \times \Sigma$, its subgroups $\underline{P}$ and $\underline{P}\langle \underline{q}\sigma\rangle$ and the module kG. □

If g is an element of G, its extended centraliser is the following subgroup of G:

$$C_G^*(g) := \left\{x \in G \;\middle|\; g^x = g \text{ or } g^{-1}\right\}.$$

We can now identify the vertices of B^*.

Proposition 14. *Suppose that B is real. Then B has a real defect class. Let $c \in G$ belong to a real defect class of B, let D be a Sylow 2-subgroup of $C_G(c)$ and let $D\langle e\rangle$ be a Sylow 2-subgroup of $C_G^*(c)$. Then $1_B \in \mathrm{Tr}_E^{\underline{G}\times\Sigma}(kG)$, for $E \le \underline{G} \times \Sigma$ if and only if $\underline{D}\langle \underline{e}\sigma\rangle \le_{\underline{G}} E$. Also $\underline{D}\langle \underline{e}\sigma\rangle$ is a vertex of B^*.*

Proof. To show that B has a real defect class, we repeat the original argument of Gow, from [4, Lemma 1.2], for the convenience of the reader. Write $1_B = \sum \lambda_K K^+$, where K runs over the conjugacy classes of G and $\lambda_K \in k$, for each class K. Then

$$1_k = \omega_B(1_B) = \sum \lambda_K \omega_B(K^+).$$

Now $\lambda_K = \lambda_{K^o}$ and $\omega_B(K^+) = \omega_B(K^{o+})$, as B is real. It follows that the contribution of a nonreal class K and its inverse class K^o to the above sum is $2\lambda_K\omega_B(K^+) = 0_k$. So there must exist a real class C such that $\lambda_C\omega_B(C^+) \neq 0_k$. Each such C is a real defect class of B.

Now fix a real defect class C of B. So $\lambda_C \neq 0_k$, using the notation of the previous paragraph. Suppose that $1_B \in \mathrm{Tr}_E^{\underline{G}\times\Sigma}(kG)$, where $E \le \underline{G} \times \Sigma$. Then Corollary 13 implies that there exists $c \in C$ and D a Sylow 2-subgroup of $C_G(c)$ and $D\langle e\rangle$ a Sylow 2-subgroup of $C_G^*(c)$, such that $\underline{D}\langle \underline{e}\sigma\rangle \leq E$.

Mackey's Theorem implies that $(B^*)^{\underline{G}\times\Sigma} \subseteq \sum \mathrm{Tr}_{P\cap(\underline{G}\times\Sigma)}^{\underline{G}\times\Sigma}(kG)$. Here P ranges over the vertices of B^*. Corollary 13 implies that each subspace $\mathrm{Tr}_{P\cap(\underline{G}\times\Sigma)}^{\underline{G}\times\Sigma}(kG)$ is an ideal of $Z^*(kG)$. But 1_B is a primitive idempotent in $Z^*(kG)$. So by Rosenberg's lemma [9, 5.1.1], there exists a vertex V of B^* such that $1_B \in \mathrm{Tr}_{V\cap\underline{G}\times\Sigma}^{\underline{G}\times\Sigma}(kG)$.

The last two paragraphs imply that $\underline{D}\langle \underline{e}\sigma\rangle \leq_{\underline{G}} V$. But $|\underline{D}\langle \underline{e}\sigma\rangle| = |V|$, as a consequence of Lemma 10. It follows that $1_B \in \mathrm{Tr}_{\underline{D}\langle \underline{e}\sigma\rangle}^{\underline{G}\times\Sigma}(kG)$, and also that $\underline{D}\langle \underline{e}\sigma\rangle$ is a vertex of B^*. □

Our Corollary shows that (ii) $\Longrightarrow$ (iii) in Theorem 2.

Corollary 15. *Suppose that there exists $g \in G$ such that $\sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(g) > 0$. Then B is real. Let $\underline{D}\langle \underline{e}\sigma\rangle$ be a vertex of B^*. Then there exists $d \in D$ such that g_2 is G-conjugate to $(de)^2$. In particular, if g can be chosen to be 2-regular, then B is strongly real.*

Proof. When B is the principal 2-block of G, the result is true. So assume otherwise. The hypothesis implies that B is real, as it forces $\nu(\chi) \neq 0$, for some $\chi \in \mathrm{Irr}(B)$.

Let (R, F, k) be a 2-modular system for G. Suppose that $\hat{B}$ is the block algebra of RG such that $B = \hat{B}/J(R)\hat{B}$. Then Lemma 8 shows that the $G \wr \Sigma$-character of $\hat{B}^*$ is $\chi_B := \sum_{\chi\in\mathrm{Irr}(B)} \chi^{\nu(\chi)}$. Now $\hat{B}^*$ and B^* have the same vertices, as both are trivial source modules [9, 4.8.9]. So $\underline{D}\langle \underline{e}\sigma\rangle$ is a vertex of $\hat{B}^*$. As $\chi_B((1_G, g)\sigma) = \sum_{\chi\in\mathrm{Irr}(B)} \nu(\chi)\chi(g)$, the hypothesis is that $\chi_B((1_G, g)\sigma) \neq 0$. It then follows from a theorem of Green [9, 4.7.4] that the 2-part of $(1_G, g)\sigma$ is contained in a vertex of $\hat{B}^*$. But $((1_G, g)\sigma)_2 = (1_G, g_2)\big(g_{2'}^{-1/2}, g_{2'}^{1/2}\big)\sigma$ and $\Big(\big(g_{2'}^{-1/2}, g_{2'}^{1/2}\big)\sigma\Big)^{(1_G, g_{2'})} = \sigma$. In particular $((1_G, g)\sigma)_2$ is $G \wr \Sigma$-conjugate to $(1_G, g_2)\sigma$. So there exists $g_1, g_2 \in G$ and $d \in D$ such that $(1_G, g_2)\sigma = (\underline{de}\sigma)^{(g_1, g_2)} = \big(g_1^{-1}deg_2, g_2^{-1}deg_1\big)\sigma$. This gives $g_2^{-1} = g_1^{-1}de$, and hence also $g_2 = [(de)^2]^{g_1}$.

Suppose that $g_2 = 1_G$. Then $(de)^2 = 1_G$. So de is an involution that belongs to $D\langle e\rangle \backslash D$. Then, using Proposition 14, we see that each real defect class of B is strongly real, whence B is strongly real. □

Let K be a field and let τ be a field automorphism of of K. Suppose that γ is a K-representation of G. Then we may form the representation γ^τ of G by applying τ to the matrix entries in $\gamma(g)$, for each $g \in G$. If M is the KG-module corresponding to γ, we let M^τ denote the KG-module corresponding to γ^τ. This construction also applies if τ is an automorphism of a subfield K_0 of K, and γ is realisable over K_0.

We use this to define the Frobenius twist of a module or character. The Frobenius automorphism Fr of k is given by $\lambda \to \lambda^2$, for $\lambda \in k$. Every $\mathbb{C}$-representation of G can be realized over $\mathbb{Q}(\zeta)$, where ζ is a primitive $|G|^{th}$ root of unity. There is a Galois automorphism Fr of $\mathbb{Q}(\zeta)$ given by $\zeta \to \zeta_2\zeta_{2'}^2$, for $\zeta \in \mathbb{Q}(\zeta)$. If M is a G-module, with Brauer or ordinary character χ, the Frobenius twist module M^{Fr} has character $\chi^{\mathrm{Fr}}\colon g \to \chi\left(g_{2'}^2 g_2\right)$, for each g in the domain of definition of χ.

Note that $\left\{\chi^{\mathrm{Fr}} \mid \chi \in \mathrm{Irr}(B)\right\}$ is the set of irreducible characters in a 2-block B^{Fr} of G. If M is an indecomposable kG-module, then M belongs to B if and only if M^{Fr} belongs to B^{Fr}.

We identify G and $\underline{G}$ and let $\underline{B}$ be the 2-block of $\underline{G}$ corresponding to B. There is a unique 2-block of $\underline{G} \times \Sigma$ that covers $\underline{B}$. We denote this block by $\underline{B} \times \Sigma$. Clearly $(\underline{B} \times \Sigma)^{\mathrm{Fr}} = \underline{B}^{\mathrm{Fr}} \times \Sigma$.

Lemma 16. *The Brauer induced block* $\left(\underline{B}^{\mathrm{Fr}} \times \Sigma\right)^{G\wr\Sigma}$ *is defined. It is the unique* 2-*block of* $G \wr \Sigma$ *that covers the block* $B \otimes B$ *of* $G \times G$.

Proof. Let D be a defect group of B. Then $\underline{D} \times \Sigma$ is a defect group of $\underline{B}^{\mathrm{Fr}} \times \Sigma$. Since $C_{G\wr\Sigma}(\underline{D}\times\Sigma) = \underline{C_G(D)}\times\Sigma$ is contained in $\underline{G}\times\Sigma$, the induced block $\left(\underline{B}^{\mathrm{Fr}}\times\Sigma\right)^{G\wr\Sigma}$ is defined [9, 5.3.6].

Let $B^{\otimes 2}$ be the unique 2-block of $G\wr\Sigma$ that covers the 2-block $B\otimes B$ of $G\times G$. So χ^{+1} belongs to $B^{\otimes 2}$, whenever $\chi \in \mathrm{Irr}(B)$.

Now $C_{G\wr\Sigma}(\sigma) = \underline{G} \times \Sigma$. Each Brauer character of $\underline{G} \times \Sigma$ can be identified with a Brauer character of $\underline{G}$. Using Brauer's second main theorem [9, 5.4.2], we have

$$\chi^{+1}\left(\underline{g}\sigma\right) = \sum_{\theta\in\mathrm{IBr}(\underline{G})} d^{\sigma}_{\chi,\theta}\theta(g), \quad \text{for all} \quad g \in G \quad \text{of odd order,} \tag{1}$$

where the $d^{\sigma}_{\chi,\theta}$ are algebraic integers with the property that $d^{\sigma}_{\chi,\theta} = 0$, unless θ belongs to a 2-block B_1 of G such that $\left(\underline{B_1} \times \Sigma\right)^{G\wr\Sigma} = B^{\otimes 2}$. On the other hand, the definition gives

$$\chi^{+1}\left(\underline{g}\sigma\right) = \chi\left(g^2\right) = \chi^{\mathrm{Fr}}(g) = \sum_{\theta\in\mathrm{IBr}(\underline{G})} d_{\chi^{\mathrm{Fr}},\theta}\theta(g), \quad \text{for all} \quad g \in G \quad \text{of odd order.} \tag{2}$$

But the irreducible Brauer characters of $\underline{G}$ are linearly independent on the 2-regular classes of $\underline{G}$. So (1) and (2) imply that $d^{\sigma}_{\chi,\theta} = d_{\chi^{\mathrm{Fr}},\theta}$, for all $\theta \in \mathrm{IBr}(\underline{G})$. As $d_{\chi^{\mathrm{Fr}},\theta} \neq 0$, for some $\theta \in \mathrm{IBr}\left(B^{\mathrm{Fr}}\right)$, we conclude that $\left(\underline{B}^{\mathrm{Fr}} \times \Sigma\right)^{G\wr\Sigma} = B^{\otimes 2}$. □

The following lemma is a key step in the proof of Theorem 2.

Lemma 17. *Restriction* $\downarrow^{\underline{G}\times\Sigma}_{\underline{G}}$ *establishes a multiplicity preserving bijection between the components of* $B^*\downarrow_{\underline{G}\times\Sigma}$ *that have a vertex containing* Σ *and the components of* $k\Omega$ *that belong to* $\underline{B}^{\mathrm{Fr}}$.

Proof. Let M be a component of $B^*\downarrow_{\underline{G}\times\Sigma}$ that has a vertex V containing Σ. Then Σ is contained in the kernel of M, as M is a trivial source module. Thus M coincides with the inflation of the indecomposable $\underline{G}$-module $M\downarrow_{\underline{G}}$ to $\underline{G}\times\Sigma$. In addition, $M\downarrow_{\underline{G}}$ has a vertex V/Σ.

The orbits of $\underline{G}\times\Sigma$ on the $G\wr\Sigma$-set G are $\{C\cup C^o \mid C$ a conjugacy class of $G\}$. Lemma 6 and Mackey's theorem imply that

$$kG\downarrow_{\underline{G}\times\Sigma} = \sum_{C,C^o} k(C\cup C^o).$$

Also $k(C\cup C^o) = k_{C_{\underline{G}\times\Sigma}(c)}\uparrow^{\underline{G}\times\Sigma}$, for each $c\in C\cup C^o$. Now Lemma 11 implies that $\Sigma\leq C_{\underline{G}\times\Sigma}(c)$ if and only if $c\in\Omega$. Then by the Krull-Schmidt theorem M is a component of $k_{C_{\underline{G}\times\Sigma}(t)}\uparrow^{\underline{G}\times\Sigma}$, for some $t\in\Omega$. But $\left(k_{C_{\underline{G}\times\Sigma}(t)}\uparrow^{\underline{G}\times\Sigma}\right)\downarrow_{\underline{G}} = k_{\underline{C_G(t)}}\uparrow^{\underline{G}}$. We conclude that $M\downarrow_{\underline{G}}$ is a component of $k\Omega$.

Let B_1 be the 2-block of G such that $\underline{B_1}^{\mathrm{Fr}}\times\Sigma$ contains M. As $\Sigma\leq V$, Lemma 5 forces $C_{G\wr\Sigma}(V)\leq\underline{G}\times\Sigma$. So by a Theorem of Nagao-Green [9, 5.3.12], the induced block $\left(\underline{B_1}^{\mathrm{Fr}}\times\Sigma\right)^{G\wr\Sigma}$ contains B^*. But $\left(\underline{B_1}^{\mathrm{Fr}}\times\Sigma\right)^{G\wr\Sigma} = B_1^{\otimes 2}$, by Lemma 16. So B^* belongs to $B_1^{\otimes 2}$. This forces the $G\times G$-module B to belong to $B_1\otimes B_1$. It follows easily that $B = B_1$.

We can reverse the above argument to show that if N is a B^{Fr}-component of $k\Omega$, then the inflation of N to $\underline{G}\times\Sigma$ is a component of $B^*\downarrow_{\underline{G}\times\Sigma}$ that has a vertex containing Σ. □

Corollaries 4 and 15 give the implication (i) $\Longrightarrow$ (iii) of Theorem 2. Our next result gives a direct proof of this and also produces some information on the vertices of the components of $k\Omega$.

Corollary 18. *Suppose that some component of $k\Omega$ belongs to B. Then B is strongly real. More precisely, let $c\in G$ belongs to a real defect class of B, let D be a Sylow 2-subgroup of $C_G(c)$, and let E be a Sylow 2-subgroup of $C_G^*(c)$ that contains D. Suppose that N is a component of $k\Omega$ that belongs to B. Then there exists $t\in\Omega\cap(E\backslash D)$ such that N has a vertex $V\leq C_D(t)$.*

Proof. Clearly N^{Fr} is also a component of $k\Omega$. Also $\underline{V}$ is a vertex of N^{Fr}, as $\underline{G}$-module. Let M be the inflation of N^{Fr} to $\underline{G}\times\Sigma$. Then Lemma 17 implies that M is a component of $B^*\downarrow_{\underline{G}\times\Sigma}$. Now M has vertex $\underline{V}\times\Sigma$. As M is a component of $B^*\downarrow_{\underline{G}\times\Sigma}$, it follows that some vertex of B^* contains $\underline{V}\times\Sigma$.

We established in Proposition 14 that $\underline{D}\langle\underline{e}\sigma\rangle$ is a vertex of B^*. Then by the previous paragraph there exists $(g_1,g_2)\in G\times G$ such that $(\underline{V}\times\Sigma)^{(g_1,g_2)}\leq\underline{D}\langle\underline{e}\sigma\rangle$. In particular $\sigma^{(g_1,g_2)} = \left(g_1^{-1}g_2,\left(g_1^{-1}g_2\right)^{-1}\right)\sigma$ belongs to $\underline{D}\langle\underline{e}\sigma\rangle$. Choose $d\in D$ such that $de = g_1^{-1}g_2$. Then $t := de$ belongs to Ω. So $E = D\langle t\rangle$ splits over D. In particular B is strongly real. Also $\underline{V}^{(g_1,g_2)}\leq C_{\underline{D}}(\underline{t}\sigma)$. So $V\leq_G C_D(t)$. □

An important special case is that of a real 2-block of defect zero. The next result was proved by R. Gow (in unpublished work), unifying results in [4] and [10]. Gow's proof used a pairing argument on L.L. Scott's 'orbital characters' defined with respect to the involution module. Our proof uses Alperin-Scott modules.

Theorem 19. *Suppose that B is a real 2-block of defect 0. Let χ be the unique irreducible character in B. Then $\nu(\chi) = +1$. Let C be a real defect class of B and let $t \in \Omega$. Then $\langle \chi{\downarrow_{C_G(t)}}, 1_{C_G(t)} \rangle = 1$ or 0, depending on whether or not t inverts an element of C.*

Proof. Note that χ is the unique principal indecomposable character in B. The fact that $\nu(\chi) = +1$ appears in [10] and independently in several other places. Here it is a simple consequence of Lemma 3. It also follows from that lemma that the G-character of $\mathbb{C}\Omega$ contains χ with multiplicity 1.

Let $c \in C$ and let $t \in \Omega$ be such that $c^t = c^{-1}$. The proof will be completed by showing that $\langle \chi{\downarrow_{C_G(t)}}, 1_{C_G(t)} \rangle \neq 0$.

Now $Z(B^*) = (B^*)^{\underline{G}\times\Sigma}$ is spanned by 1_B. Also B^* has a trivial source. So $B^*{\downarrow_{\underline{G}\times\Sigma}}$ has a unique Scott component, and this component has socle spanned by 1_B. Proposition 14 implies that $1_B \in \mathrm{Tr}_E^{\underline{G}\times\Sigma}(kG)$, for $E \leq \underline{G}\times\Sigma$ if and only if $\langle \underline{t}\sigma \rangle \leq_{\underline{G}} E$. It follows that the Scott component of $B^*{\downarrow_{\underline{G}\times\Sigma}}$ has vertex $\langle \underline{t}\sigma \rangle$.

An easy calculation shows that $\underline{C_G(t)}\times\Sigma$ coincides with the normalizer of $\langle \underline{t}\sigma \rangle$ in both $C_G(t) \wr \Sigma$ and $\underline{G} \times \Sigma$. The Green correspondence preserves Scott modules. Then, using a result of D. Burry [9, 4.4.7], and the previous paragraph, the Scott module $S(C_G(t) \wr \Sigma; \langle \underline{t}\sigma \rangle)$ has multiplicity 1 as a component of $B^*{\downarrow_{C_G(t)\wr\Sigma}}$. The restriction of $S(C_G(t) \wr \Sigma; \langle \underline{t}\sigma \rangle)$ to $C_G(t) \times C_G(t)$ has a projective Scott component. We deduce that $B{\downarrow_{C_G(t)\times C_G(t)}}$ has a Scott component.

Let $\hat{B}$ be a lift of B to a $G \times G$-module over a field of characteristic 0. Then $\hat{B}$ has character $\overline{\chi} \otimes \chi$. Thus $\langle (\overline{\chi} \otimes \chi){\downarrow_{C_G(t)\times C_G(t)}}, 1_{C_G(t)\times C_G(t)} \rangle$ is the number of Scott components of $B{\downarrow_{C_G(t)\times C_G(t)}}$. But this inner product is

$$\sum_{c_1, c_2 \in C_G(t)} \chi\left(c_1^{-1}\right) \chi(c_2) = \left\langle \chi{\downarrow_{C_G(t)}}, 1_{C_G(t)} \right\rangle^2 .$$

It then follows from the previous paragraph that $\langle \chi{\downarrow_{C_G(t)}}, 1_{C_G(t)} \rangle \neq 0$. □

We can now prove that (iii) $\implies$ (ii) in Theorem 2. This completes the proof of that theorem.

Corollary 20. *Suppose that B is strongly real. Then $\sum_{\chi \in \mathrm{Irr}(B)} \nu(\chi)\chi(1_G) > 0$.*

Proof. Let $c \in G$ belong to a real defect class C of B. Then c is strongly real. Fix a Sylow 2-subgroup D of $C_G(c)$ and a Sylow 2-subgroup E of $C_G^*(c)$ that contains

D. Write $E = D\langle t\rangle$, where $t \in \Omega$.

We need a version of Brauer's first main theorem. Let N be the normalizer of D in G and let b be the Brauer correspondent of B with respect to (G, N, D). Now b is a real 2-block of N, as $(b^o)^G = (b^G)^o = B$, and b is the unique block of kN such that $b^G = B$. If $1_B = \sum_K \lambda_K K^+$, then $1_b = \sum_K \lambda_K (K \cap C_G(D))^+$. Here K ranges over the classes of G. Also $\omega_B(K^+) = \omega_b((K \cap C_G(D))^+)$, for each conjugacy class K of G. It follows that $C_1 := C \cap C_G(D)$ is a real defect class of b.

Set $\overline{N} := N/D$ and let μ be the natural k-algebra projection $kN \to k\overline{N}$. Let $\overline{C}$ be the conjugacy class of $\overline{N}$ that contains $\bar{c} = Dc$. Then $\mu(C_1^+) = \overline{C}^+$, by [9, 5.8.9]. It follows that $\mu(1_b) \neq 0$. Write $\mu(1_b) = \sum_{i=1}^s 1_{\beta_i}$, where $\beta_1, \ldots, \beta_s$ are distinct blocks of $\overline{N}$. As $1_{b^o} = 1_b$, it follows that there is a permutation τ of $\{1, \ldots, s\}$ such that $\beta_i^o = \beta_{i\tau}$, for $i = 1, \ldots, s$. If $i \neq i\tau$, an easy argument shows that $1_{\beta_i} + 1_{\beta_{i\tau}}$ is supported on the non-real classes of $\overline{N}$. But $\overline{C}$ is a real class of $\overline{N}$ whose sum appears with non-zero multiplicity in $\mu(1_b)$. We deduce that there exists i such that β_i is a real 2-block of $\overline{N}$ and $\overline{C}^+$ appears with non-zero multiplicity in 1_{β_i}. Set $\beta := \beta_i$.

Now β has a trivial defect group, by [9, 5.8.7 (ii)], and $C_{\overline{N}}(\bar{c})$ is odd, by [9, 5.8.9 (ii)]. It follows that $\overline{C}$ is a real defect class of β. Let χ be the unique irreducible character in β. Now $\bar{t}$ is an involution in $\overline{N}$ that inverts an element of $\overline{C}$. So by Theorem 19 we have $\left\langle \chi_{C_{\overline{N}}(\bar{t})}, 1_{C_{\overline{N}}(\bar{t})} \right\rangle = 1$. The preimage of $C_{\overline{N}}(\bar{t})$ in N is $C_N(Dt) := \{n \in N \mid t^n \in Dt\}$. Inflating χ to N, we get $\langle \chi_{C_N(Dt)}, 1_{C_N(Dt)} \rangle = 1$. But $C_N(t) \leq C_N(Dt)$. So $\langle \chi_{C_N(t)}, 1_{C_N(t)} \rangle \neq 0$. Let M be the unique irreducible β-module. Then we have just shown that M is a b-composition factor of $k_{C_N(t)}\!\uparrow^N$. We deduce from Lemma 17 that $b^*\!\downarrow_{\underline{N} \times \Sigma}$ has a component with a vertex that contains Σ.

J.L. Alperin proved in [1] that b is a component of $B\!\downarrow_{N \times N}$. So b^* is a component of $B^*\!\downarrow_{N \times N}$. This and the previous paragraph show that $B^*\!\downarrow_{\underline{N} \times \Sigma}$ has a component with a vertex that contains Σ. Applying Lemma 17, we deduce that $k\Omega$ has a B-composition factor. We conclude from Corollary 4 that $\sum_{\chi \in \mathrm{Irr}(B)} \nu(\chi)\chi(1_G) > 0$. □

We conclude our paper with a small application of Theorem 2. R. Gow proved the following result in [3, 5.6]:

Proposition 21. *Let B be a real 2-block, let $c \in G$ belong to a real defect class of B, let D be a Sylow 2-subgroup of $C_G(c)$ and let $D\langle e\rangle$ be a Sylow 2-subgroup of $C_G^*(c)$. Then B contains a real-valued irreducible character of height 0 and Frobenius-Schur indicator -1 if and only if $D\langle e\rangle/D'$ does not split over D/D'.*

It is known that each real 2-block has a real-valued irreducible character of Frobenius-Schur indicator +1. So Theorem 2 and Proposition 21 combine to give:

Corollary 22. *Let B, D and $D\langle e\rangle$ be as in* Proposition 21. *Suppose that $D\langle e\rangle/D'$ splits over D/D' but $D\langle e\rangle$ does not split over D. Then B contains a real-valued irreducible character of height greater that 0 and Frobenius-Schur indicator -1.*

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Mathematics Department
National University of Ireland-Maynooth
Co. Kildare, Ireland
e-mail: John.Murray@maths.may.ie