Ballico, E.
Osaka J. Math.
43 (2006), 179–181

CURVES IN PROJECTIVE SPACES AND THEIR INDEX OF REGULARITY

EDOARDO BALLICO

(Received August 27, 2004, revised April 20, 2005)

Abstract

For all integers $n \geq 3$ we show the existence of many triples (d, g, ρ) such that there is a smooth non-degenerate curve $C \subset \mathbf{P}^n$ with degree d, genus g and index of regularity ρ. The curve C lies in a smooth $K3$ surface $S \subset \mathbf{P}^n$.

1. Index of regularity

Let $C \subset \mathbf{P}^n$ be a curve, i.e. a locally Cohen-Macaulay pure one-dimensional closed subscheme. Set $\rho(C) := \min\{t \colon h^1(\mathbf{P}^n, \mathcal{I}_C(x)) = 0$ for every $x \geq t\}$. We will call $\rho(C)$ the index of regularity of C. Since the old works of Castelnuovo, the integer $\rho(C)$ is considered a fundamental invariant of C ([5], [2]). In all cases we will consider in this paper we will have $h^1(C, \mathcal{O}_C(\rho - 1)) = 0$ and hence by Castelnuovo-Mumford lemma in this case the integer $\rho(C)$ will be also the regularity index of the minimal free resolution of C ([2]): another very good reason to consider it a fundamental invariant of C. Thus for any fixed integer $n \geq 3$ it seems nice to show the existence of many triples (d, g, ρ) such that there is a smooth non-degenerate curve $C \subset \mathbf{P}^n$ with degree d, genus g and index of regularity ρ. A weaker, but very important problem, classical problems is to find at least "almost all" pairs (d, g) that may appear as (degree, genus) of a smooth non-degenerate curve $C \subset \mathbf{P}^n$. For this classical problem (when $n = 3$) S. Mori used a $K3$ surface ([4]). Later, A.L. Knutsen extended Mori's idea to the case $n \geq 4$. Using Knutsen's paper it was possible to construct curves C such that certain cohomology groups $h^1(\mathbf{P}^n, \mathcal{I}_C(x))$ vanish ([1]). Here we adapt the proofs in [1] to get results on the index of regularity.

Theorem 1. *Fix integers d, g, n such that $n \geq 3$ and $0 \leq d - n < g < d^2/(4n - 4) - (n-1)/4$. Set $r := \lfloor (d - \sqrt{d^2 - (4n-4)g}/(2n-2) \rfloor$, $d_0 := d - (2n-2)r$ and $g_0 := (n-1)r^2 - dr + g$. Then $r \geq 1$ and $0 \leq g_0 \leq d_0 - n$. There is a smooth and arithmetically Cohen-Macaulay degree $2n-2$ $K3$ surface $S \subset \mathbf{P}^n$ with the following properties. Set $H := \mathcal{O}_S(1)$. There is a smooth and connected curve $C_0 \subset S$ such that* $\deg(C_0) = d_0$, $p_a(C_0) = g_0$, $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, $h^0(S, \mathcal{O}_S(H - C_0)) = h^0(S, \mathcal{O}_S(C_0 - H)) = 0$, $\mathrm{Pic}(S)$

2000 Mathematics Subject Classification. Primary 14H50; Secondary 14N50.
The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

is freely generated by the classes of H *and* C_0*, and the general element of* $|C_0 + rH|$ *is a smooth and connected non-degenerate curve with degree* d *and genus* g*. We have* $e(C_0) = 0$ *if* $g_0 > 0$ *and* $e(C_0) = -1$ *if* $g_0 = 0$*. Take any* $C \in |C_0 + rH|$*. Then* $\rho(C) = \rho(C_0) + r$ *and* $e(C) = e(C_0) + r$*.* C *is arithmetically normal if and only if* C_0 *is projectively normal and this is the case if and only if* $d_0 = g_0 + n$ *and* $h^1(\mathbf{P}^n, \mathcal{I}_{C_0}(2)) = 0$*. If* $d_0 = g_0 + n$ *and* $n > g_0$*, then* C_0 *is projectively normal.*

REMARK 1. Use the notation of Theorem 1. The existence of S was proved in [1], proof of Th.1.4. By [1], Th.1.4, we have $h^1(\mathbf{P}^n, \mathcal{I}_C(r+1)) = d_0 - g_0 - n$ and $h^1(\mathbf{P}^n, \mathcal{I}_C(t)) = 0$ for every integer t such that $0 \le t \le r$.

By Theorem 1 the computation of the index of regularity $\rho(C)$ of C is reduced to the computation of the integer $\rho(C_0)$. Since $d \gg d_0$, the following remark may be useful.

REMARK 2. Let $C \subset \mathbf{P}^m$ be an integral degree d non-degenerate curve. If $m = 2$, then $\rho(C) = 0$. However, if $m = 2$, then $h^1(C, \mathcal{O}_C(t)) = 0$ if and only if $t \ge d - 2$. Now assume $m \ge 3$. By [2] we have $\rho(C) \le d + 1 - m$ and $\rho(C) = d + 1 - m$ if and only if C is smooth and rational and either $d \le m + 1$ or $d \ge m + 2$ and C has a $(d + 2 - m)$-secant line. Furthermore, $h^1(C, \mathcal{O}_C(z)) = 0$ for all $z \ge d - m$.

We work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$.

Proof of Theorem 1. The existence of the pair (S, C_0) was checked in [1], proof of Th.1.4 (see in particular the last two lines of that proof for the critical condition $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$). Since $h^1(C_0, \mathcal{O}_{C_0}(1)) = 0$, we have $e(C_0) \le 0$. Hence $e(C_0) = 0$ if $g_0 > 0$ and $e(C_0) \in \{-2, -1\}$ if $g_0 = 0$. Since $h^0(S, \mathcal{O}_S(H - C_0)) = 0$, C_0 is not a line and hence $e(C_0) = -1$ if $g_0 = 0$. The construction of the pair (S, C_0) used in an essential way the construction of many curves in suitable $K3$ surfaces due to S. Mori ([4]) for $n = 3$ and to A.L. Knutsen ([3]) for arbitrary n. Fix an integer $a \ge 0$ and any $T \in |C_0 + aH|$. For all integers t we have the following exact sequence

$$0 \to \mathcal{O}_S((t-a)H - C_0) \to \mathcal{O}_S(tH) \to \mathcal{O}_T(t) \to 0 \tag{1}$$

If $t \ge a$ and $(t, a) \ne (0, 0)$, then $h^1(S, \mathcal{O}_S(tH)) = h^2(S, \mathcal{O}_S(tH)) = 0$ and hence $h^1(T, \mathcal{O}_T(t)) = h^2(S, \mathcal{O}_S((t-a)H - C_0)) = h^0(S, \mathcal{O}_S(C_0 + (a-t)H))$. From this relation for $a = 0$ and $a = r$ we get $e(C) = e(T) = e(C_0) + r$. Since this relation is obvious for $r = 0$, we do not need the case $(t, a) = (0, 0)$. By [1], Th.1.4, (its proof does not require the smoothness of C) we have $h^1(\mathbf{P}^n, \mathcal{I}_T(t)) = 0$ if $0 \le t \le r$ and $h^1(\mathbf{P}^n, \mathcal{I}_T(r+1)) = d_0 - g_0 - n$. Now assume $a \in \{0, r\}$ and take an arbitrary integer $t \ge r + 2$. Since S is projectively normal, $h^1(\mathbf{P}^n, \mathcal{I}_T(t)) = h^1(S, \mathcal{O}_S((t-a)H - C_0))$. Hence $\rho(C) = \rho(C_0) + r$. The projective normality of a degree d_0 linearly normal embedding of C_0 if $d_0 \geq 2g_0 + 1$ was proved by D. Mumford ([5], Cor. at p.55). □

REMARK 3. The proof of Theorem 1 shows that $\rho(C)$ is the minimal integer t such that $h^1(S, \mathcal{O}_S((t-r-1)H - C_0)) \neq 0$.

References

[1] E. Ballico, N. Chiarli and S. Greco: *On the existence of k-normal curves of given degree and genus in projective spaces*, Collect. Math. **55** (2004), 269–277.

[2] L. Gruson, R. Lazarsfeld and C. Peskine: *On a theorem of Castenuovo, and the equations defining space curves*, Invent. Math. **72** (1983), 491–506.

[3] A.L. Knutsen: *Smooth curves on projective $K3$ surfaces*, Math. Scand. **90** (2000), 215–231.

[4] S. Mori: *On degree and genera of curves on smooth quartic surfaces in* $\mathbf{P}^3$, Nagoya Math. J. **96** (1984), 127–132.

[5] D. Mumford: *Varieties defined by quadratic equations*; in Questions on Algebraic Varieties, Cremonese, Rome, 1970, 30–100.

Dept. of Mathematics
University of Trento
38050 Povo (TN), Italy
e-mail: ballico@science.unitn.it