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A NOTE ON COMPACT SOLVMANIFOLDS WITH KÄHLER STRUCTURES

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Abstract

In this note we show that a compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus. We can show in particular that a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus, which is known as the Benson-Gordon's conjecture.

1. Introduction

We know that the existence of Kähler structure on a compact complex manifold imposes certain homological or even homotopical restrictions on its underlining topological manifold. Hodge theory is of central importance in this line. There have been recently certain extensions and progresses in this area of research. Among them is the field of Kähler groups, in which the main subject to study is the fundamental group of a compact Kähler manifold (see [1]). Once there was a conjecture that a non-abelian, finitely generated and torsion-free nilpotent group (which is the fundamental group of a nilmanifold) can not be a Kähler group, which is a generalized assertion of the result [9, 14] that a non-toral nilmanifold admits no Kähler structures. A counter-example to this conjecture was given by Campana [12]. Later a detailed study of solvable Kähler groups was done by Arapura and Nori [3]; they showed in particular that a solvable Kähler group must be almost nilpotent, that is, it has a nilpotent subgroup of finite index. On the other hand, the author stated in the paper [15] a general conjecture on compact Kählerian solvmanifolds: a compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which is also a complex torus bundle over a complex torus; and showed under some restriction that the conjecture is valid.

In this note we will see that the above conjecture can be proved without any restriction, based on the result (mentioned above) by Arapura and Nori [3], and applying the argument being used in the proof of the main theorem on the author's paper [15]. We also see that the Benson-Gordon's conjecture on Kähler structures on a class of compact solvmanifolds [10] (so-called solvmanifolds of completely solvable type) can

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be proved as a special case of our main result.

In this note we mean by a solvmanifold (nilmanifold) a compact homogeneous space of solvable Lie group, that is, a compact differentiable manifold on which a connected solvable (nilpotent) Lie group G acts transitively. We can assume, by taking the universal covering group $\widetilde{G}$ of G, that a solvmanifold M is of the form $\widetilde{G}/D$, where $\widetilde{G}$ is a simply connected solvable Lie group and D is a closed subgroup of $\widetilde{G}$ (which contains no non-trivial connected normal subgroup of $\widetilde{G}$). It should be noted that unless M is a nilmanifold, a closed subgroup D may not be a discrete subgroup (a lattice) of G. However, it is known (due to Auslander [4]) that a solvmanifold in general has a solvmanifold $\widetilde{G}/\Gamma$ with discrete isotropy subgroup Γ as a finite (normal) covering.

We recall some terminologies which we use in this note. A solvmanifold $M = G/\Gamma$, where Γ is a discrete subgroup of a simply connected solvable Lie group G, is *of completely solvable type*, if the adjoint representation of the Lie algebra $\mathfrak{g}$ of G has only real eigenvalues; and *of rigid type* (or *of type* (*R*)), in the sense of Auslander [5], if the adjoint representation of $\mathfrak{g}$ has only pure imaginary (including 0) eigenvalues. It is clear that M is both of completely solvable and of rigid type if and only if $\mathfrak{g}$ is nilpotent, that is, M is a nilmanifold. We can see in the proof of main theorem that a Kählerian solvmanifold is of rigid type, and not of completely solvable type unless it is a complex torus. This gives a proof of the Benson-Gordon's conjecture as a special case of the theorem.

We state now our main theorem in the most general form:

Main Theorem. *A compact solvmanifold admits a Kähler structure if and only if it is a finite quotient of a complex torus which has a structure of a complex torus bundle over a complex torus. In particular, a compact solvmanifold of completely solvable type has a Kähler structure if and only if it is a complex torus.*

2. Proof of main theorem

Let M be a compact solvmanifold of dimension $2m$ which admits a Kähler structure. We will first show that M is a finite quotient of a complex torus. We can assume, taking a finite covering if necessary, that M is of the form G/Γ, where Γ is a lattice of a simply connected solvable Lie group G. By the result of Arapura and Nori, we know that the fundamental group Γ of M is almost nilpotent, that is, Γ contains a nilpotent subgroup Δ of finite index. Then we can find a normal nilpotent subgroup Δ' of Γ, which is commensurable with Δ, and thus defines another lattice of G. Therefore $M' = G/\Delta'$ is a finite normal covering of M, which is a solvmanifold with the fundamental group Δ', a finitely generated torsion-free nilpotent group. According to the well-known theorem of Mostow [18], M' is diffeomorphic to a nilmanifold. Since M' has a canonical Kähler structure induced from M, it follows from the result on Kählerian nilmanifolds [9, 14] that M' must be a complex torus. Hence M is a finite quotient of a complex torus.

We will show that M has a structure of a complex torus bundle over a complex torus. We follow the argument in the proof of the main theorem on the paper [15] to which we refer for full detail. Let Γ be the fundamental group of M. Then we can express Γ as the extension of a finitely generated and torsion-free nilpotent group N of rank $2l$ by the free abelian group of rank $2k$, where we can assume $2k$ is the first Betti number b_1 of M (we have $2k \leq b_1$ in general) [7] and $m = k + l$:

$$0 \to N \to \Gamma \to \mathbf{Z}^{2k} \to 0$$

Since M is also a finite quotient of a torus T^{2m}, Γ contains a maximal normal free abelian subgroup Δ of rank $2m$ with finite index in Γ. We can see that N must be free abelian of rank $2l$. In fact, since $N \cap \Delta$ is a abelian subgroup of N with finite index, it follows that the real completion $\tilde{N}$ of N is abelian, and thus N is also abelian. Therefore we have the following

$$0 \to \mathbf{Z}^{2l} \to \Gamma \to \mathbf{Z}^{2k} \to 0,$$

where $\Delta = \mathbf{Z}^{2l} \times s_1\mathbf{Z} \times s_2\mathbf{Z} \times \cdots \times s_{2k}\mathbf{Z}$, and $H = \mathbf{Z}/s_1\mathbf{Z} \times \mathbf{Z}/s_2\mathbf{Z} \times \cdots \times \mathbf{Z}/s_{2k}\mathbf{Z}$ (some of $\mathbf{Z}/s_i\mathbf{Z}$ may be trivial) is the holonomy group of Γ. We have now that $M = \mathbf{C}^m/\Gamma$, where Γ is a Bieberbach group with holomomy group H. Since the action of H on $\mathbf{C}^l/\mathbf{Z}^{2l}$ is holomorphic, we see that M is a holomorphic fiber bundle over the complex torus $\mathbf{C}^k/\mathbf{Z}^{2k}$ with fiber the complex torus $\mathbf{C}^l/\mathbf{Z}^{2l}$.

Conversely, let M be a finite quotient of a complex torus which is also a complex torus bundle over a complex torus. The fundamental group Γ of M is a Bieberbach group which is expressed as the extension of a free abelian group $\mathbf{Z}^{2l}$ by another free abelian group $\mathbf{Z}^{2k}$. As observed before, since the action of $\mathbf{Z}^{2k}$ on $\mathbf{Z}^{2l}$ is actually the action of the finite abelian holonomy group H of Γ on $\mathbf{Z}^{2l}$, and the action of H on the fiber is holomorphic, we may assume that the action of $\mathbf{Z}^{2k}$ is in $\mathbf{U}(l)$ and thus extendable to the action of $\mathbf{R}^{2k}$ on $\mathbf{R}^{2l}$, defining a structure of solvmanifold on M of the form G/Γ, where Γ is a lattice of a simply connected solvable Lie group G. In fact, considering $\mathbf{R}^{2l}$ as $\mathbf{C}^l$, we have $G = \mathbf{C}^l \rtimes \mathbf{R}^{2k}$ with the action $\phi\colon \mathbf{R}^{2k} \to \mathrm{Aut}(\mathbf{C}^l)$ defined by

$$\phi(t_j)((z_1, z_2, \ldots, z_l)) = \left(e^{\sqrt{-1}\theta_1^j t} z_1, e^{\sqrt{-1}\theta_2^j t} z_2, \ldots, e^{\sqrt{-1}\theta_l^j t} z_l\right),$$

where $t_j = te_j$ (e_j: the j-th unit vector in $\mathbf{R}^{2k}$), and $e^{\sqrt{-1}\theta_i^j}$ is the primitive s_j-th root of unity, $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, 2k$.

As seen in the first part of the proof, we can always take $2k$ as the first Betti number of M, and makes the fibration the Albanese map into the Albanese torus. Let $\mathfrak{g}$ be the Lie algebra of G; then we can express $\mathfrak{g}$ as a vector space over $\mathbf{R}$ having a basis $\{X_1, X_2, \ldots, X_{2l}, X_{2l+1}, \ldots, X_{2l+2k}\}$ for which the bracket multiplications are defined by the following:

$$[X_{2l+i}, X_{2j-1}] = -X_{2j}, [X_{2l+i}, X_{2j}] = X_{2j-1}$$

for all i with $s_i \neq 1$, $i = 1, 2, \dots, 2k$, $j = 1, 2, \dots, l$, and all other brackets vanish. It is clear that $\mathfrak{g}$ is of rigid type, and is of completely solvable type if and only if $\mathfrak{g}$ is abelian, that is, M is a complex torus. This completes the proof of the theorem.

REMARK. 1. A flat solvmanifold is a solvmanifold with flat Riemannian metric (which may not be invariant by the Lie group action). It is known [6, 17] that an abstract group Γ is the fundamental group of a flat solvmanifold if and only if Γ is an extension of a free abelian group $\mathbf{Z}^l$ by another free abelian group $\mathbf{Z}^k$ where the action of $\mathbf{Z}^k$ on $\mathbf{Z}^l$ is finite, and the extension corresponds to a torsion element of $H^2(\mathbf{Z}^k, \mathbf{Z}^l)$. We can check these conditions for the class of Kählerian solvmanifolds we have determined in this paper.
2. A four-dimensional solvmanifold with Kähler structure is nothing but a hyperelliptic surface. It is not hard to classify hyperelliptic surfaces as solvmanifolds [16].
3. According to a result of Auslander and Szczarba [7], a solvmanifold $M = G/D$ has *the canonical torus fibration* over the torus G/ND of dimension b_1 (the first Betti number of M) with fiber a nilmanfold, where N is the nilradical of G (the maximal connected normal subgroup of G). Since a solvmanifold (in general) is an Eilenberg-Macline space, for a Kählerian solvmanifold M, we can see that the canonical torus fibration and the Albanese map are homotopic. However it is not apriori clear that they actually coincide (up to translation). Since that is the key point in the proof of the conjecture on Kählerian solvmanifolds by Arapura [2], the proof is incomplete. It should be also noted that by the theorem of Grauert-Fischer, a fibration (a proper surjective holomorphic map) from a compact Kähler manifold to a complex torus with fibers complex tori is locally trivial. This is well known for Kähler surfaces ([8], Chap. III), and also known to be valid in general [13].
4. There is a recent result of Brudnyi [11], from which we can directly see that a Kählerian solvmanifold must be a finite quotient of a complex torus.
5. An abstract group Γ is the fundamental group of a solvmanifold M if and only if Γ is a *Wang group* (which is an extension of a finitely generated torsion-free nilpotent group by a free abelian group of finite rank) [19]. We denote by M_Γ the solvmanifold with the fundamental group Γ (which is uniquely determined due to Mostow). We have then that Γ' is a subgroup of Γ with finite index if and only if $M_{\Gamma'}$ is a finite covering of M_Γ. In perticular Γ is almost nilpotent if and only if M_Γ is a finite quotient of a nilmanifold. Furthermore, due to Auslander [5], expressing M_Γ as G/D with $\Gamma = D/D_0$ (where D_0 denotes the identity component of D), Γ is almost nilpotent if and only if the Lie algebra $\mathfrak{g}$ of G is of rigid type. In perticular, from this result, together with the result on solvable Kähler groups [3] and that on Kählerian nilmanifolds [9, 14], we can derive another proof for the Benson-Gordon's conjecture.

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