Blair, D.E. and Turgut Vanli, A.
Osaka J. Math.
43 (2006), 193–200

CORRECTED ENERGY OF DISTRIBUTIONS FOR 3-SASAKIAN AND NORMAL COMPLEX CONTACT MANIFOLDS

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(Received September 6, 2004)

Abstract

In this paper we show that the natural fibrations on 3-Sasakian manifolds and on normal complex contact metric manifolds are minima of the corrected energy of the corresponding distributions.

1. Introduction

In [6], Chacón, Naveira and Weston introduced the energy $\mathcal{E}(\mathcal{V})$ of a q-dimensional distribution on a Riemannian manifold (M, g). They studied the first and second variation of the energy and as an application showed that the Hopf fibration $S^3 \hookrightarrow S^{4n+3} \to \mathbb{H}P^n$ is an unstable critical point. The corresponding result in the case of the energy of a vector field for the Hopf fibration $S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n$ is due to C.M. Wood [14]. Wood showed that for $n > 1$, the critical point is unstable; for $n = 1$ Brito [4] showed that this Hopf fibration is a minima.

Subsequently in [5], Chacón and Naveira introduced a corrected energy $\mathcal{D}(\mathcal{V})$ for a q-dimensional distribution on a Riemannian manifold (M, g) and proved that $\mathcal{D}(\mathcal{V})$ is $\geq$ the integral of the sum of the mixed sectional curvatures associated to a compatible basis. As a single application they showed that the Hopf fibration $S^3 \hookrightarrow S^{4n+3} \to \mathbb{H}P^n$ is a minimum of $\mathcal{D}(\mathcal{V})$. In the present paper we show that this application can be greatly generalized to the natural fibrations on 3-Sasakian manifolds and on normal complex contact metric manifolds.

2. Geometry of distributions

Let (M^n, g) be a compact oriented Riemannian manifold with a q-dimensional distribution or subbundle $\mathcal{V}$ and let $\mathcal{H}$ denote the orthogonal complementary distribution of dimension $p = n - q$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis on M^n such that $\{e_1, \ldots, e_p\}$ span $\mathcal{H}$ and $\{e_{p+1}, \ldots, e_n\}$ span $\mathcal{V}$ and adopt the index conventions: $1 \leq a, b \leq n$, $1 \leq i, j \leq p$, $p+1 \leq \alpha, \beta \leq n$. The second fundamental form of the horizontal distribution $\mathcal{H}$ in the direction e_α and that of the vertical distribution $\mathcal{V}$ in

2000 Mathematics Subject Classification. Primary 53C25; Secondary 53C15, 53D10, 53E99.

the direction e_i are given respectively by

$$h_{ij}^{\alpha} = -g\big(\nabla_{e_i} e_\alpha, e_j\big), \quad h_{\alpha\beta}^{i} = -g\big(\nabla_{e_\alpha} e_i, e_\beta\big).$$

The mean curvature vectors of the horizontal and vertical distributions are given respectively by

$$\vec{H}_{\mathcal{H}} = \sum_{\alpha=p+1}^{n} \left(\frac{1}{p} \sum_{i=1}^{p} h_{ii}^{\alpha} \right) e_\alpha, \quad \vec{H}_{\mathcal{V}} = \sum_{i=1}^{p} \left(\frac{1}{q} \sum_{\alpha=p+1}^{n} h_{\alpha\alpha}^{i} \right) e_i.$$

One can regard a distribution, such as $\mathcal{V}$, as a section of the Grassmann bundle, $G(q, M^n)$, of oriented q-planes in the tangent spaces of M^n. The geometry of this bundle was developed in [6]. We also view $\mathcal{V}$ as a map $\xi : M^n \to G(q, M^n)$ where $\xi(x)$ is a unit q-vector with respect to the induced metric on $\bigwedge^q(M^n)$, in particular

$$\xi(x) = e_{p+1}(x) \wedge \cdots \wedge e_n(x).$$

Note that we have chosen a local orthonormal basis; in [14] the variations of unit vector fields are through unit vector fields and the variations of distributions in [6] are through unit q-vectors. The norm of the covariant derivative of ξ is given in terms of the second fundamental forms of $\mathcal{H}$ and $\mathcal{V}$ by

$$(*) \qquad \sum_{a} \big\| \nabla_{e_a} \xi \big\|^2 = \sum_{i,j,\alpha} \big(h_{ij}^{\alpha}\big)^2 + \sum_{i,\alpha,\beta} \big(h_{\alpha\beta}^{i}\big)^2.$$

The *energy* of a distribution $\mathcal{V}$ was defined in [6] as

$$\mathcal{E}(\mathcal{V}) = \frac{1}{2} \int_M \sum_{a=1}^{n} \big\| \nabla_{e_a} \xi \big\|^2 \, d\,\mathrm{vol} + \frac{n}{2} \mathrm{vol}(M).$$

The *corrected energy* of [5] is defined by

$$\mathcal{D}(\mathcal{V}) = \int_M \left(\sum_{a} \big\| \nabla_{e_a} \xi \big\|^2 + p(p-2) \big\| \vec{H}_{\mathcal{H}} \big\|^2 + q^2 \big\| \vec{H}_{\mathcal{V}} \big\|^2 \right) d\,\mathrm{vol}.$$

The main result of [5] is the following.

Theorem A. *If $\mathcal{V}$ is integrable, then*

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{i,\alpha} c_{i\alpha} \, d\,\mathrm{vol}$$

where $c_{i\alpha}$ is the sectional curvature of the plane section spanned by $e_i \in \mathcal{H}$ and $e_\alpha \in \mathcal{V}$.

3. 3-Sasakian manifolds

By a *contact manifold* we mean a differentiable manifold M^{2n+1} together with a 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. It is well known that given η there exists a unique vector field ξ, such that $d\eta(\xi, X) = 0$ and $\eta(\xi) = 1$; ξ is called the *characteristic vector field* or *Reeb vector field* of the contact form η.

A Riemannian metric g is an *associated metric* for a contact form η if, first of all, $\eta(X) = g(X, \xi)$ and secondly, there exists a field of endomorphisms ϕ such that $\phi^2 = -I + \eta \otimes \xi$ and $d\eta(X, Y) = g(X, \phi Y)$. We refer to (ϕ, ξ, η, g) as a *contact metric structure* and to M^{2n+1} with such a structure as a *contact metric manifold.*

An *almost contact structure*, (ϕ, ξ, η), consists of a field of endomorphisms ϕ, a vector field ξ and a 1-form η such that $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$ and an *almost contact metric structure* includes a Riemannian metric satisfying the compatibility condition $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$.

The product $M^{2n+1} \times \mathbb{R}$ carries a natural almost complex structure defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right)$$

and the underlying almost contact structure is said to be *normal* if J is integrable. The normality condition can be expressed as $N = 0$ where N is defined by

$$N(X, Y) = [\phi, \phi](X, Y) + 2\, d\eta(X, Y)\xi,$$

$[\phi, \phi]$ being the Nijenhuis tensor of ϕ.

A *Sasakian manifold* is a normal contact metric manifold. In terms of the covariant derivative of ϕ with respect to the Levi-Civita connection, the Sasakian condition is

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X.$$

As is well known, from this it is easily seen that

$$\nabla_X \xi = -\phi X$$

and in turn that ξ is a Killing vector field, i.e. the contact metric structure is *K-contact*. It is also well known that on a K-contact manifold the sectional curvature of all plane sections containing ξ are equal to +1 (see e.g. [1], p.92).

A manifold admitting three almost contact structures, $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, $\alpha = 1, 2, 3$, satisfying

$$\phi_\gamma = \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta.$$
$$\xi_\gamma = \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha$$

is said to have an *almost contact* 3*-structure*. Kuo [13] showed that given such a structure there exists a Riemannian metric g compatible with each of the three almost contact structures giving us an *almost contact metric* 3*-structure* $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. If each of the three structures is Sasakian we have a 3*-Sasakian structure*. A remarkable result of Kashiwada [11] is that if each of the three almost contact metric structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a contact metric structure, then the structure is a 3-Sasakian structure. There are many 3-Sasakian manifolds aside from the sphere S^{4n+3} including several homogeneous spaces; see e.g. [1] pp.218–220 or the survey of Boyer and Galicki [3].

Using $\nabla_X \xi_\alpha = -\phi_\alpha X$ one readily obtains on a 3-Sasakian manifold that $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$. Thus the distribution $\mathcal{V}$ determined by the tri-vector $\xi = \xi_\alpha \wedge \xi_\beta \wedge \xi_\gamma$ is integrable with totally geodesic leaves. The horizontal distribution $\mathcal{H}$ is defined by $\eta_\alpha = 0$, $\alpha = 1, 2, 3$.

We now state and prove the following theorem.

Theorem 1. *The vertical distribution $\mathcal{V}$ on a compact* 3*-Sasakian manifold is a minima of the corrected energy $\mathcal{D}(\mathcal{V})$.*

Proof. The proof will be the direct computations of both sides of the inequality in Theorem A showing that they are equal. We first show that the mean curvature vectors $\vec{H}_{\mathcal{V}}$ and $\vec{H}_{\mathcal{H}}$ vanish. This will follow immediately from the following computations:

$$h^i_{\alpha\beta} = -g(\nabla_{\xi_\alpha} e_i, \xi_\beta) = g(e_i, \nabla_{\xi_\alpha}\xi_\beta) = g(e_i, -\phi_\beta\xi_\alpha) = g(e_i, \xi_\gamma) = 0,$$
$$h^\alpha_{ii} = -g(\nabla_{e_i}\xi_\alpha, e_i) = g(\phi_\alpha e_i, e_i) = 0.$$

Thus by equation $(*)$ we have for the norm of the covariant derivative of the tri-vector ξ defining $\mathcal{V}$,

$$\sum_a \left\| \nabla_{e_a}\xi \right\|^2 = \sum_{i,j,\alpha} \left(h^\alpha_{ij}\right)^2 = \sum_{i,j,\alpha} g\left(\nabla_{e_i}\xi_\alpha, e_j\right)^2$$
$$= \sum_{i,\alpha} g(-\phi_\alpha e_i, -\phi_\alpha e_i) = \sum_\alpha 4n = 12n.$$

On the other hand we have noted that on a Sasakian manifold all sectional curvatures of plane sections containg the characteristic vector field are equal to +1. Noting this for each of the three Sasakian structures and the horizontal vectors e_i, we have that each $c_{i\alpha} = +1$. Therefore

$$\sum_{i,\alpha} c_{i\alpha} = 12n$$

and we see that both sides of the inequality are equal to $12n$ times the volume of the manifold. □

4. Complex contact manifolds

A complex contact manifold is a complex manifold of odd complex dimension $2n+1$ together with an open covering $\{\mathcal{U}\}$ by coordinate neighborhoods such that

1. On each $\mathcal{U}$, there is a holomorphic 1-form θ with $\theta \wedge (d\theta)^n \neq 0$.
2. On $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$ there is a non-vanishing holomorphic function f such that $\theta' = f\theta$.

The complex contact structure determines a non-integrable distribution $\mathcal{H}$ by the equation $\theta = 0$. A complex contact structure is given by a global 1-form if and only if its first Chern class vanishes [2].

On the other hand let M be a Hermitian manifold with almost complex structure J, Hermitian metric g and an open covering by coordinate neighborhoods $\{\mathcal{U}\}$; M is called a *complex almost contact metric manifold* if it satisfies the following two conditions:

1. In each $\mathcal{U}$ there exist 1-forms u and $v = u \circ J$, with dual vector fields U and $V = -JU$ and $(1,1)$ tensor fields G and $H = GJ$ such that

$$H^2 = G^2 = -I + u \otimes U + v \otimes V$$
$$GJ = -JG, \quad GU = 0, \quad g(X, GY) = -g(GX, Y).$$

2. On $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, we have

$$u' = au - bv, \quad v' = bu + av$$
$$G' = aG - bH, \quad H' = bG + aH$$

where a and b are functions on $\mathcal{U} \cap \mathcal{U}'$ with $a^2 + b^2 = 1$.

Since u and v are dual to the vector feilds U and V, we easily see from the second condition that on $\mathcal{U} \cap \mathcal{U}'$, $U' = aU - bV$ and $V' = bU + aV$. Also since $a^2 + b^2 = 1$, $U' \wedge V' = U \wedge V$. Thus U and V determine a global vertical distribution $\mathcal{V}$ by $\xi = U \wedge V$ which is typically assumed to be integrable.

A complex contact manifold admits a complex almost contact metric structure for which the local contact form θ is $u - iv$ to within a non-vanishing complex-valued function multiple and the local tensor fields G and H are related to du and dv by

$$du(X, Y) = g(X, GY) + (\sigma \wedge v)(X, Y), \quad dv(X, Y) = g(X, HY) - (\sigma \wedge u)(X, Y)$$

where $\sigma(X) = g(\nabla_X U, V)$, ∇ being the Levi-Civita connection of g (Ishihara and Konishi [10], Foreman [7]). We refer to a complex contact metric manifold with a complex almost contact metric structure satisfying these conditions as a *complex contact metric manifold.*

Ishihara and Konishi [8], [9] introduced a notion of normality for complex contact structures. Their notion is the vanishing of the two tensor fields S and T given by

$$S(X, Y) = [G, G](X, Y) + 2g(X, GY)U - 2g(X, HY)V + 2(v(Y)HX - v(X)HY)$$
$$+ \sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX,$$

$$T(X,Y) = [H,H](X,Y) - 2g(X,GY)U + 2g(X,HY)V + 2(u(Y)GX - u(X)GY) + \sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX.$$

However this notion is too strong; among its implications is that the underlying Hermitian manifold (M,g) is Kähler. Thus while indeed one of the canonical examples of a complex contact manifold, the odd-dimensional complex projective space, is normal in this sense, the complex Heisenberg group, is not. In [12] B. Korkmaz generalized the notion of normality and we adopt her definition here. A complex contact metric structure is said to be *normal* if

$$S(X,Y) = T(X,Y) = 0, \quad \text{for every} \quad X, Y \in \mathcal{H},$$
$$S(U,X) = T(V,X) = 0, \quad \text{for every} \quad X.$$

Even though the definition appears to depend on the special nature of U and V, it respects the change in overlaps, $\mathcal{U} \cap \mathcal{U}'$, and is therefore a global notion. With this notion of normality both odd-dimensional complex projective space and the complex Heisenberg group with their standard complex contact metric structures are normal.

One important consequence of normality for us is that the sectional curvature of a plane section spanned by a vector in $\mathcal{V}$ and a vector in $\mathcal{H}$ is equal to +1 (cf. Korkmaz [12]). Another consequence of normality is that

$$(**) \qquad \nabla_X U = -GX + \sigma(X)V, \quad \nabla_X V = -HX - \sigma(X)U.$$

Theorem 2. *If M is a compact normal complex contact metric manifold, then the vertical distribution is a minima of the corrected energy, i.e.*

$$\mathcal{D}(\mathcal{V}) = \int_M \sum_{i,\alpha} c_{i\alpha}\, d\,\mathrm{vol}$$

where $c_{i\alpha}$ is the sectional curvature of the plane section spanned by $e_i \in \mathcal{H}$ and $e_\alpha \in \mathcal{V}$.

Proof. As with Theorem 1, the proof will be the direct computations of both sides of the inequality in Theorem A showing that they are equal. We first note that the integral submanifolds of $\mathcal{V}$ are totally geodesic. This follows readily from $(**)$ and the fact G and H annihilate U and V. Therefore all $h^i_{\alpha\beta}$ vanish and in particular $\vec{H}_{\mathcal{V}} = 0$. Similarly

$$h^1_{ii} = -g\left(\nabla_{e_i} U, e_i\right) = -g\left(-Ge_i + \sigma(e_i)V, e_i\right) = 0$$

and

$$h^2_{ii} = -g\left(\nabla_{e_i} V, e_i\right) = -g\left(-He_i - \sigma(e_i)U, e_i\right) = 0$$

which shows that the mean curvature vector $\vec{H}_{\mathcal{H}}$ vanishes.

Again by equation $(*)$ we have for the norm of the covariant derivative of the bivector $\xi = U \wedge V$,

$$\begin{aligned}\sum_a \|\nabla_{e_a}\xi\|^2 &= \sum_{i,j,\alpha}\left(h_{ij}^{\alpha}\right)^2 = \sum_{i,j}\left[g(\nabla_{e_i}U, e_j)^2 + g(\nabla_{e_i}V, e_j)^2\right] \\ &= \sum_{i,j}\left[g(Ge_i, e_j)^2 + g(He_i, e_j)^2\right] = \sum_i [g(Ge_i, Ge_i) + g(He_i, He_i)] \\ &= 2\sum_i g(e_i, e_i) = 8n.\end{aligned}$$

On the other hand we have noted that on a normal complex contact metric manifold all sectional curvatures of plane sections spanned by a vector in $\mathcal{V}$ and a vector in $\mathcal{H}$ are equal to +1. Noting this for each of $U, V \in \mathcal{V}$ and the horizontal vectors e_i, we have that each $c_{i\alpha} = +1$. Therefore

$$\sum_{i,\alpha} c_{i\alpha} = 8n$$

and we see that both sides of the inequality in Theorem A are equal to $8n$ times the volume of the manifold. □

References

[1] D.E. Blair: Riemannian Geometry of Contact and Symplectic Manifolds, Birkhäuser, Boston, 2002.

[2] W.M. Boothby: *Homogeneous complex contact manifolds*; in Proc. Sympos. Pure Math. III, Amer. Math. Soc., 1961, 144–154.

[3] C.P. Boyer and K.Galicki: 3-*Sasakian manifolds*; in Essays on Einstein Manifolds, Surv. Differ. Geom., VI, Int. Press, Boston, MA, 1999, 123–184.

[4] F.G.B. Brito: *Total bending of flows with mean curvature correction*, Differential Geom. Appl. **12** (2000), 157–163.

[5] P.M. Chacón and A.M. Naveira, *Corrected energy of distributions on Riemannian manifolds*, Osaka J. Math. **41** (2004), 97–105.

[6] P.M. Chacón, A.M. Naveira and J.M. Weston: *On the energy of distributions with application to quaternionic Hopf fibrations*, Monatsh. Math. **133** (2001), 281–294.

[7] B. Foreman: Variational Problems on Complex Contact Manifolds with Applications to Twistor Space Theory, Thesis, Michigan State University, 1996.

[8] S. Ishihara and M. Konishi: *Real contact* 3-*structure and complex contact structure*, Southeast Asian Bulletin of Math. **3** (1979), 151–161.

[9] S. Ishihara and M. Konishi: *Complex almost contact manifolds*, Kōdai Math. J. **3** (1980), 385–396.

[10] S. Ishihara and M. Konishi: *Complex almost contact structures in a complex contact manifold*, Kōdai Math. J. **5** (1982), 30–37.

[11] T. Kashiwada: *On a contact* 3-*structure*, Math. Z. **238** (2001), 829–832.

[12] B. Korkmaz: *Normality of complex contact manifolds*, Rocky Mountain J. Math. **30** (2000), 1343–1380.
[13] Y.-Y. Kuo: *On almost contact* 3*-structure*, Tôhoku Math. J. **22** (1970), 325–332.
[14] C.M. Wood: *On the energy of a unit vector field*, Geom. Dedicata **64** (1997), 319–330.

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