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MICROLOCAL ANALYTIC SMOOTHING EFFECTS FOR OPERATORS OF REAL PRINCIPAL TYPE

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Abstract

We are interested in the microlocal smoothing effect for operators of real principal type. On the initial value problem for a dispersive evolution equation, we study the fact that the sufficient decay of the initial data gives the smoothness of the solution. We develop the theory of the FBI transform in order to transform our operator of real principal type into a simple operator of first order. Since the smoothing effect is of global nature, our transformation is realized globally along the bicharacteristics defined from the principal symbol of the operator.

1. Introduction and the main results

Dispersive evolution equations are known to have the smoothing properties. We call these properties the smoothing effects. There are many expressions for them. We study the smoothing effects from the point of view that the smoothness of a solution to the initial value problem for a dispersive evolution equation depends on the decay of the initial data. This problem was refined and generalized in many aspects. In [4] Craig-Kappeler-Strauss studied Schrödinger evolution equations on $\mathbb{R}^n$ with asymptotically flat coefficients of the short range perturbation. The microlocal regularity of a solution increases at a point $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ when the bicharacteristics of the principal symbol through ρ_0 is not trapped backwards in a compact set and the initial data decays along the base projection of this bicharacteristics. In [7] Doi studied the relations between the global behavior of a Hamilton flow and the smoothing effects in general situations (see also [8] and [9]). In [21] Robbiano-Zuily gave the results in the analytic category similar to those given by Craig-Kappeler-Strauss [4] in C^∞ one. Robbiano-Zuily's approach is based on the theory of the FBI transform. Our aim is to generalize these results to a wider class of dispersive evolution equations following the idea given by Robbiano and Zuily.

Let m be an integer greater than or equal to 2. Let $P(x, D_x)$ be a linear differential operator of order m in $\mathbb{R}^n$ given by

$$P(x, D_x) = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha, \tag{1.1}$$

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where $i = \sqrt{-1}$, $D_{x_j} = -i\partial/\partial x_j$ and the coefficients $a_\alpha(x)$ $(|\alpha| \le m)$ are analytic in $\mathbb{R}^n$. Let $\widetilde{P} = \widetilde{P}(x, D_x) = \widetilde{P}(D_x)$ be a differential operator with real constant coefficients given by

$$\widetilde{P}(D_x) = \sum_{|\alpha|=m} \widetilde{a}_\alpha D_x^\alpha, \quad \widetilde{a}_\alpha \in \mathbb{R}. \tag{1.2}$$

In order to simplify the notation, we define $\widetilde{a}_\alpha = 0$ when $|\alpha| \le m-1$.

We assume that $P(x, D_x)$ is a perturbation of $\widetilde{P}(D_x)$ in the following sense. We can find constants $C_0 \ge 1$, $K_0 \ge 1$, $R_0 \ge 1$ and $\sigma_0 \in (0,1)$ such that for all $x \in \mathbb{R}^n$ with $|x| > R_0$ and $\beta \in (\mathbb{N} \cup \{0\})^n$ we have

$$\sum_{|\alpha|\le m} \left| D_x^\beta (a_\alpha(x) - \widetilde{a}_\alpha) \right| \le \frac{C_0 K_0^\beta \beta!}{|x|^{1+\sigma_0+|\beta|}}. \tag{1.3}$$

Let $p_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha$ be the principal symbol for $P(x, D_x)$ and $\widetilde{p}_m(x,\xi) = \widetilde{p}_m(\xi) = \sum_{|\alpha|=m} \widetilde{a}_\alpha \xi^\alpha$ be the one for $\widetilde{P}(D_x)$. The second main assumption is that $P(x, D_x)$ and $\widetilde{P}(D_x)$ are operators of real principal type in the strong sense, that is,

$$p_m(x,\xi) \quad \text{and} \quad \widetilde{p}_m(\xi) \quad \text{are real valued,} \tag{1.4}$$

$$\nabla_\xi p_m(x,\xi) \ne 0, \quad \text{when} \quad p_m(x,\xi) = 0, \ (x,\xi) \in T^*\mathbb{R}^n \setminus 0, \tag{1.5}$$

$$\nabla_\xi \widetilde{p}_m(\xi) \ne 0, \quad \text{when} \quad \widetilde{p}_m(\xi) = 0, \ \xi \in \mathbb{R}^n \setminus 0. \tag{1.6}$$

Let $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$, and let $(Y(s; y_0, \eta_0), \Theta(s; y_0, \eta_0))$ be the solution to the equation

$$\begin{cases} \dfrac{d}{ds} Y(s) = \dfrac{\partial p_m}{\partial \xi}(Y(s), \Theta(s)), & Y(0) = y_0, \\ \dfrac{d}{ds} \Theta(s) = -\dfrac{\partial p_m}{\partial x}(Y(s), \Theta(s)), & \Theta(0) = \eta_0. \end{cases} \tag{1.7}$$

This is the bicharacteristic of $p_m(x,\xi)$ passing through ρ_0. The third assumption is so-called "non-trapping forward" in the following:

$$\text{The solution} \quad (Y(s; y_0, \eta_0), \Theta(s; y_0, \eta_0)) \quad \text{exists for} \quad \forall s \in \mathbb{R}, \tag{1.8}$$

and

$$\lim_{s\to\infty} |Y(s; y_0, \eta_0)| = \infty. \tag{1.9}$$

If $a_\alpha(x)$ $(|\alpha| = m)$ are constant, then (1.8) and (1.9) are satisfied. Let $\big(\widetilde{Y}(s; y_0, \eta_0),$

$\widetilde{\Theta}(s; y_0, \eta_0)\big)$ be the solution to the equation

$$\begin{cases} \dfrac{d}{ds}\widetilde{Y}(s) = \dfrac{\partial \widetilde{p}_m}{\partial \xi}\left(\widetilde{\Theta}(s)\right), & \widetilde{Y}(0) = y_0, \\ \dfrac{d}{ds}\widetilde{\Theta}(s) = -\dfrac{\partial \widetilde{p}_m}{\partial x}\left(\widetilde{\Theta}(s)\right) = 0, & \widetilde{\Theta}(0) = \eta_0. \end{cases} \tag{1.10}$$

We can easily compute

$$\widetilde{Y}(s) = y_0 + s\frac{\partial \widetilde{p}_m}{\partial \xi}(\eta_0), \quad \widetilde{\Theta}(s) = \eta_0. \tag{1.11}$$

Let us define a class of the initial data. Let $\gamma^+_{\rho_0}$ be the bicharacteristics in the positive direction:

$$\gamma^+_{\rho_0} = \{(Y(s; y_0, \eta_0), \Theta(s; y_0, \eta_0)) \in T^*\mathbb{R}^n\,;\, 0 \le s < \infty\}. \tag{1.12}$$

For $\varepsilon_0 > 0$ we set

$$X^+_{\rho_0} = \left\{ v \in L^2(\mathbb{R}^n)\,;\, \exists \varepsilon_0 > 0,\ \exists \delta_0 > 0,\ e^{\delta_0 |x|^{1/(m-1)}} v(x) \in L^2\left(\Gamma^+_{\varepsilon_0, \rho_0}\right) \right\}, \tag{1.13}$$

where

$$\Gamma^+_{\varepsilon_0, \rho_0} = \bigcup_{s \ge 0} \{x \in \mathbb{R}^n\,;\, |x - Y(s; y_0, \eta_0)| < \varepsilon_0(1 + |s|)\}. \tag{1.14}$$

Let $u_0 \in L^2(\mathbb{R}^n)$ and $u(t, \cdot) \in C((-\infty, 0]; L^2(\mathbb{R}^n))$ be a solution to the initial value problem

$$\begin{cases} D_t u + P(x, D_x) u = 0, & t < 0, \ x \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x), \end{cases} \tag{1.15}$$

where $D_t = -i\partial/\partial t$. The next theorem is our main result in this paper.

Theorem 1.1. *Let* $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$. *Let* $P(x, D_x)$ *and* $\widetilde{P}(D_x)$ *be differential operators defined in* (1.1) *and* (1.2), *respectively. Assume from* (1.3) *to* (1.6), (1.8) *and* (1.9). *Moreover if* $m \ge 3$, *we assume that there exists a positive constant* M *such that the coefficients of the principal part of* $P(x, D_x)$ *are constant for* $|x| > M$ *and there exists a constant* $s_0 > 0$ *such that for* $s > s_0$ *the matrix*

$$\left(\frac{\partial^2 \widetilde{p}_m}{\partial \xi_j \partial \xi_k} (\Theta(s; y_0, \eta_0)) \right)_{1 \le j,k \le n}, \tag{1.16}$$

is invertible. If $u(t, \cdot)$ *is a solution of* (1.15) *and* $t < 0$, *then* $u_0 \in X^+_{\rho_0}$ *implies* $\rho_0 \notin WF_A[u(t, \cdot)]$.

Theorem 1.1 is the direct extension of the results given by Robbiano and Zuily in [21], who considered Schrödinger operators with variable coefficients under the similar assumptions as ours. They proposed the method based on the FBI transform (Fourier-Bros-Iagolnitzer transform), which had been introduced systematically in Sjöstrand [26]. The microlocal regularity of a distribution can be tested by its FBI transform. Another aspect of the FBI transform is the Fourier Integral Operator with a complex phase function. Thanks to Egorov principle we can transform the first order operator $P(x, D_x) = P_1(x, D_x)$ of real principal type into D_{z_n} by using a Fourier Integral Operator. Robbiano and Zuily proposed the analogy of Egorov principle via the FBI transform for the Laplacian $P(x, D_x) = P_2(x, D) = |D_x|^2$. This analogy in a microlocal sense had already been known in [26], however, Robbiano and Zuily proved it globally.

In the preceding paper [23], the author attempted to extend their method to particular operators of higher order namely when the spatial dimension was equal to one and their principal symbol had constant coefficients. Even in this simple case of higher order there were some difficulties that came from estimating the derivatives of the phase function globally in the FBI transform. In fact, the author showed the key estimate which was valid only in a restricted region in comparison with that of [21]. In the present paper we follow the argument [23]. We generalize "the out-going point" for operators of real principal type which was introduced for the Laplacian in [21]. The out-going point plays an important role in estimating the bicharacteristics defined from the principal symbol of operators of real principal type. We also study the resolvent $\mathrm{G}^{-1} = \left(\mathrm{Id} - i z_n \nabla_\xi^2 \widetilde{p}_m(\xi)\right)^{-1}$ for the Hessian matrix $\nabla_\xi^2 \widetilde{p}_m(\xi)$ when $\operatorname{Re} z_n > -\varepsilon_1$, $|\operatorname{Im} z_n| < \varepsilon_1$. The careful observations for the Hessian matrix $\nabla_\xi^2 \widetilde{p}_m(\xi)$ and its resolvent G^{-1}, which did not appear explicitly in [21] and [23], enable us to obtain the estimates of the derivatives of the phase function (see Lemma 4.2 and 4.3). We note that, in the second order cases, the condition of real principal type implies that the Hessian matrix $\mathrm{A} = \nabla_\xi^2 \widetilde{p}_2(\xi)$ is invertible.

In this paper we consider the class of the initial data which decreases outside. In [19] Morimoto, Robbiano and Zuily considered the class of the initial data based on some Gevrey class. In [22] another class based on the oscillating data was studied. Other approaches by integral transformations can be seen in [16] and [20], where different classes of operators from ours are discussed. We also note the two results concerning our assumptions from (1.4) to (1.6). Chihara [3] proved the estimate expressing the smoothing effects for the operators of real principal type. In [15] Hoshiro showed the counter parts of Chihara's results. These two results show sufficiency and necessity between the estimate expressing the smoothing effects and the real principal type conditions.

The plan of this paper is as follows: In Section 2 we define the FBI transform and the analytic wave front sets and recall the relation between them. We construct the out-going point. The results of this section show that our theorem follows from the special case that ρ_0 is an out-going point. Our main idea is to transform the original operator $P(x, D_x)$ into D_{z_n}. This idea is known as Egorov principle in the theory of the Fourier Integral Operators. However the order m of our operator $P(x, D_x)$ is different from D_{z_n}. We realize this transformation by using the FBI transform which has a large parameter λ. The parameter λ is used in order to balance the difference of orders of operators. We construct the phase function in Section 4 and the amplitude functions in Section 5. Since the phenomena of the smoothing effects are of global nature, we have to do this transformation globally. So we study the global behaviors of the Hamilton flows in Section 3 and the global properties of the phase functions in Section 4. In Section 6 we show the main theorem (Theorem 1.1) by using the results obtained in previous sections. We note that the additional assumptions for $m \geq 3$ in Theorem 1.1 are used only when we estimate the second derivatives of the phase function. Except for Lemma 4.3 and Section 6, we can consider the general cases without assuming that the coefficients $a_\alpha(x)$ of the principal part of the operator $P(x, D_x)$ are constant when $m \geq 3$.

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2. The locally uniform analytic wave front set and the FBI transform

There are many characterizations of the analytic wave front set. In [1] Bony showed that they coincide. In the present paper we use the characterization by the FBI transform. Let us recall the definition of the locally uniform analytic wave front set introduced in [21].

Let $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$, $z_0 \in \mathbb{C}^n$, and let $\varphi = \varphi(z, x)$ be a holomorphic function in a neighborhood W of (z_0, y_0) in $\mathbb{C}^n \times \mathbb{C}^n$ satisfying the properties from (2.1) to (2.3)

$$\frac{\partial \varphi}{\partial x}(z_0, y_0) = -\eta_0, \tag{2.1}$$

$$\operatorname{Im} \frac{\partial^2 \varphi}{\partial x^2}(z_0, y_0) \quad \text{is a positive definite matrix,} \tag{2.2}$$

$$\det\left(\frac{\partial^2 \varphi}{\partial z \partial x}(z_0, y_0)\right) \neq 0. \tag{2.3}$$

From (2.1) we have $\operatorname{Im}(\partial\varphi/\partial x)(z_0, y_0) = 0$. So we can set the function defined in a neighborhood U_{z_0} of z_0 in $\mathbb{C}^n$ by

$$\Phi(z) = \sup_{x \in V_{y_0}} [-\operatorname{Im} \varphi(z, x)], \tag{2.4}$$

where V_{y_0} is a neighborhood of y_0 in $\mathbb{R}^n$.

Let $a = a(z, x, \lambda) = \sum_{k\geq 0} a_k(z, x)\lambda^{-k}$ be an analytic symbol of order 0, elliptic in a neighborhood of (z_0, y_0). Namely, $a_k(z, x)$ are holomorphic in a neighborhood of (z_0, y_0) in $\mathbb{C}^n \times \mathbb{C}^n$, and there exists a constant $C > 0$ such that for all $k \in (\mathbb{N} \cup \{0\})$ we have

$$|a_k(z, x)| \leq C^{k+1} k!, \tag{2.5}$$

and $a_0(z_0, y_0) \neq 0$.

Let $\chi = \chi(x) \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function with support in a neighborhood of y_0 satisfying $0 \leq \chi \leq 1$, and $\chi = 1$ near y_0.

Assume that $u(t, \cdot)$ is a family of distributions depending on a real parameter (time) t. The FBI transform of $u(t, \cdot)$ (with respect to the space variables) is defined through

$$\begin{aligned} Tu(t, z, \lambda) &= \langle u(t, \cdot), \chi(\cdot) e^{i\lambda\varphi(z,\cdot)} a(z, \cdot, \lambda)\rangle_{\mathcal{D}'(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} e^{i\lambda\varphi(z,x)} a(z, x, \lambda)\chi(x) u(t, x)\, dx, \end{aligned} \tag{2.6}$$

for $\lambda \geq 1$.

We define the analytic wave front set which has the local uniformness with respect to t by using the FBI transform (see Definition 1.1 in [21]).

DEFINITION 2.1. Let $t_0 \in \mathbb{R}$, $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$. We say that $\rho_0 \notin \widetilde{WF_A}[u(t_0, \cdot)]$ if and only if there exist a phase function φ satisfying the properties from (2.1) to (2.3), an analytic symbol $a = a(z, x, \lambda)$ of order 0, elliptic in a neighborhood of (z_0, y_0), a cutoff function χ, a neighborhood U_{z_0}, and positive constants $C, \mu_0, \lambda_0, \varepsilon_0$ such that

$$|Tu(t, z, \lambda)| \leq C e^{\lambda\Phi(z) - \mu_0\lambda}, \tag{2.7}$$

for $\forall z \in U_{z_0}$, $\forall \lambda \geq \lambda_0$, and $\forall t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.

This definition is independent of the choice of φ, a, χ which satisfy the conditions above.

When we consider the initial value problems for dispersive operators, the usual analytic wave front set does not nicely propagate, so we introduce the locally uniform analytic wave front set $\widetilde{WF_A}[u(t_0, \cdot)]$. Thanks to the uniformness, we know the propagation theorem for the locally uniform analytic wave front set. Let $u \in C(\mathbb{R}; L^2(\mathbb{R}^n))$ be the solution of the equation $D_t u + P(x, D_x)u = 0$. The next lemma means the propagation of the locally uniform analytic wave front set.

Lemma 2.1. *Let $\rho = (y, \eta) \in T^*\mathbb{R}^n \setminus 0$, and let $\gamma_\rho = \{(Y(s; y, \eta), \Theta(s; y, \eta)) \in T^*\mathbb{R}^n, s \in \mathbb{R}\}$ be the bicharacteristics of $p_m(x, \xi)$ passing through ρ. Let $t_0 \in \mathbb{R}$. If*

$\rho \notin \widetilde{WF_A}[u(t_0, \cdot)]$, *then* $\widetilde{WF_A}[u(t_0, \cdot)] \cap \gamma_\rho = \phi$.

This lemma is essentially of local nature. We can arrange the proof given by Robbiano and Zuily in [21] (see Theorem 6.1 and its proof in [21]).

On account of Lemma 2.1, it suffices to prove Theorem 1.1 with $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ replaced by the special point of γ_{ρ_0}, called "out-going point" in [21]. The next lemma shows how to construct the out-going point:

Lemma 2.2. *Let* $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ *and let*

$$\gamma^+_{\rho_0} = \{(Y(s; y_0, \eta_0), \Theta(s; y_0, \eta_0)) \in T^*\mathbb{R}^n \,;\, 0 \le s < \infty\} \tag{2.8}$$

be the forward bicharacteristics of $p_m(x, \xi)$ *passing through* $\rho_0 = (y_0, \eta_0)$. *Assume*

$$\lim_{s\to\infty} |Y(s; y_0, \eta_0)| = \infty.$$

Then for any $R > 0$ *large enough, there exists an "out-going point"* $\rho_1 = (y_1, \eta_1) \in \gamma^+_{\rho_0}$ *such that*

$$\begin{aligned} &|y_1| > 2R, \\ &\left\langle y_1, \nabla_\xi \widetilde{p}_m(\eta_1)\right\rangle = \sum_{l=1}^n y_{1l} \frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta_1) \ge 0. \end{aligned} \tag{2.9}$$

When $P(x, D_x) = \widetilde{P}(D_x) = \sum_{j=1}^n |D_j|^2$ (flat Laplacian), the point $\rho_1 = (y_1, \eta_1)$ satisfies $y_1 \cdot \eta_1 \ge 0$. Before giving the proof of Lemma 2.2, we remark on the next fundamental property for operators of principal type. It follows from (1.5) that

$$\nabla_\xi p_m(x, \xi) \ne 0, \quad \text{for} \quad (x, \xi) \in T^*\mathbb{R}^n \setminus 0. \tag{2.10}$$

Indeed, since $p_m(x, \xi)$ is homogeneous of degree m with respect to ξ, we have the Euler identity $\sum_{j=1}^n \xi_j (\partial p_m / \partial \xi_j)(x, \xi) = m p_m(x, \xi)$, which shows that $p_m(x, \xi) = 0$ if $\nabla_\xi p_m(x, \xi) = 0$. In the same way we have $\nabla_\xi \widetilde{p}_m(\xi) \ne 0$ for all $\xi \ne 0$ by (1.6). Hence there exists a positive constant $\delta > 0$ such that

$$|\nabla_\xi \widetilde{p}_m(\xi)|^2 \ge \delta |\xi|^{2(m-1)}, \quad \text{for} \quad \forall \xi \in \mathbb{R}^n. \tag{2.11}$$

Proof of Lemma 2.2. Let $R > 2R_0$ (R_0 is introduced in (1.3)). Since $\lim_{s\to\infty} |Y(s; y_0, \eta_0)| = \infty$, we can find $s_1 > 0$ such that $|Y(s; y_0, \eta_0)| > 2R$, if $s_1 < s < \infty$. Moreover we have

$$\Theta(s; y_0, \eta_0) \ne 0 \quad \text{for} \quad s_1 < s < \infty. \tag{2.12}$$

We check this fact. Assume $\Theta(s_2; y_0, \eta_0) = 0$ with $s_1 < s_2 < \infty$. Since $|Y(s)| =$

$|Y(s;y_0,\eta_0)| > 2R > 4R_0$, the coefficients $|\nabla_x a_\alpha(Y(s))|$ are uniformly bounded functions for $s_1 < s < \infty$, the equation

$$\frac{d}{ds}\Theta(s) = -\nabla_x p_m(Y(s),\Theta(s)), \quad \Theta(s_1) = 0,$$

has a unique solution $\Theta(s) = \Theta(s;y_0,\eta_0) = 0$ for $s_1 < s < \infty$. This shows $(d/ds)Y(s;y_0,\eta_0) = 0$, which contradicts $\lim_{s\to\infty}|Y(s;y_0,\eta_0)| = \infty$.

For $Y(s) = Y(s;y_0,\eta_0)$, $\Theta(s) = \Theta(s;y_0,\eta_0)$, we have

(2.13)

$$\begin{aligned}
&\frac{d}{ds}\left\{\sum_{l=1}^{n} Y_l(s)\frac{\partial \widetilde{p}_m}{\partial \xi_l}(\Theta(s))\right\}\\
&= \sum_{l=1}^{n}\left\{\frac{dY_l}{ds}(s)\frac{\partial \widetilde{p}_m}{\partial \xi_l}(Y(s)) + Y_l(s)\sum_{k=1}^{n}\frac{\partial^2 \widetilde{p}_m}{\partial \xi_k\partial \xi_l}(\Theta(s))\frac{d\Theta_k}{ds}\right\}\\
&= \sum_{l=1}^{n}\frac{\partial p_m}{\partial \xi_l}(Y(s),\Theta(s))\frac{\partial \widetilde{p}_m}{\partial \xi_l}(\Theta(s)) + Y_l(s)\sum_{k=1}^{n}\frac{\partial^2 \widetilde{p}_m}{\partial \xi_k\partial \xi_l}(\Theta(s))\left(-\frac{\partial p_m}{\partial y_k}(Y(s),\Theta(s))\right)\\
&= I_1 + I_2.
\end{aligned}$$

Since I_1, I_2 are homogeneous of degree $2(m-1)$ with respect to $\Theta(s)$, we may first assume $|\Theta(s)| = 1$. We have

$$\begin{aligned}
I_1 &= \sum_{l=1}^{n}\left|\frac{\partial \widetilde{p}_m}{\partial \xi_l}(\Theta(s))\right|^2 + \sum_{l=1}^{n}\left\{\frac{\partial p_m}{\partial \xi_l}(Y(s),\Theta(s)) - \frac{\partial \widetilde{p}_m}{\partial \xi_l}(\Theta(s))\right\}\frac{\partial \widetilde{p}_m}{\partial \xi_l}(\Theta(s))\\
&= \left|\nabla_\xi \widetilde{p}_m(\Theta(s))\right|^2 + \sum_{l=1}^{n}\sum_{|\alpha|=m}\{a_\alpha(Y(s)) - \widetilde{a}_\alpha\}\frac{\partial}{\partial \xi_l}\xi^\alpha\Big|_{\xi=\Theta(s)}\sum_{|\alpha|=m}\widetilde{a}_\alpha\frac{\partial}{\partial \xi_l}\xi^\alpha\Big|_{\xi=\Theta(s)}.
\end{aligned}$$

By (1.3) and (2.11) we have

$$I_1 \ge \delta - C_{m,n}\frac{C_0}{|Y(s)|^{1+\sigma_0}} \ge \frac{\delta}{2},$$

if R is large enough. We also obtain

$$\begin{aligned}
|I_2| &\le \sum_{l=1}^{k=1}\left|\frac{\partial^2 \widetilde{p}_m}{\partial \xi_k\partial \xi_l}(\Theta(s))\right||Y_l(s)|\left|\sum_{|\alpha|\le m}\frac{\partial a_\alpha}{\partial x_k}(Y(s))\,\Theta(s)^\alpha\right|\\
&\le \sum_{l=1}^{k=1}\left|\frac{\partial^2 \widetilde{p}_m}{\partial \xi_k\partial \xi_l}(\Theta(s))\right|\sum_{|\alpha|=m}|Y_l(s)|\frac{C_0K_0}{|Y(s)|^{2+\sigma_0}}|\Theta(s)|^{|\alpha|}\\
&\le C_m\frac{C_0K_0}{|Y(s)|^{1+\sigma_0}} \le \frac{\delta}{4},
\end{aligned}$$

if R is large enough. Combining the estimates above, we have

$$I_1 + I_2 \geq \frac{\delta}{4}, \quad \text{when} \quad |\Theta(s)| = 1.$$

On the other hand,

$$\begin{aligned}\frac{d}{ds}\,|\Theta(s)|^{2(m-1)} &= \frac{d}{ds}\left\{\sum_{j=1}^{n}\Theta_j(s)^2\right\}^{m-1}\\ &= (m-1)\left\{\sum_{j=1}^{n}\Theta_j(s)^2\right\}^{m-2}\sum_{j=1}^{n}2\Theta_j(s)\frac{d}{ds}\Theta_j(s)\\ &= 2(m-1)|\Theta(s)|^{2(m-2)}\sum_{j=1}^{n}\Theta_j(s)\frac{\partial p_m}{\partial x_j}(Y(s),\Theta(s)).\end{aligned}$$

Since

$$\begin{aligned}&\frac{d}{ds}\left\{\frac{\left\langle Y(s), \nabla_\xi \widetilde{p}_m\,(\Theta(s))\right\rangle}{|\Theta(s)|^{2(m-1)}}\right\}\\ &= \frac{1}{|\Theta(s)|^{2(m-1)}}\frac{d}{ds}\left\langle Y(s), \nabla_\xi \widetilde{p}_m\,(\Theta(s))\right\rangle\\ &\quad - \frac{1}{|\Theta(s)|^{4(m-1)}}\left\langle Y(s), \nabla_\xi \widetilde{p}_m\,(\Theta(s))\right\rangle\frac{d}{ds}\,|\Theta(s)|^{2(m-1)},\end{aligned}$$

is homogeneous of degree 0 with respect to $\Theta(s)$, and we have, as I_2 is estimated, on $|\Theta(s)| = 1$

$$\begin{aligned}&\left|\left\langle Y(s), \nabla_\xi \widetilde{p}_m\,(\Theta(s))\right\rangle\frac{d}{ds}\,|\Theta(s)|^{2(m-1)}\right|\\ &\leq |Y(s)|\left|\nabla_\xi \widetilde{p}_m(\Theta(s))\right| C\,|\Theta(s)|^{2(m-2)}|\Theta(s)|\frac{C_0K_0}{|Y(s)|^{2+\sigma_0}}|\Theta(s)|^m\\ &\leq \frac{C}{|Y(s)|^{1+\sigma_0}},\end{aligned}$$

we obtain

$$\frac{d}{ds}\left\{\frac{\left\langle Y(s), \nabla_\xi \widetilde{p}_m\,(\Theta(s))\right\rangle}{|\Theta(s)|^{2(m-1)}}\right\} \geq \frac{\delta}{8}, \quad \text{for} \quad s_1 < s < \infty. \tag{2.14}$$

Therefore there exists a positive constant s_3 with $s_3 > s_1$ such that for $s_3 < s < \infty$

$$\frac{\left\langle Y(s), \nabla_\xi \widetilde{p}_m\,(\Theta(s))\right\rangle}{|\Theta(s)|^{2(m-1)}} \geq \frac{\delta}{8}s - \frac{\left\langle Y(s_1), \nabla_\xi \widetilde{p}_m\,(\Theta(s_1))\right\rangle}{|\Theta(s_1)|^{2(m-1)}} \geq \frac{\delta}{16}s.$$

We can take

$$\rho_1 = (Y(s_3), \Theta(s_3)) = (Y(s_3; y_0, \eta_0), \Theta(s_3; y_0, \eta_0)). \tag{2.15}$$

The proof of Lemma 2.2 is completed. □

In the case that $P(x, D_x)$ is an elliptic operator, we have

$$C_1|\Theta(s)|^m \leq |p_m(Y(s), \Theta(s))| = |p_m(Y(0), \Theta(0))| \leq C_2|\Theta(s)|^m.$$

It follows from (1.7) that $(Y(s; y_0, \eta_0), \Theta(s; y_0, \eta_0))$ exists globally in $s \in \mathbb{R}$. In Section 3 we will prove the estimates for bicharacteristics from the outgoing point for operators of real principal type precisely. Writing

$$p_2(x, \xi) = \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k = \langle A\xi, \xi\rangle,$$

where $\mathrm{A} = \left(a_{jk}(x)\right)_{1\leq j,k\leq n}$, we have $\nabla_\xi p_2(x, \xi) = 2\mathrm{A}\xi$. If $\mathrm{A}^{-1} = 2\left(\nabla_\xi^2 p_2\right)^{-1}$ does not exist, there exists $\xi_0 \neq 0$ such that $\mathrm{A}\xi_0 = 0$, which contradicts the fact $\nabla_\xi p_2(x, \xi) \neq 0$ for all $\xi \neq 0$. In the second order cases, the condition of real principal type implies that the Hessian matrix $2\mathrm{A} = \nabla_\xi^2 p_2$ is invertible.

3. Global estimates for bicharacteristics

In order to construct the phase function and the amplitude function globally, we study the global behavior of the bicharacteristics. If we consider the operators whose principal parts are independent of x, we know well about them from (1.11). In the variable coefficients cases we can also get the precise estimates for the bicharacteristics starting from the generalized outgoing point $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ with $|y_0| > 2R$, $\langle y_0, \nabla_\xi \widetilde{p}_m(\eta_0)\rangle \geq 0$. By the condition (1.3) the coefficients $a_\alpha(x)$ of $P(x, D_x)$ can be extended as holomorphic functions in the set,

$$\Omega_1 = \left\{x \in \mathbb{C}^n\,;\, |\operatorname{Re} x| > R_0,\ \ K_0|\operatorname{Im} x| < \frac{1}{3}|\operatorname{Re} x|\right\}, \tag{3.1}$$

and they satisfy

$$\sum_{|\alpha|\leq m} \left|D_x^\beta\left(a_\alpha(x) - \widetilde{a}_\alpha\right)\right| \leq \frac{3C_0(2K_0)^{|\beta|}\beta!}{|\operatorname{Re} x|^{1+\sigma_0+|\beta|}}, \quad \text{in} \quad \Omega_1. \tag{3.2}$$

We take the constants $C_1 = 3C_0$, $K_1 = 2K_0$ in (3.2) and we write C_0, K_0 instead of C_1, K_1 respectively. As in [21] we study the bicharacteristics of $p_m(x, \xi)$ by comparing with those of $\widetilde{p}_m(\xi)$.

Lemma 3.1. *We assume the hypothesis given in* Theorem 1.1. *Then we can find* $\varepsilon^* > 0$, $R^* \geq 1$ *such that for all* $0 < \varepsilon \leq \varepsilon^*$, $R \geq R^*$, $(y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ *with* $|y_0| > 2R$, $|\eta_0| = 1$, $\langle y_0, \nabla_\xi \widetilde{p}_m(\eta_0)\rangle \geq 0$ *and* $(y, \eta) \in \mathbb{C}^n \times \mathbb{C}^n$ *with* $|y - y_0| < \varepsilon$, $|\eta - \eta_0| < \varepsilon$, *the system*

$$\begin{cases} \dfrac{d}{ds} Y(s) = \dfrac{\partial p_m}{\partial \xi}(Y(s), \Theta(s)), & Y(0) = y, \\ \dfrac{d}{ds}\Theta(s) = -\dfrac{\partial p_m}{\partial x}(Y(s), \Theta(s)), & \Theta(0) = \eta, \end{cases} \tag{3.3}$$

has a unique solution which is a global one with respect to $s \in \mathbb{R}$ *and holomorphic in* (y, η). *Moreover if we define* $(Z(s), \zeta(s), H(s))$ *as*

$$\begin{cases} Z(s) = Y(s) - \widetilde{Y}(s), \\ \zeta(s) = \Theta(s) - \widetilde{\Theta}(s), \\ H(s) = Z(s) - \displaystyle\int_0^s \left\{\nabla_\xi \widetilde{p}_m(\zeta(\tau) + \eta) - \nabla_\xi \widetilde{p}_m(\eta)\right\} d\tau, \end{cases} \tag{3.4}$$

then we have

$$|Z(s)| \leq A_1 \frac{C_0 K_0}{R^{1+\sigma_0}} s |\eta|^{m-1}, \tag{3.5}$$

$$|\zeta(s)| \leq B_1 \frac{C_0 K_0}{R^{1+\sigma_0}} |\eta|, \tag{3.6}$$

$$|H(s)| \leq D_1 \frac{C_0 K_0}{R^{\sigma_0}}, \tag{3.7}$$

where A_1, B_1 *and* D_1 *are the constants depending only on* m, n *and* σ_0.

The assumption $|\eta_0| = 1$ is not essential. For simplicity we assume this property in Lemma 3.1.

Proof of Lemma 3.1. We take R so large and $\varepsilon > 0$ so small that $|y| > 2R$ and $(1/2)|\eta_0| \leq |\eta| \leq (3/2)|\eta_0|$. From (1.7), (1.10) and (3.4)

$$\begin{aligned} \frac{d}{ds} Z_l(s) &= \frac{d}{ds} Y_l(s) - \frac{d}{ds}\widetilde{Y}_l(s) \\ &= \frac{\partial p_m}{\partial \xi_l}(Y(s), \Theta(s)) - \frac{\partial \widetilde{p}_m}{\partial \xi_l}\left(\widetilde{\Theta}(s)\right) \\ &= \sum_{|\alpha|=m} a_\alpha(Y(s)) \frac{\partial}{\partial \xi_l}\xi^\alpha\bigg|_{\xi=\Theta(s)} - \sum_{|\alpha|=m} \widetilde{a}_\alpha \frac{\partial}{\partial \xi_l}\xi^\alpha\bigg|_{\xi=\eta} \\ &= \sum_{|\alpha|=m} b_\alpha(Y(s)) \frac{\partial}{\partial \xi_l}\xi^\alpha\bigg|_{\xi=\Theta(s)} + \sum_{|\alpha|=m} \widetilde{a}_\alpha \left(\frac{\partial}{\partial \xi_l}\xi^\alpha\bigg|_{\xi=\Theta(s)} - \frac{\partial}{\partial \xi_l}\xi^\alpha\bigg|_{\xi=\eta}\right), \end{aligned}$$

where $b_\alpha(x) = a_\alpha(x) - \widetilde{a}_\alpha$ $(|\alpha| = m)$. In the same way

$$\begin{aligned}\frac{d}{ds}\zeta_l(s) &= \frac{d}{ds}\Theta_l(s) - \frac{d}{ds}\widetilde{\Theta}_l(s)\\ &= -\frac{\partial p_m}{\partial x_l}(Y(s), \Theta(s))\\ &= -\sum_{|\alpha|=m}\frac{\partial b_\alpha}{\partial x_l}(Y(s))\,\xi^\alpha\Big|_{\xi=\Theta(s)}.\end{aligned}$$

Then $(Z(s), \zeta(s))$ satisfies the differential equations,

(3.8)
$$\begin{cases}\dfrac{d}{ds}Z_l(s) = \displaystyle\sum_{|\alpha|=m} b_\alpha(Y(s))\frac{\partial}{\partial\xi_l}\xi^\alpha\Big|_{\xi=\zeta(s)+\eta} + \sum_{|\alpha|=m}\widetilde{a}_\alpha\left(\frac{\partial}{\partial\xi_l}\xi^\alpha\Big|_{\xi=\zeta(s)+\eta} - \frac{\partial}{\partial\xi_l}\xi^\alpha\Big|_{\xi=\eta}\right),\\ \dfrac{d}{ds}\zeta_l(s) = -\displaystyle\sum_{|\alpha|=m}\frac{\partial b_\alpha}{\partial x_l}(Y(s))(\zeta(s)+\eta)^\alpha,\\ Z_l(0) = 0, \quad \zeta_l(0) = 0, \qquad 1 \le l \le n,\end{cases}$$

where $Y(s) = Z(s) + (y + s\nabla_\xi\widetilde{p}_m(\eta))$.

(i) STEP 1. (3.8) is well defined for small s: If $\varepsilon > 0, s > 0, |Z(s)|$ are small enough, then the coefficients $b_\alpha(Y(s))$, $(\partial b_\alpha/\partial x_l)(Y(s))$ are well defined. For $0 < s \le 1$

$$\begin{aligned}\operatorname{Re} Y(s) &= \operatorname{Re} y + s\operatorname{Re}\nabla_\xi\widetilde{p}_m(\eta) + \operatorname{Re} Z(s)\\ &= y_0 + (\operatorname{Re} y - y_0) + s\nabla_\xi\widetilde{p}_m(\eta_0) + s\left(\operatorname{Re}\nabla_\xi\widetilde{p}_m(\eta) - \nabla_\xi\widetilde{p}_m(\eta_0)\right) + \operatorname{Re} Z(s).\end{aligned}$$

We have

$$|\operatorname{Re} Y(s)| \ge \left|y_0 + s\nabla_\xi\widetilde{p}_m(\eta_0)\right| - |\operatorname{Re} y - y_0| - s\left|\operatorname{Re}\nabla_\xi\widetilde{p}_m(\eta) - \nabla_\xi\widetilde{p}_m(\eta_0)\right| - |Z(s)|.$$

Since ρ_0 is a outgoing point, that is, $\langle y_0, \nabla_\xi\widetilde{p}_m(\eta_0)\rangle = \sum_{l=1}^n y_{0l}(\partial\widetilde{p}_m/\partial\xi_l)(\eta_0) \ge 0$, $|y_0| > 2R$, then

$$\begin{aligned}\left|y_0 + s\nabla_\xi\widetilde{p}_m(\eta_0)\right|^2 &= |y_0|^2 + s^2\left|\nabla_\xi\widetilde{p}_m(\eta_0)\right|^2 + 2s\left\langle y_0, \nabla_\xi\widetilde{p}_m(\eta_0)\right\rangle\\ &\ge 4R^2 + s^2|\nabla\widetilde{p}_m(\eta_0)|^2 \qquad (3.9)\\ &\ge \frac{1}{2}\left(2R + s\left|\nabla_\xi\widetilde{p}_m(\eta_0)\right|\right)^2.\end{aligned}$$

Since we have

$$\begin{aligned}&|\operatorname{Re} y - y_0| < \varepsilon,\\ &\left|\operatorname{Re}\nabla_\xi\widetilde{p}_m(\eta) - \nabla_\xi\widetilde{p}_m(\eta_0)\right| \le C_{m,n}|\eta - \eta_0|\,|\eta_0|^{m-2},\end{aligned}$$

and

$$\begin{aligned}\operatorname{Im} Y(s) &= \operatorname{Im} y + s \operatorname{Im} \nabla_\xi \widetilde{p}_m(\eta) + \operatorname{Im} Z(s),\\ |\operatorname{Im} y| &= |\operatorname{Im} y - \operatorname{Im} y_0| < \varepsilon,\\ \left|\operatorname{Im} \nabla_\xi \widetilde{p}_m(\eta)\right| &\le \left|\nabla_\xi \widetilde{p}_m(\eta) - \nabla_\xi \widetilde{p}_m(\eta_0)\right| \le C_{m,n}|\eta - \eta_0|\,|\eta_0|^{m-2},\end{aligned}$$

we obtain

$$\begin{aligned}&|\operatorname{Re} Y(s)| - 3K_0\,|\operatorname{Im} Y(s)|\\ &\ge \sqrt{2}R - (1+3K_0)\left(\varepsilon + C_{m,n}|\eta-\eta_0|\,|\eta_0|^{m-2} + |Z(s)|\right).\end{aligned}$$

We take $\varepsilon > 0$ so small enough that we can get

$$|\operatorname{Re} Y(s)| - 3K_0\,|\operatorname{Im} Y(s)| \ge 0. \tag{3.10}$$

From this fact there exists a positive constant $s_1 > 0$ such that the initial value problem (3.8) has a unique local solution $(Z(s), \zeta(s))$ on $[0, s_1]$.
(ii) STEP 2. There exists a positive constant s_3 such that on $[0, s_3]$ $(Z(s), \zeta(s))$ satisfies the estimates from (3.5) to (3.7): and the continuity of ζ and $\zeta(0) = 0$ mean that we can find a constant $s_2 = s_2(C_0, K_0, m, n, \eta_0, R)$ with $s_1 > s_2 > 0$ such that

$$|\zeta_l(s)| \le \frac{C_0 K_0}{R^{1+\sigma_0}}|\eta|, \quad \text{for} \quad 0 \le s \le s_2.$$

From (3.2) and (3.8)

$$\begin{aligned}\left|\dot{Z}_l(0)\right| &\le \left|\sum_{|\alpha|=m} b_\alpha(y)\frac{\partial}{\partial \xi_l}\xi^\alpha\bigg|_{\xi=\eta}\right|\\ &\le \frac{C_0}{(2R)^{1+\sigma_0}} C_1(m,n)|\eta|^{m-1}\\ &\le C_2\frac{C_0}{R^{1+\sigma_0}}|\eta|^{m-1},\end{aligned}$$

where $C_2 = C_2(m, n, \sigma_0)$ is a constant which depends only on m, n and σ_0. It follows from $Z_l(0) = 0$ and the continuity of $Z(s)$ that

$$|Z_l(s)| \le \int_0^s \left|\dot{Z}_l(\tau)\right| d\tau \le 2C_2\frac{C_0}{R^{1+\sigma_0}}|\eta|^{m-1}s.$$

If we take A_1, B_1 as $A_1 > 2C_2, B_1 > C_2$ respectively, we get the local estimates for $Z(s)$ and $\zeta(s)$.
(iii) STEP 3. global behavior: Let us define an interval I. We call $\widetilde{s} \in I$ when the solution $(Z(s), \zeta(s))$ exists on $[0, \widetilde{s}\,]$ and the estimates (3.5), (3.6) and (3.7) are satisfied on $[0, \widetilde{s}\,]$. If $\sup I = \infty$, this shows that the proof of Lemma 3.1 is completed. Step 1

and Step 2 deduce $s_3 \in I$. Assume $\sup I = s^* < \infty$. We show that this is the contradiction. For any $s_0 < s^*$ the system (3.8) has a solution $(Z(s), \zeta(s))$ which satisfies the estimates from (3.5) to (3.7) on $[0, s_0]$. As in Step 1 we have

$$\begin{cases} |\mathrm{Re}\, Y(s)| \geq \dfrac{1}{\sqrt{2}} \left(2R + s \left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|\right) - \varepsilon - C_{m,n} s |\eta - \eta_0| \, |\eta_0|^{m-2} - |Z(s)|, \\ |\mathrm{Im}\, Y(s)| \leq \varepsilon + C_{m,n} s |\eta - \eta_0| \, |\eta_0|^{m-2} + |Z(s)|. \end{cases}$$

It follows from (3.5) that

$$\begin{aligned} &|\mathrm{Re}\, Y(s)| \\ &\geq \left(\sqrt{2}R - \varepsilon\right) + s \left(\frac{1}{\sqrt{2}} \left|\nabla_\xi \widetilde{p}_m(\eta_0)\right| - C_{m,n} |\eta - \eta_0| \, |\eta_0|^{m-2} - A_1 \frac{C_0 K_0}{R^{1+\sigma_0}} |\eta|^{m-1} \right). \end{aligned}$$

If we take ε small and R large, then we get

$$|\mathrm{Re}\, Y(s)| \geq R + \frac{1}{2} s \left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|. \tag{3.11}$$

From (3.8) and (3.11) we have

$$\begin{aligned} |\zeta(s)| &\leq \int_0^s \left|\dot{\zeta}_l(\tau)\right| d\tau \\ &\leq \int_0^s \left| \sum_{|\alpha|=m} \frac{\partial b_\alpha}{\partial x_l} (Y(\tau))(\zeta(\tau) + \eta)^\alpha \right| d\tau \\ &\leq \int_0^s \frac{C_0 K_0}{|\mathrm{Re}\, Y(s)|^{2+\sigma_0}} (|\zeta(\tau)| + |\eta|)^m d\tau \\ &\leq \frac{C_0 K_0}{R^{2+\sigma_0}} \int_0^s \frac{1}{\left(1 + \tau/(2R) \left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|\right)^{2+\sigma_0}} (|\zeta(\tau)| + |\eta|)^m \, d\tau. \end{aligned}$$

Let us set

$$a(\tau) = \frac{1}{\left(1 + \tau/(2R) \left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|\right)^{2+\sigma_0}}, \quad \Phi(s) = \int_0^s a(\tau)(|\zeta(\tau)| + |\eta|)^m \, d\tau. \tag{3.12}$$

By the definition we have $\Phi(s) = 0$, and

$$\frac{d}{ds} \Phi(s) = a(s)(|\zeta(s)| + |\eta|)^m, \quad |\zeta_l(s)| \leq \frac{C_0 K_0}{R^{2+\sigma_0}} \Phi(s). \tag{3.13}$$

We can write

$$|\zeta(s)| = \left(\sum_{j=1}^n |\zeta_l(s)| \right)^{1/2} \leq \sqrt{n} \frac{C_0 K_0}{R^{2+\sigma_0}} \Phi(s) = N(R)\Phi(s).$$

We shall solve the differential inequality

$$
\begin{aligned}
\frac{d}{ds}\Phi(s) &\le a(s)\,(N(R)\Phi(s) + |\eta|)^m \\
&= a(s)N(R)^m \left(\Phi(s) + \frac{|\eta|}{N(R)}\right)^m .
\end{aligned}
\tag{3.14}
$$

Integrating (3.14), we get

$$
\left(\frac{|\eta|}{N(R)}\right)^{-(m-1)} - \left(\Phi(s) + \frac{|\eta|}{N(R)}\right)^{-(m-1)} \le (m-1)N(R)^m \int_0^s a(\tau)\,d\tau .
$$

We set $L = |\eta|/N(R)$ and $M(s) = (m-1)N(R)^m \int_0^s a(\tau)\,d\tau$. From (3.14)

$$
\frac{1}{(\Phi(s)+L)^{m-1}} \ge \frac{1 - M(s)L^{m-1}}{L^{m-1}}.
$$

Here we can show

$$
\begin{aligned}
\int_0^s a(\tau)\,d\tau &= \int_0^s \frac{1}{\left(1 + \tau/(2R)\left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|\right)^{2+\sigma_0}}\,d\tau \\
&\le \frac{2R}{\left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|}\int_0^\infty \frac{1}{(1+\tau)^{2+\sigma_0}}\,d\tau \\
&= \frac{2CR}{\left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
M(s)L^{m-1} &= (m-1)N(R)^m \int_0^s a(\tau)\,d\tau \left(\frac{|\eta|}{N(R)}\right)^{m-1} \\
&= (m-1)\sqrt{n}\frac{C_0K_0}{R^{2+\sigma_0}}|\eta|^{m-1}\int_0^s a(\tau)\,d\tau \\
&\le (m-1)\sqrt{n}\frac{C_0K_0}{R^{2+\sigma_0}}|\eta|^{m-1}\frac{2CR}{\left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|} \\
&\le 2CM(m-1)\sqrt{n}\frac{C_0K_0}{R^{1+\sigma_0}},
\end{aligned}
$$

where M is the positive constant defined by

$$
M = \sup \frac{|\eta|^{m-1}}{\left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|} \le \left(\frac{3}{2}\right)^{m-1} \sup_{\xi \in S^{n-1}} \frac{1}{\left|\nabla_\xi \widetilde{p}_m(\xi)\right|}.
$$

We note that M is independent of η_0 and $M(s)L^{m-1}$ is sufficiently small because R is large enough. Since $1 - M(s)L^{m-1} > 0$, we have

$$\begin{aligned}\Phi(s) &\le \left(\frac{L^{m-1}}{1 - M(s)L^{m-1}}\right)^{1/(m-1)} - L \\ &\le \left\{\left(1 - M(s)L^{m-1}\right)^{-1/(m-1)} - 1\right\} L \\ &\le \frac{2}{m-1} M(s)L^{m-1}L.\end{aligned}$$

Therefore we can show the estimate

$$\begin{aligned}|\zeta_l(s)| &\le N(R)\Phi(s) \\ &\le N(R)\frac{2}{m-1} 2CM(m-1)\sqrt{n}\frac{C_0K_0}{R^{1+\sigma_0}}\frac{|\eta|}{N(R)} \\ &\le B_2\frac{C_0K_0}{R^{1+\sigma_0}}|\eta|,\end{aligned}$$

where $B_2 = 4\sqrt{n}CM$. Next we give the estimate for $Z(s)$,

$$\begin{aligned}|Z_l(s)| &\le \int_0^s \left|\dot{Z}_l(\tau)\right| d\tau \\ &\le \int_0^s \left| \sum_{|\alpha|=m} b_\alpha(Y(\tau))\frac{\partial}{\partial \xi_l}\xi^\alpha \Big|_{\xi=\zeta(\tau)+\eta} \right| d\tau \\ &\quad + \int_0^s \left| \sum_{|\alpha|=m} \widetilde{a}_\alpha \left\{ \frac{\partial}{\partial \xi_l}\xi^\alpha \Big|_{\xi=\zeta(\tau)+\eta} - \frac{\partial}{\partial \xi_l}\xi^\alpha \Big|_{\xi=\eta} \right\} \right| d\tau \\ &\le \int_0^s \frac{C_0}{|\mathrm{Re}\, Y(\tau)|^{1+\sigma_0}} C_{m,n}(|\zeta(\tau)| + |\eta|)^{m-1}\, d\tau \\ &\quad + \int_0^s \sup_{|\alpha|=m} |\widetilde{a}_\alpha|\, C_{m,n} \sum_{l=0}^{m-2} |\zeta(\tau)|^{m-1-l}|\eta|^l d\tau \\ &\le \frac{C_0}{R^{1+\sigma_0}} C_{m,n} \int_0^s \left(B_2\frac{C_0K_0}{R^{1+\sigma_0}}|\eta| + |\eta|\right)^{m-1} d\tau \\ &\quad + C_{m,n} \sup_{|\alpha|=m} |\widetilde{a}_\alpha| \sum_{l=0}^{m-2} \left(B_2\frac{C_0K_0}{R^{1+\sigma_0}}\right)^{m-1-l} |\eta|^{m-1}s \\ &\le A_2\frac{C_0K_0}{R^{1+\sigma_0}}|\eta|^{m-1}s,\end{aligned}$$

where A_2 is a constant, and $C_0K_0/R^{1+\sigma_0} \le 1$ if R is large enough. If our constants A_1, B_1 are taken as $A_1 = 2A_2$, $B_1 = B_2$, then the system (3.8) can be solved again and there exists $\delta > 0$ such that $s^* + \delta \in I$. This fact contradicts $\sup I = s^* < \infty$. Thus we obtained Lemma 3.1 except the estimate (3.7). The details of these arguments can be seen in Section 2 in [21].

Let us show (3.7). Since we have

$$\begin{aligned}\frac{d}{ds}H(s) &= \frac{d}{ds}Z(s) - \left\{\nabla_\xi \widetilde{p}_m(\zeta(s)+\eta) - \nabla_\xi \widetilde{p}_m(\eta)\right\} \\ &= \nabla_\xi \sum_{|\alpha|=m} b_\alpha\left(Y(s)\right)\left(\zeta(s)+\eta\right)^\alpha ,\end{aligned}$$

using (3.5) and (3.6), we get

$$\left|\frac{d}{ds}H(s)\right| \le C\frac{K_0|\eta_0|^{m-1}}{\left(R + s(1/2)\left|\nabla_\xi \widetilde{p}_m(\eta_0)\right|\right)^{1+\sigma_0}}.$$

Since $H(0) = 0$, we have the desired estimates (3.7). This ends the proof of Lemma 3.1. □

We can extend the variable s in $s \in \mathbb{C}$ with $\operatorname{Re} s > -\rho$ and $|\operatorname{Im} s| < \rho$:

Lemma 3.2. *We can find $\varepsilon^* > 0$, $R^* \ge 1$ $\rho > 0$ such that for all $0 < \varepsilon \le \varepsilon^*$, $R \ge R^*$, $(y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ with $|y_0| > 2R$, $|\eta_0| = 1$, $\left\langle y_0, \nabla_\xi \widetilde{p}_m(\eta_0)\right\rangle \ge 0$ and $(y, \eta) \in \mathbb{C}^n \times \mathbb{C}^n$ with $|y - y_0| < \varepsilon$, $|\eta - \eta_0| < \varepsilon$, the system*

$$\text{(3.15)}\qquad \begin{cases} \dfrac{d}{ds}Y(s) = \dfrac{\partial p_m}{\partial \xi}(Y(s), \Theta(s)), & Y(0) = y, \\ \dfrac{d}{ds}\Theta(s) = -\dfrac{\partial p_m}{\partial x}(Y(s), \Theta(s)), & \Theta(0) = \eta, \end{cases}$$

has a unique solution which is holomorphic in the set

$$\text{(3.16)}\qquad \begin{aligned}\Omega = \{&(s, y, \eta) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n ; \\ &|y - y_0| < \varepsilon, |\eta - \eta_0| < \varepsilon, -\rho < \operatorname{Re} s < +\infty, |\operatorname{Im} s| < \rho\},\end{aligned}$$

and if we define $(Z(s), \zeta(s), H(s))$ as

$$\text{(3.17)}\qquad \begin{cases} Z(s) = Y(s) - \widetilde{Y}(s), \\ \zeta(s) = \Theta(s) - \widetilde{\Theta}(s), \\ H(s) = Z(s) - \displaystyle\int_0^s \left\{\nabla_\xi \widetilde{p}_m(\zeta(\tau)+\eta) - \nabla_\xi \widetilde{p}_m(\eta)\right\} d\tau, \end{cases}$$

then we have

$$\text{(3.18)}\qquad \begin{cases} |Z(s)| \le A_1'(C_0, K_0, |\eta_0|)\dfrac{1}{R^{1+\sigma_0}}\max(1, |s|), \\ |\zeta(s)| \le B_1'(C_0, K_0, |\eta_0|)\dfrac{1}{R^{1+\sigma_0}}, \\ |H(s)| \le D_1'(C_0, K_0, |\eta_0|)\dfrac{1}{R^{\sigma_0}}. \end{cases}$$

Proof of Lemma 3.2. We can apply the following well known result about the unique existence of the holomorphic solution on the initial value problem for the ordinary differential equation. Let $(z_0, w_0) = (z_0, w_{01}, \dots, w_{0d}) \in \mathbb{C} \times \mathbb{C}^d$ and $D = \{(z, w) \in \mathbb{C} \times \mathbb{C}\,;\, |z - z_0| < A, |w_j - w_{0j}| < B,\ (1 \le j \le d)\}$. Let $f : D \to \mathbb{C}^d$ be a holomorphic function on D and $\sup_D |f_j| \le M < \infty$. Let $\rho = A\{1 - \exp(-B/((1+d)MA))\}$. Then the Cauchy problem

$$\begin{cases} \dfrac{dw}{dz} = f(z, w(z)), \\ w(z_0) = w_0, \end{cases}$$

has a unique solution which is holomorphic in $\Delta = \{z \in \mathbb{C}\,;\, |z - z_0| < \rho\}$ and $|w_j(z) - w_{0j}| < B$ for $z \in \Delta$. In our case we choose A and B as

$$A = \frac{B}{(2n+1)M \log 2}, \quad B = 1,$$

where M is defined later. Let s_0 be in $\{0 \le s < \infty\}$, and $(Y(s), \theta(s))$ be the solution defined in Lemma 3.1. For $(Y, \theta) \in D = \{(Y, \Theta) \in \mathbb{C}^n \times \mathbb{C}^n\,;\, |Y - Y(s_0)| + |\Theta - \Theta(s_0)| < B\}$, we have

$$\begin{aligned} |Y| &\ge \left|\widetilde{Y}(s_0)\right| - |Z(s_0)| - B \ge \left|y + s_0 \nabla_\xi \widetilde{p}_m(\eta)\right| - A_1 \frac{C_0 K_0}{R^{1+\sigma_0}} s_0 |\eta|^{m-1} - 1, \\ |\Theta| &\le |\Theta(s_0)| + B \le \left|\widetilde{\Theta}(s)\right| + |\zeta(s_0)| + B \le |\eta| + \frac{C_0}{R^{1+\sigma_0}}|\eta| + 1 \\ &\le C(C_0)|\eta_0| + 1. \end{aligned}$$

This implies that the coefficients $a_\alpha(x)$ are holomorphic in D. Since

$$|\nabla_\xi p_m(Y, \theta)| \le \left|\sum_{|\alpha|=m} a_\alpha(Y)\right| \left|\nabla_\xi \xi^\alpha|_{\xi=\Theta}\right|,$$

and

$$|-\nabla_x p_m(Y, \theta)| \le \left|\sum_{|\alpha|=m} (\nabla_x a_\alpha)(Y)\right| |\Theta^\alpha|,$$

it follows from (3.2) and the estimates above that there exists a positive constant $M = M(n, C_0, K_0, |\eta_0|)$, independent of $s_0 \in [0, \infty)$ and R, such that

$$\left|\nabla_\xi p_m(Y, \theta)\right| \le M, \quad |-\nabla_x p_m(Y, \theta)| \le M,$$

for $(Y, \Theta) \in D$. For $s_0 \in [0, \infty)$, $(Y(s), \Theta(s))$ can be extended as holomorphic functions in $\{s \in \mathbb{C}\,;\, |s - s_0| < \rho\}$, where ρ has been defined by A, B, M as above, that is, independent of s_0 and R. Therefore (3.17) has a unique holomorphic solution in

$$J = \left\{ s \in \mathbb{C};\ \operatorname{Re} s > -\frac{1}{10}A,\ \ |\operatorname{Im} s| < \frac{1}{10}A \right\}.$$

Once the existence of solution to (3.15) is proved, the same argument to get the estimates (3.16) works well. Here we show the estimate for $|\zeta(s; y, \eta)|$. As used in the proof of Lemma 3.1, the differential inequality

$$\frac{d}{dt} f(t) \le C(t)(f(t) + \beta)^m, \quad f(0) = 0, \quad \beta > 0,$$

implies

$$f(t) \le 2\beta^m \int_0^t C(\tau)\, d\tau, \quad \text{for small} \quad t > 0.$$

Since $\zeta(s; y, \eta)$ is holomorphic with respect to $s \in J$, the equation for $\zeta(s)$ in (3.8) gives

$$|\zeta_l(s; y, \eta) - \zeta_l(\operatorname{Re} s; y, \eta)| \le \sum_{|\alpha|=m} \left| \int_\gamma \frac{\partial b_\alpha}{\partial x_l}(Y(\tau))(\zeta(\tau) + \eta)^\alpha\, d\tau \right|,$$

where $\gamma = \{\tau \in \mathbb{C};\ \tau = \operatorname{Re} s + it \operatorname{sgn} s,\ \ 0 \le t \le \operatorname{Im} s\}$, $\operatorname{Im} s = s/|s|$. Since (3.2) and (3.11) show

$$\left| \frac{\partial b_\alpha}{\partial x_l}(Y(\tau)) \right| \le \frac{C_0 K_0}{(R/2)^{2+\sigma_0}},$$

we obtain

$$\begin{aligned} &|\zeta(\operatorname{Re} s + i \operatorname{Im} s; y, \eta) - \zeta(\operatorname{Re} s; y, \eta)| \\ &\le C(n, m, \sigma_0) \frac{C_0 K_0}{R_0^{2+\sigma_0}} \int_0^{|\operatorname{Im} s|} (|\zeta(\operatorname{Re} s + it; y, \eta)| + |\eta|)^m\, dt. \end{aligned}$$

By (3.6), we have $|\zeta(\operatorname{Re} s; y, \eta)| \le B_1 C_0 K_0 |\eta| / R^{1+\sigma_0}$. Since the differential inequality

$$|g(t) - g(0)| \le D \int_0^t (|g(\tau)| + F)^m\, d\tau = D f(t),$$

implies

$$\frac{df}{dt} = (|g(t) + F|)^m \le D^m \left(f(t) + \frac{|g(0)| + F}{D} \right)^m,$$

we have

$$\begin{aligned}|g(t)| &\le |g(0)| + Df(t)\\ &\le |g(0)| + 2D\left(\frac{|g(0)|+F}{D}\right)^m \int_0^t D^m\,d\tau\\ &= |g(0)| + 2D(|g(0)|+F)^m t\end{aligned}$$

for small $t > 0$. Hence

$$\begin{aligned}|\zeta(\operatorname{Re} s + i\operatorname{Im} s; y,\eta)| &\le |\zeta(\operatorname{Re} s; y,\eta)| + 2C\frac{C_0K_0}{R^{2+\sigma_0}}(|\zeta(\operatorname{Re} s; y,\eta)| + |\eta|)^m|\operatorname{Im} s|\\ &\le B_1'(n,m,C_0,K_0,\sigma_0,|\eta_0|)\frac{1}{R^{1+\sigma_0}}.\end{aligned}$$

For $Z(s)$ it follows from (3.5) and (3.8) that we obtain

$$\begin{aligned}|Z_l(s)| &\le |Z_l(\operatorname{Re} s)| + \left|\int_\gamma \frac{dZ_l}{d\tau}(\tau)\,d\tau\right|\\ &\le |Z_l(\operatorname{Re} s)| + \int_0^{|\operatorname{Im} s|}\left|\sum_{|\alpha|=m} b_\alpha\,(Y(\operatorname{Re} s+it))\frac{\partial}{\partial\xi_l}\xi^\alpha\Big|_{\xi=\zeta(\operatorname{Re} s+it)+\eta}\right|dt\\ &\quad+\int_0^{|\operatorname{Im} s|}\left|\sum_{|\alpha|=m}\widetilde{a}_\alpha\left\{\frac{\partial}{\partial\xi_l}\xi^\alpha\Big|_{\xi=\zeta(\operatorname{Re} s+it)+\eta}-\frac{\partial}{\partial\xi_l}\xi^\alpha\Big|_{\xi=\eta}\right\}\right|dt\\ &\le |Z_l(\operatorname{Re} s)| + \int_0^{|\operatorname{Im} s|}\frac{C_0}{|\operatorname{Re} Y(\operatorname{Re} s+it)|^{1+\sigma_0}}C_{m,n}\,(|\zeta(\operatorname{Re} s+it)|+|\eta|)^{m-1}\,dt\\ &\quad+\int_0^{|\operatorname{Im} s|}\sup_{|\alpha|=m}|\widetilde{a}_\alpha|C_{m,n}\sum_{l=0}^{m-2}|\zeta(\operatorname{Re} s+it)|^{m-1-l}|\eta|^l\,dt\\ &\le |Z_l(\operatorname{Re} s)| + \frac{C_0}{R^{1+\sigma_0}}C_{m,n}\int_0^{|\operatorname{Im} s|}\left(B_1'\frac{1}{R^{1+\sigma_0}}+|\eta|\right)^{m-1}d\tau\\ &\quad+C_{m,n}\sup_{|\alpha|=m}|\widetilde{a}_\alpha|\sum_{l=0}^{m-2}\left(B_1'\frac{1}{R^{1+\sigma_0}}\right)^{m-1-l}|\eta|^{m-1}\,|\operatorname{Im} s|\\ &\le A_1'(n,m,C_0,K_0,\sigma_0,|\eta_0|)\frac{1}{R^{1+\sigma_0}}|s|.\end{aligned}$$

The estimate for $H(s)$ is also proved as Lemma 3.1. Some details can be seen in [21]. □

We use the estimate for $H(s)$ only when $m = 2$. In this case we have

$$\left|Z(s) - s\left(\nabla_\xi^2\widetilde{p}_m(\eta_0)\right)\zeta(s)\right| \le D_1'\frac{1}{R^{\sigma_0}}. \tag{3.19}$$

We use (3.19) in estimating the imaginary part of the second derivative of the phase function in Section 4.

4. Construction of the phase function

Thanks to Lemma 2.1 and Lemma 2.2 it suffices to prove Theorem 1.1 in the special case, $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ with $|y_0| > 2R$, $\langle y_0, \nabla_\xi \widetilde{p}_m(\eta_0)\rangle \geq 0$ and $\det \nabla_\xi^2 \widetilde{p}_m(\eta_0) \neq 0$. Since $P(x, D_x)$ is the operator of real principal type, we may assume $\partial_{\xi_n} p(y_0, \eta_0) \neq 0$. We set $z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$, $z_0 = (z_0', 0) = (y_0' - i\eta_0', 0)$.

As we explained in the introduction, one reason to use the FBI transform is to transform the original operator $P(x, D_x)$ into D_{z_n}. To realize this transformation, we define a kind of FBI transform

$$Su(t, z, \lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(z,x)} f(z, x, \lambda)\chi_1(z, x)u(t, x)\, dx, \tag{4.1}$$

where the phase function $\varphi = \varphi(z, x)$, the amplitude function $f = f(z, x, \lambda)$ and the cutoff function $\chi_1 = \chi_1(z, x)$ will be given later. In this section the phase function φ will be determined. The amplitude function and the cutoff function will be determined in other sections.

Let us define

$$\begin{aligned} I(t, z, \lambda) &= \frac{1}{\lambda} D_{z_n} Su - \frac{1}{\lambda^m} SP(x, D_x)u \\ &= \int_{\mathbb{R}} \left(\frac{1}{\lambda} D_{z_n} - \frac{1}{\lambda^m} {}^tP(x, D_x)\right)(e^{i\lambda\varphi} f\chi_1)u(t, x)\, dx, \end{aligned} \tag{4.2}$$

where ${}^tP(x, D_x)$ is the transposed operator of $P(x, D_x)$. The coefficients of ${}^tP(x, D_x)$ also satisfy the condition (3.2). Since χ_1 is a cutoff function, we first pay attention to the term $F(z, x, \lambda)$:

$$F(z, x, \lambda) = \left(\frac{1}{\lambda} D_{z_n} - \frac{1}{\lambda^m} {}^tP(x, D_x)\right)(e^{i\lambda\varphi} f). \tag{4.3}$$

We have

$$\begin{aligned} & e^{-i\lambda\varphi} F(z, x, \lambda) \\ &= \frac{1}{\lambda}\left(D_{z_n} + \lambda\frac{\partial\varphi}{\partial z_n}\right) f - \frac{1}{\lambda^m} {}^tP(x, D_x + \lambda\nabla_x\varphi) f \\ &= \left\{\frac{\partial\varphi}{\partial z_n} - {}^tp_m(x, \nabla_x\varphi)\right\} f \\ &\quad + \frac{1}{i\lambda}\left\{\frac{\partial}{\partial z_n} - \sum_{l=1}^{n} \frac{\partial^t p_m}{\partial\xi_l}(x, \nabla_x\varphi)\frac{\partial}{\partial x_l} - d\left(z, x, \nabla_x\varphi, \nabla_x^2\varphi\right)\right\} f \\ &\quad + \frac{1}{\lambda^2}(\cdots) f + \cdots + \frac{1}{\lambda^m} {}^tP(x, D_x) f. \end{aligned} \tag{4.4}$$

The next theorem and its proof show the existence and the properties of the phase function which is the solution of the eikonal equation.

Theorem 4.1. *There exist positive constants* ε_1 *and* ε_2 *with* $0 < \varepsilon_1 < \varepsilon_2$ *and a holomorphic function* $\varphi = \varphi(z, x)$ *in the set*

(4.5)
$$E = \{(z', z_n, x', x_n) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}\,; \quad |z' - z'_0| < \varepsilon_1,\ \operatorname{Re} z_n > -\varepsilon_1,\ |\operatorname{Im} z_n| < \varepsilon_1,\ |x - Y(z_n; y_0, \eta_0)| < \varepsilon_2(1 + |z_n|)\},$$

such that

$$\frac{\partial\varphi}{\partial z_n}(z, x) = p_m\left(x, -\frac{\partial\varphi}{\partial x}(z, x)\right) \quad in \quad E, \tag{4.6}$$

$$\frac{\partial\varphi}{\partial x}(z_0, y_0) = -\eta_0, \tag{4.7}$$

$$\operatorname{Im}\frac{\partial^2\varphi}{\partial x^2}(z_0, y_0) \quad \textit{is a positive definite symmetric matrix}, \tag{4.8}$$

$$\det\frac{\partial^2\varphi}{\partial z\partial x}(z_0, y_0) \neq 0. \tag{4.9}$$

Proof of Theorem 4.1. Let us introduce the holomorphic function $\varphi_0 = \varphi_0(z', y)$: $\mathbb{C}^{n-1} \times \mathbb{C}^n \to \mathbb{C}$ by

$$\varphi_0(z', y) = \frac{i}{2}\left(z' - y'\right)^2 - \eta_{0n} y_n + \frac{i}{2}(y_n - y_{0n})^2. \tag{4.10}$$

We note

$$\frac{\partial\varphi_0}{\partial z'} = i(z' - y'), \quad \frac{\partial\varphi_0}{\partial y'} = -i(z' - y'), \quad \frac{\partial\varphi_0}{\partial y_n} = -\eta_{0n} + i(y_n - y_{0n}).$$

We set the symbol

$$q(z, x, \zeta, \xi) = \zeta_n - {}^t p_m(x, \xi), \tag{4.11}$$

where ${}^t p_m(x, \xi) = p_m(x, -\xi)$, and set the submanifold

$$\begin{aligned}\Lambda_0 &= \left\{(Z(0), X(0), G(0), F(0)) \in \mathbb{C}^{4n}\right\} \\ &= \left\{\left((z', 0), (y', y_n), \left(\frac{\partial\varphi_0}{\partial z'}(z', y), {}^t p_m\left(y, \frac{\partial\varphi_0}{\partial y}(z', y)\right)\right),\right.\right. \\ &\qquad \left.\left.\left(\frac{\partial\varphi_0}{\partial y'}(z', y), \frac{\partial\varphi_0}{\partial y_n}(z', y)\right)\right) \in \mathbb{C}^{4n}\,;\ \left|z' - z'_0\right| < \varepsilon_1,\ |y - y_0| < \varepsilon_3\right\},\end{aligned} \tag{4.12}$$

where ε_1 and ε_3 are small positive constants determined later. We introduce the submanifold of $\mathbb{C}^{4n}$,

$$\begin{aligned}\Lambda &= \left\{(Z(s), X(s), G(s), F(s)) \in \mathbb{C}^{4n}\right\}\\ &= \bigcup_{\mathrm{Re}\, s > -\varepsilon_1, |\mathrm{Im}\, s| < \varepsilon_1} \exp(isH_q)\Lambda_0,\end{aligned}$$

by the solution of the differential equations

$$\begin{cases} \dfrac{d}{ds} Z(s) = \dfrac{\partial q}{\partial \zeta}(Z(s), X(s), G(s), F(s)), \\ \dfrac{d}{ds} X(s) = \dfrac{\partial q}{\partial \xi}(Z(s), X(s), G(s), F(s)), \\ \dfrac{d}{ds} G(s) = -\dfrac{\partial q}{\partial z}(Z(s), X(s), G(s), F(s)), \\ \dfrac{d}{ds} F(s) = -\dfrac{\partial q}{\partial x}(Z(s), X(s), G(s), F(s)), \end{cases} \tag{4.13}$$

with the initial data $(Z(0), X(0), G(0), F(0))$ given in (4.12). Since the symbol is given in (4.11), we have

$$\begin{cases} Z'(s) = Z'(0) = z', \quad Z_n(s) = s, \\ G(s) = G(0), \end{cases} \tag{4.14}$$

and

$$\begin{cases} \dfrac{d}{ds} X(s) = -\dfrac{\partial\, {}^t p_m}{\partial \xi}(X(s), F(s)), \quad X(0) = y, \\ \dfrac{d}{ds} F(s) = \dfrac{\partial\, {}^t p_m}{\partial x}(X(s), F(s)), \quad F(0) = \dfrac{\partial \varphi_0}{\partial y}(z', y). \end{cases} \tag{4.15}$$

Since ${}^t p_m(x, \xi) = p_m(x, -\xi)$, we have

$$\begin{cases} \dfrac{d}{ds} X(s) = \dfrac{\partial p_m}{\partial \xi}(X(s), -F(s)), \quad X(0) = y, \\ \dfrac{d}{ds} F(s) = \dfrac{\partial p_m}{\partial x}(X(s), -F(s)), \quad F(0) = \dfrac{\partial \varphi_0}{\partial y}(z', y). \end{cases} \tag{4.16}$$

If we set $X(s) = Y(s; y, \eta)$, and $F(s) = -\Theta(s; y, \eta)$ with $\eta = -(\partial\varphi_0/\partial y)(z', y)$, then $(Y(s), \Theta(s))$ satisfies the differential equations (3.15) given in Section 3. For $(z', y) \in \mathbb{C}^{n-1} \times \mathbb{C}^n$ with $\left|z' - z'_0\right| < \varepsilon_1$, $|y - y_0| < \varepsilon_3$,

$$|\eta - \eta_0| = \left|-\frac{\partial \varphi_0}{\partial y}(z', y) - \eta_0\right| = \left| i \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} - i \begin{pmatrix} y' - y'_0 \\ y_n - y_{0n} \end{pmatrix} \right| < \varepsilon_1 + \varepsilon_3.$$

We can apply the Lemma 3.2 by taking ε_1 and ε_3 small enough. We set $s = z_n$ and define

$$O = \{(z, y) \in \mathbb{C}^n \times \mathbb{C}^n ; \quad |z' - z'_0| < \varepsilon_1, \ |y - y_0| < \varepsilon_3, \ \operatorname{Re} z_n > -\varepsilon_1, \ |\operatorname{Im} z_n| < \varepsilon_1\}, \tag{4.17}$$

and

$$\Lambda = \left\{(z, Y(z_n; y, \eta), G(0), -\Theta(z_n; y, \eta)) \in \mathbb{C}^{4n} ; \eta = -\frac{\partial \varphi_0}{\partial y}(z', y), \ (z, y) \in O\right\}. \tag{4.18}$$

We also set

$$E = \left\{\left(z', z_n, x', x_n\right) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} ; \left|z' - z'_0\right| < \varepsilon_1, \ \operatorname{Re} z_n > -\varepsilon_1, \ |\operatorname{Im} z_n| < \varepsilon_1, \ |x - Y(z_n; y_0, \eta_0)| < \varepsilon_2(1 + |z_n|)\right\}, \tag{4.19}$$

For the two form $\sigma = d\zeta \wedge dz + d\xi \wedge dx$, we have

$$\sigma|_{\Lambda_0} = 0. \tag{4.20}$$

The dimension of the submanifold Λ_0 in $\mathbb{C}^{4n}$ is equal to $2n-1$. Since H_q is transverse to Λ_0, the dimension of the submanifold Λ of $\mathbb{C}^{4n}$ is equal to $2n$, and we obtain

$$\sigma|_{\Lambda} = 0. \tag{4.21}$$

This shows that the submanifold Λ of $\mathbb{C}^{4n}$ is a Lagrangean submanifold.

Lemma 4.1. *If the constant ε_3 which satisfies $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3$ is small enough, then there exists a holomorphic function $\varphi = \varphi(z, x)$ in E such that*

$$\Lambda = \left\{\left(z, x, \frac{\partial \varphi}{\partial z}(z, x), \frac{\partial \varphi}{\partial x}(z, x)\right) \in \mathbb{C}^{4n} ; (z, x) \in E\right\}. \tag{4.22}$$

Proof of Lemma 4.1. Let $\pi : \Lambda \to \mathbb{C}^n \times \mathbb{C}^n$ be the projection map of Λ on the base, that is, $\pi(\lambda) = (z, Y(z_n; y, \eta))$ for $\lambda \in \Lambda$, where $\eta = \eta(z', y) = -(\partial\varphi_0/\partial y)(z', y)$. Since Λ is a Lagrangean submanifold of $\mathbb{C}^{4n}$, it suffices to prove that π is bijective and that $d\pi(\lambda)$ is surjective. Let us prove that for fixed $z \in \mathbb{C}^n$ with $\left|z' - z'_0\right| < \varepsilon_1$, $\operatorname{Re} z_n > -\varepsilon_1$, $|\operatorname{Im} z_n| < \varepsilon_1$ and $x \in \mathbb{C}^n$ with $|x - Y(z_n; y_0, \eta_0)| < \varepsilon_2(1 + |z_n|)$, there exists a unique $y \in \mathbb{C}^n$ with $|y - y_0| < \varepsilon_3$ such that $x = Y(z_n; y, -(\partial\varphi_0/\partial y)(z', y))$. By following [21] and [23], we prove this fact. We have

(4.23)
$$
\begin{aligned}
&Y\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right) - Y(z_n; y_0, \eta_0)\\
&= \widetilde{Y}\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right) - \widetilde{Y}(z_n; y_0, \eta_0) + Z\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right) - Z(z_n; y_0, \eta_0)\\
&= \left(y + z_n \frac{\partial \widetilde{p}_m}{\partial \xi}(\eta)\right) - \left(y_0 + z_n \frac{\partial \widetilde{p}_m}{\partial \xi}(\eta_0)\right) + Z^1 - Z^0,
\end{aligned}
$$

where $Z^1 = Z(z_n; y, \eta)$, $Z^0 = Z(z_n; y_0, \eta_0)$. From (4.10) we have

$$
\begin{aligned}
\eta - \eta_0 &= -\frac{\partial\varphi_0}{\partial y}(z', y) - \eta_0\\
&= \begin{pmatrix} i(z' - y') \\ \eta_{0n} - i(y_n - y_{0n}) \end{pmatrix} - \begin{pmatrix} \eta_0' \\ \eta_{0n} \end{pmatrix}\\
&= i\begin{pmatrix} z' - z_0' \\ 0 \end{pmatrix} - i\begin{pmatrix} y' - y_0' \\ y_n - y_{0n} \end{pmatrix},
\end{aligned}
$$

where $z_0' = y_0' - i\eta_0'$. Since the l th element is written as

$$
\begin{aligned}
Y_l\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right) &= \widetilde{Y}_l\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right) + Z_l\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right)\\
&= y_l + z_n \frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta) + Z_l^1,
\end{aligned}
$$

we get from (4.23)

$$
\begin{aligned}
&Y_l\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right) - Y_l(z_n; y_0, \eta_0)\\
&= (y_l - y_{0l}) + z_n\left(\frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta) - \frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta_0)\right) + Z_l^1 - Z_l^0.
\end{aligned}
$$

Applying the Taylor's expansion

$$
\frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta) - \frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta_0)
$$

$$= \sum_{k=1}^{n} (\eta_k - \eta_{0k}) \frac{\partial^2 \widetilde{p}_m}{\partial \xi_k \partial \xi_l}(\eta_0)$$
$$+ 2 \sum_{|\gamma|=2} \frac{1}{\gamma!}(\eta - \eta_0)^\gamma \int_0^1 (1-\theta) \left(\frac{\partial}{\partial \xi}\right)^\gamma \left(\frac{\partial \widetilde{p}_m}{\partial \xi_l}\right) (\eta_0 + \theta(\eta - \eta_0))\, d\theta$$
$$= \left\langle \eta - \eta_0, \nabla_\xi \frac{\partial \widetilde{p}_m}{\partial \xi_l}(\eta_0) \right\rangle$$
$$+ 2 \sum_{|\gamma|=2} \frac{1}{\gamma!}(\eta - \eta_0)^\gamma \int_0^1 (1-\theta) \left(\frac{\partial}{\partial \xi}\right)^\gamma \left(\frac{\partial \widetilde{p}_m}{\partial \xi_l}\right) (\eta_0 + \theta(\eta - \eta_0))\, d\theta,$$

it follows from (4.23) that we obtain

$$Y\left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y)\right) - Y(z_n; y_0, \eta_0)$$
$$= (y - y_0) + z_n \left(\nabla_\xi \widetilde{p}_m(\eta) - \nabla_\xi \widetilde{p}_m(\eta_0)\right) + Z^1 - Z^0$$
$$= (y - y_0) + z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0) \left(i \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} - i(y - y_0) \right)$$
$$+ 2 z_n \sum_{|\gamma|=2} \frac{1}{\gamma!}(\eta - \eta_0)^\gamma \int_0^1 (1-\theta) \left(\frac{\partial}{\partial \xi}\right)^\gamma \nabla_\xi \widetilde{p}_m(\eta_0 + \theta(\eta - \eta_0))\, d\theta$$
$$+ Z^1 - Z^0$$
$$= \left(\mathrm{Id} - i z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0)\right)(y - y_0) + i z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0) \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix}$$
$$+ 2 z_n \sum_{|\gamma|=2} \frac{1}{\gamma!}(\eta - \eta_0)^\gamma \int_0^1 (1-\theta) \left(\frac{\partial}{\partial \xi}\right)^\gamma \nabla_\xi \widetilde{p}_m\, (\eta_0 + \theta(\eta - \eta_0))\, d\theta$$
$$+ Z^1 - Z^0,$$

where A $= \nabla_\xi^2 \widetilde{p}_m(\eta_0)$ is the Hessian matrix of $\widetilde{p}_m(\xi)$ at $\xi = \eta_0$ and Id is the $n \times n$ identity matrix. The equation $x = Y(z_n, y, -(\partial \varphi_0 / \partial y)(z', y))$ is equivalent to the equation

$$x - Y(z_n; y_0, \eta_0)$$
$$= \left(\mathrm{Id} - i z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0)\right)(y - y_0) + i z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0) \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix}$$
$$+ 2 z_n \sum_{|\gamma|=2} \frac{1}{\gamma!}(\eta - \eta_0)^\gamma \int_0^1 (1-\theta) \left(\frac{\partial}{\partial \xi}\right)^\gamma \nabla_\xi \widetilde{p}_m\, (\eta_0 + \theta(\eta - \eta_0))\, d\theta \tag{4.24}$$
$$+ Z(z_n; y, \eta) - Z(z_n; y_0, \eta_0).$$

Let us define the map H as

$$
\begin{aligned}
H(y) = & y_0 + \left(\mathrm{Id} - i z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0)\right)^{-1} \\
& \Bigg\{ x - Y(z_n; y_0, \eta_0) - i z_n \nabla_\xi^2 \widetilde{p}_m(\eta_0) \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} \\
& - 2 z_n \sum_{|\gamma|=2} \frac{1}{\gamma!} (\eta - \eta_0)^\gamma \int_0^1 (1-\theta) \left(\frac{\partial}{\partial \xi}\right)^\gamma \nabla_\xi \widetilde{p}_m\left(\eta_0 + \theta(\eta - \eta_0)\right) d\theta \\
& - Z(z_n; y, \eta) + Z(z_n; y_0, \eta_0) \Bigg\}.
\end{aligned}
\tag{4.25}
$$

Let us prove that if $\varepsilon_3 > 0$ is small enough, then H maps the ball $B(y_0, \varepsilon_3)$ into itself and is a contraction map, which means the bijectivity of the projection map π. Since the matrix $\mathrm{A} = \nabla_\xi^2 \widetilde{p}_m(\eta_0)$ is real symmetric, $|\mathrm{Im}\, z_n| < \varepsilon_1$ and $\det \mathrm{A} \neq 0$, the matrix $\mathrm{G} = \mathrm{Id} - i z_n \mathrm{A}$ has the inverse matrix G^{-1} and we have

$$
C_1 \frac{1}{\left(1 + |\mathrm{Re}\, z_n|^2\right)^{1/2}} \leq \|\mathrm{G}^{-1}\|_{\mathcal{L}(\mathbb{C}^n;\mathbb{C}^n)} \leq C_2 \frac{1}{\left(1 + |\mathrm{Re}\, z_n|^2\right)^{1/2}},
\tag{4.26}
$$

where $\|\mathrm{G}^{-1}\|_{\mathcal{L}(\mathbb{C}^n;\mathbb{C}^n)}$ is the operator norm of G^{-1}. The last inequality comes from $\det \mathrm{A} \neq 0$. From (4.25) and (4.26) we have

$$
\begin{aligned}
|H(y) - y_0| \leq & \frac{C}{1 + |z_n|} \Bigg(\varepsilon_2 (1 + |z_n|) + C \varepsilon_1 |z_n| \\
& \qquad + C \varepsilon_3^2 |z_n| + C(K_0, \eta_0) \frac{1}{R^{1+\sigma_0}} (1 + |z_n|) \Bigg) \\
\leq & \ \varepsilon_3, \quad (0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3).
\end{aligned}
\tag{4.27}
$$

Since $Z(z_n; y, \eta(z', y))$ is holomorphic in y, we also obtain

(4.28)

$$
\begin{aligned}
|H(y_1) - H(y_2)| \leq & \ C \frac{1}{1 + |z_n|} \left(C \varepsilon_3 |z_n| \, |y_1 - y_2| + C(K_0, \eta_0) \frac{1}{R^{1+\sigma_0}} (1 + |z_n|) |y_1 - y_2| \right) \\
\leq & \ k |y_1 - y_2|, \quad 0 < k < 1.
\end{aligned}
$$

It follows from (4.27) and (4.28) that H is a contraction map on $B(y_0, \varepsilon_3)$.

We shall show the surjectivity of the differential map $d\pi(\lambda) \colon T_\lambda \Lambda \to T_{\pi(\lambda)} E$. We write (4.24) as

$$
\begin{aligned}
& x - Y(z_n; y_0, \eta_0) \\
& = G(y - y_0) + i z_n A \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} + 2 z_n W(\eta) + Z(z_n; y, \eta) - Z(z_n; y_0, \eta_0),
\end{aligned}
\tag{4.29}
$$

where

$$W(\eta)=\sum_{|\gamma|=2}\frac{1}{\gamma!}(\eta-\eta_0)^\gamma\int_0^1(1-\theta)\left(\frac{\partial}{\partial\xi}\right)^\gamma\nabla_\xi\widetilde{p}_m\,(\eta_0+\theta(\eta-\eta_0))\,d\theta.$$

Using this notation, we differentiate the function $Y(z_n;y,\eta(z',y))=y+z_n\nabla_\xi\widetilde{p}(\eta(z',y))+Z^1$ with respect to y. We get

$$\frac{\partial Y}{\partial y}=\mathrm{G\,Id}+2z_n\frac{\partial}{\partial y}W(\eta)+\frac{\partial}{\partial y}Z\left(z_n;y,\eta\left(z',y\right)\right).$$

Since we have

$$\begin{aligned}\frac{\partial W_l}{\partial y_k}&=\sum_{|\gamma|=2}\frac{1}{\gamma!}\left\{\frac{\partial}{\partial y_k}(\eta-\eta_0)^\gamma\right\}\int_0^1(1-\theta)\left(\frac{\partial}{\partial\xi}\right)^\gamma\frac{\partial\widetilde{p}_m}{\partial\xi_l}\,(\eta_0+\theta(\eta-\eta_0))\,d\theta\\&\quad+\sum_{|\gamma|=2}\frac{1}{\gamma!}(\eta-\eta_0)^\gamma\int_0^1(1-\theta)\left(\frac{\partial}{\partial\xi}\right)^\gamma\frac{\partial^2\widetilde{p}_m}{\partial\xi_k\partial\xi_l}\,(\eta_0+\theta(\eta-\eta_0))\,\theta\frac{\partial}{\partial y_k}\eta_l\,d\theta.\end{aligned}$$

and (3.18), we have

$$\left|\det\left(\frac{\partial Y}{\partial y}(z_n;y,\eta)\right)\right|\geq C(1+|z_n|)^n,\quad (z,y)\in\mathcal{O}. \tag{4.30}$$

It follows from (4.30) that the proof of the surjectivity of the differential map $d\pi(\lambda)\colon T_\lambda\Lambda\to T_{\pi(\lambda)}E$ is completed. □

We note that we can choose the constants ε_j $(j=1,2,3)$ as $\varepsilon_3^2<C\varepsilon_2<\varepsilon_3$, moreover, ε_1 and ε_3 are independent each other. Though we do not use this fact in this paper, it is useful in considering the global properties of the phase function $\varphi=\varphi(z,x)$.

End of the proof of Theorem 4.1: Lemma 4.1 directly shows the property (4.6). The proof for this part can be seen in [23] (see the argument after Lemma 3.1 in [23]). The properties from (4.7) to (4.9) can also be obtained by following the arguments in [21]. Since we have

$$\begin{aligned}&x=Y\left(z_n;y,-\frac{\partial\varphi_0}{\partial y}(z',y)\right),\\&\frac{\partial\varphi}{\partial z}(z,x)=G(0)=\left(\frac{\partial\varphi_0}{\partial z'}(z',y),{}^t p_m\left(y,\frac{\partial\varphi_0}{\partial y}(z',y)\right)\right),\\&\frac{\partial\varphi}{\partial x}(z,x)=F(z_n)=\Theta\left(z_n;y,-\frac{\partial\varphi_0}{\partial y}(z',y)\right),\end{aligned}$$

we have the properties from (4.6) to (4.9). Especially, we can obtain the equation

$$\frac{\partial^2\varphi}{\partial z_n\partial x_n}(z_0,y_0)=\frac{\partial p_m}{\partial x_n}(y_0,\eta_0)-i\frac{\partial p_m}{\partial\xi_n}(y_0,\eta_0),$$

which was shown in [21]. Since p_m is real valued and $\partial_{\xi_n} p_m(y_0, \eta_0) \neq 0$, which are the conditions of real principal type, we can show the property (4.9). The details about this part is found after Corollary 3.6 in [21]. Therefore we have proved Theorem 4.1. □

Next we study the global properties of the phase functions which are used for the construction of the amplitude function globally along the bicharacteristics. The proof of Lemma 4.1 shows that the map $x = Y(z_n; y, \eta(z', y))$ has the inverse map $\kappa : E \to \pi^{-1}(E)$ such that $y = \kappa(z, x)$. The function κ is holomorphic and $\kappa(z_0, y_0) = y_0$. From the construction of κ we have

$$|\kappa(z, x) - y_0| < \varepsilon_3, \quad (z, x) \in E. \tag{4.31}$$

The equation (4.29) implies

$$\begin{aligned} \kappa(z, x) - y_0 = G^{-1} \Bigg\{ & x - Y(z_n; y_0, \eta_0) - i z_n A \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} \\ & - 2z_n W(\eta)|_{y=\kappa(z,x)} - Z(z_n; y, \eta)|_{y=\kappa(z,x)} + Z(z_n; y_0, \eta_0) \Bigg\} . \end{aligned} \tag{4.32}$$

and

$$\mathrm{Id} = G \nabla_x \kappa(z, x) + 2z_n \nabla_x \left(W(\eta)|_{y=\kappa(z,x)} \right) + \nabla_x \left\{ Z \left(z_n, y, -\frac{\partial \varphi_0}{\partial y}(z', y) \right) \Bigg|_{y=\kappa(z,x)} \right\} . \tag{4.33}$$

It follows from (4.32), the estimates on $Z(s)$ and the Cauchy formula that for $(z, x) \in E$ we have

$$\begin{cases} \left| \dfrac{\partial \kappa}{\partial x_j}(z, x) \right| \leq \dfrac{C}{(1 + |z_n|)}, \\ \left| \dfrac{\partial^2 \kappa}{\partial x_j \partial x_k}(z, x) \right| \leq \dfrac{C}{(1 + |z_n|)^2}. \end{cases} \tag{4.34}$$

Since the Lagrangean manifold Λ is expressed in the two ways, (4.18) and (4.22), we have

$$\begin{aligned} \frac{\partial \varphi}{\partial x}(z, x) &= -\Theta \left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y) \right) \Bigg|_{y=\kappa(z,x)} \\ &= - \left\{ -\frac{\partial \varphi_0}{\partial y}(z', y) + \zeta \left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y) \right) \right\} \Bigg|_{y=\kappa(z,x)} \\ &= -\eta_0 - i \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} + i(y - y_0) \Big|_{y=\kappa(z,x)} - \zeta \left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y) \right) \Bigg|_{y=\kappa(z,x)} . \end{aligned}$$

We also obtain

$$\frac{\partial^2\varphi}{\partial x^2} = \nabla_x^2\varphi(z,x) = i\nabla_x\kappa(z,x) - \nabla_x\left\{\zeta\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z',y)\right)\Big|_{y=\kappa(z,x)}\right\}.$$

By the equation (4.33) we have

(4.35)

$$\begin{aligned}
&\frac{\partial^2\varphi}{\partial x^2}(z,x)\\
&= iG^{-1}\left[\mathrm{Id} - 2z_n\nabla_x\left(W(\eta)|_{y=\kappa(z,x)}\right) - \nabla_x\left\{Z\left(z_n, y, -\frac{\partial\varphi_0}{\partial y}(z',y)\right)\Big|_{y=\kappa(z,x)}\right\}\right]\\
&\quad - \nabla_x\left\{\zeta\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z',y)\right)\Big|_{y=\kappa(z,x)}\right\}\\
&= iG^{-1} - i2z_nG^{-1}\nabla_x\left(W(\eta)|_{y=\kappa(z,x)}\right)\\
&\quad - iG^{-1}\nabla_x\left\{Z\left(z_n, y, -\frac{\partial\varphi_0}{\partial y}(z',y)\right)\Big|_{y=\kappa(z,x)} - z_n\mathrm{A}\zeta\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z',y)\right)\Big|_{y=\kappa(z,x)}\right\}\\
&\quad - \mathrm{G}^{-1}\nabla_x\zeta\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z',y)\right)\Big|_{y=\kappa(z,x)}.
\end{aligned}$$

First we consider the case $m=2$. Since $\partial_\xi^\alpha\widetilde{p}_2(\xi)=0$ for $|\alpha|=3$, we have $W(\eta)=0$ for $m=2$. We prove that there exists a constant $C>0$ such that

$$\left\langle \mathrm{Im}(i\mathrm{G}^{-1})v, v\right\rangle \geq \frac{C}{(1+|z_n|)^2}|v|^2 \quad \text{for} \quad v\in\mathbb{C}^n. \tag{4.36}$$

Since $\mathrm{G}|_{z_n=0}=\mathrm{Id}$, it suffices to show (4.36) when $|z_n|\geq\delta_0$. Using the Hessian matrix $\mathrm{A}=\nabla_\xi^2\widetilde{p}_2(\eta_0)$ $(\det\mathrm{A}\neq 0)$, the matrix G is defined by

$$\begin{aligned}
\mathrm{G} &= \mathrm{Id} - iz_n\mathrm{A} = \{\mathrm{Id} + (\mathrm{Im}\, z_n)\mathrm{A}\} - i(\mathrm{Re}\, z_n)\mathrm{A}\\
&= (\mathrm{Re}\, z_n)\mathrm{A}\left[(\mathrm{Re}\, z_n)^{-1}\mathrm{A}^{-1}\{\mathrm{Id}+(\mathrm{Im}\, z_n)\mathrm{A}\} - i\,\mathrm{Id}\right].
\end{aligned}$$

Writing

$$\mathrm{M}_1 = \mathrm{Id} + (\mathrm{Im}\, z_n)\mathrm{A}, \quad \mathrm{M}_2 = (\mathrm{Re}\, z_n)^{-1}\mathrm{A}^{-1}\mathrm{M}_1,$$

we have

$$\mathrm{G} = (\mathrm{Re}\, z_n)\mathrm{A}(\mathrm{M}_2 - i\,\mathrm{Id}).$$

We note that the matrices A, M_1 and M_2 are commutable each other. Using

$$(\mathrm{M}_2 - i\,\mathrm{Id})(\mathrm{M}_2 + i\,\mathrm{Id}) = \mathrm{M}_2^2 + \mathrm{Id},$$

we obtain

$$\begin{aligned} G^{-1} &= (\operatorname{Re} z_n)^{-1} A^{-1} (M_2 - i \operatorname{Id})^{-1} \\ &= (\operatorname{Re} z_n)^{-1} A^{-1} (M_2 + i \operatorname{Id}) \left(M_2^2 + \operatorname{Id}\right)^{-1}. \end{aligned}$$

Since A, M_1 and M_2 are real symmetric matrices, we have

$$\operatorname{Im}\left(i G^{-1}\right) = \operatorname{Re}\left(G^{-1}\right) = (\operatorname{Re} z_n)^{-1} A^{-1} M_2 \left(M_2^2 + \operatorname{Id}\right)^{-1}. \tag{4.37}$$

Since A is a real symmetric matrix, there exists the matrix T with $\det T \neq 0$ such that $T^{-1}AT = D_1$, where D_1 is the diagonal matrix associated with the eigenvalues of A. We note that each eigenvalue of A is real and not zero. Using the common matrix T, the matrices A^{-1}, M_1 and M_2 are diagonalized as

$$\begin{aligned} &T^{-1}A^{-1}T = D_1^{-1}, \quad T^{-1}M_1T = \operatorname{Id} + (\operatorname{Im} z_n)D_1 = D_2, \\ &T^{-1}M_2T = (\operatorname{Re} z_n)^{-1}T^{-1}ATT^{-1}M_1T = (\operatorname{Re} z_n)^{-1}D_1^{-1}D_2 = (\operatorname{Re} z_n)^{-1}D_3. \end{aligned}$$

We also have

$$T^{-1}\left(M_2^2 + \operatorname{Id}\right)T = D_3^2 + \operatorname{Id} = D_4.$$

This consideration shows that $\operatorname{Im}(iG^{-1})$ is also diagonalized as

$$T^{-1} \operatorname{Im}\left(i G^{-1}\right) T = (\operatorname{Re} z_n)^{-2} D_1^{-1} D_3 D_4^{-1}. \tag{4.38}$$

We denote $d_{(1),j}$ $(1 \le j \le n)$ eigenvalues of A. In other words $d_{(1),j}$ are diagonal elements of the diagonal matrix D_1. For D_2, D_3 and D_4 we can define $d_{(2),j}, d_{(3),j}$ and $d_{(4),j}$, respectively. From $\det A \neq 0$, we have $d_{(1),j} \neq 0$. The calculation above shows

$$d_{(2),j} = 1 + (\operatorname{Im} z_n) d_{(1),j}, \quad d_{(3),j} = \left(d_{(1),j}\right)^{-1} d_{(2),j}, \quad d_{(4),j} = 1 + \left(d_{(3),j}\right)^2.$$

We have $d_{(2),j} \ge 1/2$ by making $|\operatorname{Im} z_n|$ small enough. From (4.38) the eigenvalues ν_j of the real symmetric matrix $\operatorname{Im}(iG^{-1})$ are given as

$$\nu_j = (\operatorname{Re} z_n)^{-2} \left(d_{(1),j}\right)^{-1} d_{(3),j} \left(d_{(4),j}\right)^{-1} = (\operatorname{Re} z_n)^{-2} \left(d_{(1),j}\right)^{-2} d_{(2),j} \left(d_{(4),j}\right)^{-1}.$$

It follows from $d_{(1),j} \neq 0$ and the expressions of $d_{(k),j}$ $(k = 2, 3, 4)$ that there exists a positive constant C such that $\nu_j \ge C(\operatorname{Re} z_n)^{-2}$. When $(z, x) \in E$ with $|z_n| \ge \delta_0$, our quadratic form can be estimated as

$$\left\langle \operatorname{Im}\left(i G^{-1}\right) v, v \right\rangle \ge \left(\min_{1 \le j \le n} \mu_j\right) |v|^2 \ge \frac{C}{(1 + |z_n|)^2} |v|^2 \quad \text{for} \quad v \in \mathbb{C}^n. \tag{4.39}$$

On the other hand (3.19), (4.26), (4.34) and Cauchy formula show

(4.40)

$$\begin{aligned}&\left|G^{-1}\nabla_x\left\{Z\left(z_n, y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right)\Big|_{y=\kappa(z,x)} - z_n \mathrm{A}\zeta\left(z_n; y, -\frac{\partial\varphi_0}{\partial y}(z', y)\right)\Big|_{y=\kappa(z,x)}\right\}\right|\\ &\leq C\|\mathrm{G}^{-1}\|\,|Z(z_n) - z_n\mathrm{A}\zeta(z_n)|\left|\frac{\partial\kappa}{\partial x}\right|\\ &\leq \frac{C(K_0, |\eta_0|)}{R^{\sigma_0}}\frac{1}{(1+|z_n|)^2}.\end{aligned}$$

In the same way we estimate the other term in (4.35). Now we proved the next Lemma:

Lemma 4.2. *When $m = 2$, there exists a positive constant C such that for all $(z, x) \in E$,*

$$\mathrm{Im}\,\frac{\partial^2\varphi}{\partial x^2}(z, x) \geq \frac{C}{(1+|z_n|)^2}\,\mathrm{Id}\,. \tag{4.41}$$

Next we study the case $m \geq 3$. Since we consider the operators with the principal parts associated with constant coefficients, we note that

$$Y(z_n; y, \eta) = \widetilde{Y}(z_n; y, \eta) = y + z_n\nabla_\xi\widetilde{p}_m(\eta). \tag{4.42}$$

We showed the existence and the uniqueness of the holomorphic map $y = \kappa(z, x)$ with $y_0 = \kappa(z_0, \eta_0)$, which is the inverse of (4.42). We define

$$X(z, x) = \eta\left(z', y\right)\big|_{y=\kappa(z,x)} = \eta_0 + i\begin{pmatrix} z' - z'_0 \\ 0\end{pmatrix} - i\left(\kappa(z, x) - y_0\right).$$

Writing $x = Y(z_n; y, \eta(z', y))$ by using $X(z, x)$, we have

$$x = z_n\nabla_\xi\widetilde{p}_m(X(z, x)) + iX(z, x) + y_0 - i\eta_0 + \begin{pmatrix} z' - z'_0 \\ 0\end{pmatrix}. \tag{4.43}$$

On the other hand we have

$$\frac{\partial\varphi}{\partial x}(z, x) = -\Theta\left(z_n; y, \eta\left(z', y\right)\right)\big|_{y=\kappa(z,x)} = -\eta\left(z', y\right)\big|_{y=\kappa(z,x)} = -X(z, x). \tag{4.44}$$

Differentiating (4.43) and (4.44), we have

$$\begin{aligned}\mathrm{Id} &= \nabla_x\left\{z_n\nabla_\xi\widetilde{p}_m\left(X(z, x)\right) + iX(z, x)\right\}\\ &= \left\{z_n\nabla_\xi^2\widetilde{p}_m\left(X(z, x)\right) + i\,\mathrm{Id}\right\}\nabla_x X(z, x),\end{aligned}$$

and

$$\nabla_x^2 \varphi(z,x) = \frac{\partial^2 \varphi}{\partial x^2}(z,x) = -\nabla_x X(z,x) = -i\left(\mathrm{Id} - i z_n \nabla_\xi^2 \widetilde{p}_m\,(X(z,x))\right)^{-1}.$$

We introduce the notation as the case $m = 2$

$$\begin{cases} \mathrm{G}_1 = \mathrm{Id} - i z_n \mathrm{A}(\eta), \\ \mathrm{A}(\eta) = \nabla_\xi^2 \widetilde{p}_m\,(X(z,x)). \end{cases}$$

The following arguments were exploited in [23] when $n = 1$. The set E was defined in Theorem 4.1 as

(4.45)
$$E = \Big\{ \left(z', z_n, x', x_n\right) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}\,; \\ \left|z' - z_0'\right| < \varepsilon_1,\ \mathrm{Re}\, z_n > -\varepsilon_1,\ |\mathrm{Im}\, z_n| < \varepsilon_1,\ |x - Y(z_n; y_0, \eta_0)| < \varepsilon_2(1 + |z_n|) \Big\}.$$

We set

(4.46)
$$E_1 = \left\{ \left(z', z_n, x', x_n\right) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{R}^{n-1} \times \mathbb{R}\,; \right. \\ \left. \left|z' - z_0'\right| < \varepsilon_1,\ \mathrm{Re}\, z_n > -\varepsilon_1,\ |\mathrm{Im}\, z_n| < \varepsilon_1,\ |x - Y(z_n; y_0, \eta_0)| < \frac{1}{2}\varepsilon_2(1 + |z_n|) \right\}.$$

and

(4.47)
$$\widetilde{E} = \bigcup_{(r_1, r_0)} \Big\{ \left(z', z_n, x', x_n\right) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{R}^{n-1} \times \mathbb{R}\,; \\ \left|z' - z_0'\right| < \varepsilon_1,\ \mathrm{Re}\, z_n > -\varepsilon_1,\ |\mathrm{Im}\, z_n| < \varepsilon_1,\ x = Y(z_n; y_0, \eta_0) + z_n r_1 + r_0 \Big\},$$

where the union for (r_1, r_0) is taken in $r_1 \in \mathbb{R}^n, r_0 \in \mathbb{C}^n$ with $|r_1| < \varepsilon_2, |r_0| < \varepsilon_2$, respectively. We know $E_1 \subset \widetilde{E}$ $(\subset (E \cap \{x \in \mathbb{R}^n\}))$. Let us prove the next estimate which is used in Section 6 to show the main theorem. There exists $C > 0$ such that we have

$$\mathrm{Im}\, \frac{\partial^2 \varphi}{\partial x^2}(z,x) \geq C \frac{1}{(1 + |z_n|)^2}\, \mathrm{Id}, \quad \text{for} \quad (z,x) \in \widetilde{E}. \tag{4.48}$$

The precise statement will be found in Lemma 4.3.

For $(z,x) \in \widetilde{E}$ the equation (4.43) becomes

$$z_n \nabla_\xi \widetilde{p}_m(X(z,x)) + i X(z,x) + y_0 - i\eta_0 \\ + \begin{pmatrix} z' - z_0' \\ 0 \end{pmatrix} - \{Y(z_n; y_0, \eta_0) + z_n r_1 + r_0\} = 0. \tag{4.49}$$

We set $t = z_n^{-1}$. Let us consider the next system of equations

$$
\begin{aligned}
F\left(W, t, z', r_1, r_0\right) &= \nabla_\xi \widetilde{p}_m(W) + itW - \left(\nabla_\xi \widetilde{p}_m(\eta_0) + r_1\right) \\
&\quad + t\left\{\begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} - i\eta_0 - r_0\right\} = 0,
\end{aligned} \tag{4.50}
$$

for $(W, t, z', r_0, r_1) \in \mathbb{C}^n \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^n \times \mathbb{C}^n$. Each elements of $F(W, t, z', r_0, r_1)$ is holomorphic function of (W, t, z', r_0, r_1), and $F(\eta_0, 0, z'_0, 0, 0) = 0$. Thanks to the implicit function theorem in holomorphic function category (for example see [14]), there exists a positive constant μ_0, and holomorphic functions $\psi = (\psi_1, \dots, \psi_n)$ such that $W = \psi(t, z', r_1, r_0)$ and $\eta_0 = \psi(0, z'_0, 0, 0)$ and $F(\psi(t, z', r_1, r_0), t, z', r_0, r_1) = 0$ for $(t, z', r_1, r_0) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}^n \times \mathbb{C}^n$ with $|t|, |z' - z'_0|, |r_1|, |r_0| < \mu_0$. Especially $\psi(t, z', r_1, r_0)$ is holomorphic in t. We can write

$$
W = \psi\left(t, z', r_1, r_0\right) = W_0\left(z', r_1, r_0\right) + tW_1\left(z', r_1, r_0\right) + R_2\left(t, z', r_1, r_0\right), \tag{4.51}
$$

and there exists a constant C such that for $|z' - z'_0| \le (1/2)\mu_0$, $|r_0| \le (1/2)\mu_0$ and $|r_1| \le (1/2)\mu_0$ we have

$$
\left|W - W_0\left(z', r_1, r_0\right) - tW_1\left(z', r_1, r_0\right)\right| \le C|t|^2, \quad |t| \le \frac{1}{2}\mu_0. \tag{4.52}
$$

So we can get the expansion

$$
X(z, x) = X_0 + \frac{1}{z_n}X_1 + Q_2, \tag{4.53}
$$

for $z_n \in \mathbb{C}$ with $|\operatorname{Im} z_n| < \varepsilon_1$ and $|z_n| > M$, where $M = 2\mu_0^{-1}$. The remainder term Q_2 is small uniformly. Let us find $X_0 = X_0(z', r_1, r_0)$ and $X_1 = X_1(z', r_1, r_0)$. Since we have

$$
\nabla_\xi \widetilde{p}_m(X(z, x)) = \nabla_\xi \widetilde{p}_m(X_0) + \nabla_\xi^2 \widetilde{p}_m(X_0)\left(\frac{1}{z_n}X_1 + \widetilde{R}_2\right) + \widetilde{R}_3,
$$

where R_2 and R_3 are in $\mathcal{O}(|z_n|^{-2})$, we obtain from (4.49)

$$
\begin{aligned}
&z_n \nabla_\xi \widetilde{p}_m(X_0) + \nabla_\xi^2 \widetilde{p}_m(X_0)\left(X_1 + z_n\widetilde{R}_2\right) + z_n\widetilde{R}_3 + i\left(X_0 + \frac{1}{z_n}X_1 + \widetilde{R}_2\right) \\
&- z_n\left(\nabla_\xi \widetilde{p}_m(\eta_0) + r_1\right) + \begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} - i\eta_0 - r_0 = 0.
\end{aligned}
$$

From (4.52) the remainder terms are in $\mathcal{O}(|z_n|^{-1})$. From this expansion we obtain

$$
\begin{aligned}
&\nabla_\xi \widetilde{p}_m(X_0) = \nabla_\xi \widetilde{p}_m(\eta_0) + r_1, \\
&\nabla_\xi^2 \widetilde{p}_m(X_0)X_1 + i\left(X_0 - i\begin{pmatrix} z' - z'_0 \\ 0 \end{pmatrix} - \eta_0 + ir_0\right) = 0.
\end{aligned}
$$

Now we restrict r_1 to $\mathbb{R}^n$. Since r_1 is a real vector and $\widetilde{p}_m(\xi)$ is real, there exist $\mu_1 > 0$ and $X_0 = X_0(\eta_0, r_1) \in \mathbb{R}^n$ by the implicit function theorem such that $X_1 \in \mathbb{C}^n$ with $|X_1| \le C\mu_1$ is also obtained as

$$X_1 = -i\nabla_\xi^2 \widetilde{p}_m(X_0)^{-1}\left(X_0 - i\begin{pmatrix} z' - z_0' \\ 0 \end{pmatrix} - \eta_0 + ir_0\right).$$

If $(z, x) \in \widetilde{E}$, we have

$$\begin{aligned} A(\eta) = \nabla_\xi^2 \widetilde{p}_m\left(X(z, x)\right) &= \nabla_\xi^2 \widetilde{p}_m(X_0) + \frac{1}{z_n}\left\langle X_1, \nabla_\xi\left(\nabla_\xi^2 \widetilde{p}_m\right)(X_0)\right\rangle + \widetilde{R}_2' \\ &= A_0 + \frac{1}{z_n}A_1 + \widetilde{R}_2', \end{aligned}$$

where $A_0 = \nabla_\xi \widetilde{p}_m(X_0)$ is the real symmetric matrix with $\det A_0 \ne 0$ and $|\operatorname{Im} A_1| < C\mu_1$. Writing

$$\mathrm{G} = \mathrm{Id} - iz_n\mathrm{A}(\eta) = -iz_n\left\{\mathrm{Id} + i\frac{1}{z_n}\mathrm{A}(\eta)^{-1}\right\}\mathrm{A}(\eta),$$

we have

$$\mathrm{G}^{-1} = i\frac{1}{z_n}\mathrm{A}(\eta)^{-1}\left\{\mathrm{Id} + i\frac{1}{z_n}\mathrm{A}(\eta)^{-1}\right\}^{-1}.$$

Since

$$\mathrm{A}(\eta) = \mathrm{A}_0\left\{\mathrm{Id} + \frac{1}{z_n}\mathrm{A}_0^{-1}\mathrm{A}_1 + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^2\right)\right\},$$

we obtain from the Neumann series

$$\begin{aligned} \mathrm{A}(\eta)^{-1} &= \left\{\mathrm{Id} + \frac{1}{z_n}\mathrm{A}_0^{-1}\mathrm{A}_1 + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^2\right)\right\}^{-1}\mathrm{A}_0^{-1} \\ &= \left\{\mathrm{Id} - \frac{1}{z_n}\mathrm{A}_0^{-1}\mathrm{A}_1 + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^2\right)\right\}\mathrm{A}_0^{-1} \\ &= \mathrm{A}_0^{-1} - \frac{1}{z_n}\mathrm{A}_0^{-1}\mathrm{A}_1\mathrm{A}_0^{-1} + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^2\right). \end{aligned}$$

From this expression we have

$$\begin{aligned} G^{-1} &= i\frac{1}{z_n}A(\eta)^{-1}\left\{\mathrm{Id} - i\frac{1}{z_n}A(\eta)^{-1} + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^2\right)\right\} \\ &= i\frac{1}{z_n}A(\eta)^{-1} + \frac{1}{z_n^2}\left(A(\eta)^{-1}\right)^2 + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^3\right) \\ &= i\frac{1}{z_n}A_0^{-1} + \frac{1}{z_n^2}\left\{\left(A_0^{-1}\right)^2 - iA_0^{-1}A_1A_0^{-1}\right\} + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^3\right). \end{aligned}$$

Since A_0^{-1} is also the real symmetric matrix with $\det A_0^{-1} \neq 0$, we have

$$\begin{aligned} &\left(i\frac{1}{z_n}A_0^{-1}\right) + \left(i\frac{1}{z_n}A_0^{-1}\right)^* = i\frac{1}{z_n}A_0^{-1} - i\frac{1}{\overline{z_n}}\left(A_0^{-1}\right)^* \\ &= \frac{2\,\mathrm{Im}\,z_n}{|z_n|^2}A_0^{-1}, \end{aligned}$$

and

$$\begin{aligned} &\left\{\frac{1}{z_n^2}\left(A_0^{-1}\right)^2\right\} + \left\{\frac{1}{z_n^2}\left(A_0^{-1}\right)^2\right\}^* = \left(\frac{1}{z_n^2} + \frac{1}{\overline{z_n}^2}\right)\left(A_0^{-1}\right)^2 \\ &= \frac{2\left\{(\mathrm{Re}\,z_n)^2 - (\mathrm{Im}\,z_n)^2\right\}}{|z_n|^4}\left(A_0^{-1}\right)^2. \end{aligned}$$

By writing $B = A_0^{-1}A_1A_0^{-1}$, it follows from $B^* = A_0^{-1}A_1^*A_0^{-1}$ that we have

$$\begin{aligned} &\left\{\frac{1}{z_n^2}(-iB)\right\} + \left\{\frac{1}{z_n^2}(-iB)\right\}^* \\ &= -i\frac{1}{z_n^2}B + i\frac{1}{\overline{z_n}^2}B^* \\ &= -i\frac{\overline{z_n}^2}{|z_n|^4}A_0^{-1}A_1A_0^{-1} + i\frac{z_n^2}{|z_n|^4}A_0^{-1}A_1^*A_0^{-1} \\ &= -i\frac{(\mathrm{Re}\,z_n)^2}{|z_n|^4}A_0^{-1}\left(A_1 - A_1^*\right)A_0^{-1} + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^3\right) \\ &= \frac{2(\mathrm{Re}\,z_n)^2}{|z_n|^4}A_0^{-1}\left(\mathrm{Im}\,A_1\right)A_0^{-1} + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^3\right). \end{aligned}$$

The positivity of the matrix $\left(A_0^{-1}\right)^2$ shows

$$\begin{aligned}
&\left\langle \operatorname{Im} \frac{\partial^2\varphi}{\partial x^2}(z,x)v,\, v\right\rangle = \left\langle \left(\operatorname{Re} G^{-1}\right) v,\, v\right\rangle \\
&= \left\langle \frac{1}{2}\left\{G^{-1} + \left(G^{-1}\right)^*\right\} v,\, v\right\rangle \\
&\geq -\frac{|\operatorname{Im} z_n|}{|z_n|^2}\left\langle A_0^{-1}v,\, v\right\rangle + \frac{(\operatorname{Re} z_n)^2}{|z_n|^4}\left\langle \left(A_0^{-1}\right)^2 v,\, v\right\rangle \\
&\quad - \frac{(\operatorname{Re} z_n)^2}{|z_n|^4}\left\langle A_0^{-1}(\operatorname{Im} A_1)A_0^{-1}\right\rangle + \mathcal{O}\left(\left|\frac{1}{z_n}\right|^3\right)\langle v,\, v\rangle \\
&\geq \frac{1}{2}\frac{(\operatorname{Re} z_n)^2}{|z_n|^4} C\langle v,\, v\rangle.
\end{aligned} \tag{4.54}$$

Therefore we prove that there exists a positive constant $C > 0$ such that for $(z,x) \in \widetilde{E}$, $|z_n| > M$ we have

$$\left\langle \operatorname{Im} \frac{\partial^2\varphi}{\partial x^2}(z,x)v,\, v\right\rangle \geq \frac{C}{1+|z_n|^2}\langle v,\, v\rangle. \tag{4.55}$$

The positive constant $M = M(\mu_1)$ is independent of $\varepsilon_1, \varepsilon_2$ and ε_3. We consider the case $M \geq |z_n|$. For $(z,x) \in \widetilde{E}$ we have $|X(z,x) - \eta_0| < \varepsilon_3$. Since we have

$$\left\| z_n \nabla_\xi^2 \widetilde{p}_m(X(z,x)) - (\operatorname{Re} z_n)\nabla_\xi^2 \widetilde{p}_m(\eta_0)\right\| < \varepsilon_1 + \varepsilon_3,$$

for small $\varepsilon_1 > 0$ and $\varepsilon_3 > 0$ with $\varepsilon_1 < \mu_1$, $\varepsilon_3 < \mu_1$, it follows from the expression $\nabla_x^2\varphi(z,x) = -i\left(\operatorname{Id} - i z_n \nabla_\xi^2 \widetilde{p}_m(X(z,x))\right)^{-1}$ that we obtain

$$\left\langle \operatorname{Im} \frac{\partial^2\varphi}{\partial x^2}(z,x)v,\, v\right\rangle \geq \frac{1}{2}\|v\|^2, \tag{4.56}$$

for $(z,x) \in \widetilde{E}$ with $|z_n| \leq M$. Thus we have the following lemma:

Lemma 4.3. *Let $m \geq 3$. We assume that the coefficients of the principal part of $P(x, D_x)$ are independent of x. Then there exists a positive constant C such that for $(z,x) \in E_1$ $(\subset \mathbb{C}^n \times \mathbb{R}^n)$ we have*

$$\operatorname{Im} \frac{\partial^2\varphi}{\partial x^2}(z,x) \geq \frac{C}{(1+|z_n|)^2}\operatorname{Id}. \tag{4.57}$$

Let $(z,x) \in E_1 \subset \mathbb{C}^n \times \mathbb{R}^n$. We set $g(z,x) = \operatorname{Im}(\partial\varphi/\partial x)(z,x)$. Since $g\left(\left(z_0', \operatorname{Re} z_n\right), Y(\operatorname{Re} z_n; y_0, \eta_0)\right) = \operatorname{Im}\Theta(\operatorname{Re} z_n; y_0, \eta_0) = 0$, it follows from Lemma 4.2

or Lemma 4.3 that there exists a function $x(z)\colon \mathbb{C}^n \to \mathbb{R}^n$ such that

$$g\,(z, x(z)) = \operatorname{Im} \frac{\partial \varphi}{\partial x}\,(z, x(z)) = 0,$$
$$x\left(z_0', \operatorname{Re} z_n\right) = Y(\operatorname{Re} z_n; y_0, \eta_0).$$

The existence and uniqueness of $x(z)$ come from the repeated use of the implicit function theorem. In order to justify this argument, we show the estimate

$$|x(z) - Y(\operatorname{Re} z_n; y_0, \eta_0)| < \frac{1}{4}\varepsilon_2(1 + |z_n|), \tag{4.58}$$

which guarantees the repeated use of (4.57).

Since $x(z)$ is real valued and φ is holomorphic, the differentiation of the equation

$$0 = \operatorname{Im} \frac{\partial \varphi}{\partial x}\,(z, x(z)) = \frac{1}{2i}\left(\frac{\partial \varphi}{\partial x}\,(z, x(z)) - \overline{\frac{\partial \varphi}{\partial x}\,(z, x(z))}\right),$$

implies

$$\frac{\partial x(z)}{\partial z} = -\frac{1}{2i}\left(\operatorname{Im} \frac{\partial^2 \varphi}{\partial x^2}\,(z, x(z))\right)^{-1} \frac{\partial^2 \varphi}{\partial z \partial x}\,(z, x(z))\,.$$

Since Lemma 4.2 and Lemma 4.3 show

$$\left\| \left(\operatorname{Im} \frac{\partial^2 \varphi}{\partial x^2}\,(z, x(z))\right)^{-1} \right\| \le C(1 + |z_n|)^2,$$

and the differentiation of the equation

$$\frac{\partial \varphi}{\partial z}(z, x) = G(0) = \left(\frac{\partial \varphi_0}{\partial z'}(z', y),\, p_m\left(y, -\frac{\partial \varphi_0}{\partial y}(z', y)\right)\right)\Bigg|_{y=\kappa(z,x)},$$

we obtain from (4.34) that for $z \in \mathbb{C}^n$ with $\left|z' - z_0'\right| < \varepsilon_1$, $\operatorname{Re} z_n > -\varepsilon_1$ and $|\operatorname{Im} z_n| < \varepsilon_1$

$$\left\| \frac{\partial x(z)}{\partial z} \right\| \le C(1 + |z_n|).$$

By the Taylor expansion

$$x(z) - x\left(z_0', \operatorname{Re} z_n\right) = \frac{\partial x}{\partial z}(z^*) \begin{pmatrix} z' - z_0' \\ z_n - \operatorname{Re} z_n \end{pmatrix},$$

we have

$$|x(z) - Y(\operatorname{Re} z_n; y_0, \eta_0)| < \varepsilon_1(1 + |z_n|).$$

By the choices of ε_1 and ε_2 with $0 < 4\varepsilon_1 < \varepsilon_2$, we obtain (4.58) and the argument about the repeated use of the implicit function theorem has been justified.

For $z \in \mathbb{C}^n$ with $\left|z' - z_0'\right| < \varepsilon_1$, $\operatorname{Re} z_n > -\varepsilon_1$ and $|\operatorname{Im} z_n| < \varepsilon_1$ we define

$$\Phi(z) = -\operatorname{Im} \varphi(z, x(z)). \tag{4.59}$$

We can show

$$\frac{\partial}{\partial \operatorname{Re} z} \Phi(z) = 0. \tag{4.60}$$

We do not repeat the proof for (4.60) (see Lemma 3.8 in [21]).

5. Solving the transport equations

In the previous section we constructed the phase function globally along the bicharacteristics. Now we solve the transport equations. We construct the amplitude function

$$f(z, x, \lambda) = \sum_{k=0}^{\infty} \frac{1}{\lambda^k} f_k(z, x), \tag{5.1}$$

as an analytic symbol of order zero, elliptic near the support of the cutoff function along the bicharacteristics.

Theorem 5.1. *Let* $F(z, x, \lambda)$ *be introduced in* (4.3), *and* $\Phi(z)$ *be defined in* (4.59). *There exist* $r_\infty > 0$ *and an analytic symbol* $f = f(z, x, \lambda)$ *of order zero, elliptic, defined in the set*

$$E = \Big\{ \left(z', z_n, x', x_n\right) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}; \\ \left|z' - z_0'\right| < \frac{1}{2} r_\infty,\ \operatorname{Re} z_n > -\frac{1}{2} r_\infty,\ |\operatorname{Im} z_n| < \frac{1}{2} r_\infty, \\ |x - Y(z_n; y_0, \eta_0)| < \frac{1}{2} r_\infty (1 + |z_n|) \Big\},$$

such that we have

$$|F(z, x, \lambda)| \leq C e^{\lambda \Phi(z) - \mu_0 \lambda} (1 + |z_n|)^{N_0}, \tag{5.2}$$

where $\mu_0 > 0$ *and* $N_0 \in \mathbb{N}$.

In order to prove Theorem 5.1, we change the variables and the unknown function f. We define the change of variables $\Psi \colon \mathcal{O} \to E$ by

$$(z, x) = \left(z, Y\left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y)\right)\right) \quad \text{for} \quad (z, y) \in \mathcal{O}, \tag{5.3}$$

where the set $\mathcal{O}$ is given by

(5.4)
$$O = \left\{(z, y) \in \mathbb{C}^n \times \mathbb{C}^n\, ;\ \left|z' - z_0'\right| < \varepsilon_1,\ \operatorname{Re} z_n > -\varepsilon_1,\ |\operatorname{Im} z_n| < \varepsilon_1,\ |y - y_0| < \varepsilon_3\right\},$$

and the set E is given by (4.5). We define

$$\begin{aligned} g(z, y, \lambda) &= \sum_{k=0}^{\infty} \frac{1}{\lambda^k} g_k(z, y) = f \circ \Psi \\ &= f\left(z, Y\left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y)\right)\right) \\ &= \sum_{k=0}^{\infty} \frac{1}{\lambda^k} f_k\left(z, Y\left(z_n; y, -\frac{\partial \varphi_0}{\partial y}(z', y)\right)\right). \end{aligned} \tag{5.5}$$

Since we have from (4.16)

$$\frac{\partial}{\partial z_n} g(z, y) = \left\{\frac{\partial}{\partial z_n} - \sum_{l=1}^{n} \frac{\partial^t p_m}{\partial \xi_l}\left(x, \frac{\partial \varphi}{\partial x}(z, x)\right)\frac{\partial}{\partial x_l}\right\} f(z, x)\Bigg|_{x=Y(z_n; y, \eta(z', y))}, \tag{5.6}$$

we obtain

$$i\lambda e^{-i\lambda\varphi} F \circ \Psi = \left(\frac{\partial}{\partial z_n} + d(z, y)\right) g - \sum_{l=1}^{m-1} \frac{1}{\lambda^l} Q_l(z, y, D_y) g, \tag{5.7}$$

where

$$\begin{cases} |d(z, y)| \le \dfrac{C}{1 + |z_n|}, \\ Q_l(z, y, D_y) = \displaystyle\sum_{|\alpha| \le l+1} q_\alpha^l(z, y) D_y^\alpha, \\ \left|q_\alpha^l(z, y)\right| \le \dfrac{C_{l,\alpha}}{(1 + |z_n|)^{1+\sigma_0}}. \end{cases} \tag{5.8}$$

Next we change the unknown functions $g_k(z, y)$ into $h_k(z, y)$ by

$$\begin{cases} A(z, y) = \partial_{z_n}^{-1} d(z, y) = z_n \displaystyle\int_0^1 d(z', t z_n, y)\, dt, \\ h_k(z, y) = e^{A(z, y)} g_k(z, y). \end{cases}$$

Then the analytic symbol $h = \sum \lambda^{-k} h_k(z, y)$ should be constructed by the equations

$$\begin{cases} \dfrac{\partial h_0}{\partial z_n} = 0, \quad h_0|_{z_n=0} = 1, \\ \dfrac{\partial h_k}{\partial z_n} = \displaystyle\sum_{l=1}^{M} W_l(z, y, D_y) h_{k-l}, \quad h_k|_{z_n=0} = 0, \quad (k \ge 1), \end{cases} \tag{5.9}$$

where

$$M = \begin{cases} k, & (k \leq m-1), \\ m-1, & (k \geq m), \end{cases}$$

and

$$\begin{cases} W_l = e^A Q_l e^{-A} = \displaystyle\sum_{|\alpha| \leq l+1} w_\alpha^l(z, y) D_y^\alpha, \\ \left| w_\alpha^l(z, y) \right| \leq \dfrac{C_{j,\alpha}}{(1+|z_n|)^{1+\sigma_0}} (\log(1+|z_n|))^{l+1-|\alpha|}. \end{cases} \tag{5.10}$$

The solutions of the equations (5.9) are given by

$$\begin{cases} h_0(z, y) = 1, \\ h_k(z, y) = z_n \displaystyle\int_0^1 \sum_{l=1}^{M} W_l h_{k-l}(z', t z_n, y)\, dt, \quad (k \geq 1). \end{cases} \tag{5.11}$$

Let us check the estimates

$$|g_k(z, y)| \leq C_0 C_1^k k^k, \tag{5.12}$$

which shows that the existence and the boundedness of the symbol $g(z, y, \lambda)$. The technique, "nested open set," introduced in [21] and [26] is useful in proving these estimates (see also Section 3 in [23]). For $j \in (0 \cup \mathbb{N})$ we define the sequences $\{s_j\}$, $\{R_j\}$, $\{r_j\}$ by

$$\begin{cases} s_0 = 0, \quad R_0 = 2^{(1/4)\sigma_0}, & r_0 \quad \text{given in} \quad (0, \varepsilon_1), \\ s_j = 2^j, \quad R_j = 2^{j(1+(1/2)\sigma_0)}, & j \geq 1, \\ r_j = r_{j-1} - \dfrac{r_{j-1}}{R_{j-1}}(s_j - s_{j-1}), & j \geq 1. \end{cases} \tag{5.13}$$

As proved in [21], the monotone decreasing sequence $\{r_j\}_{j=0}^\infty$ converges to $r_\infty > 0$. We define the open set

$$\Omega_t^j = \left\{(z, y) \in \mathbb{C}^n \times \mathbb{C}^n \,;\, \left|z' - z_0'\right| + M_j(z_n) + |y - y_0| < r_j - t, \ \operatorname{Re} z_n \geq s_j \right\}, \tag{5.14}$$

where $t \in (0, r_j]$ and

$$M_j(z_n) = \frac{r_j}{R_j} \left|\operatorname{Re} z_n - s_j\right| + |\operatorname{Im} z_n|.$$

We note that $(z', s_j, y) \in \Omega_t^j$ implies $(z', s_j, y) \in \Omega_t^{j-1}$ for $j \geq 1$. Let ρ be a positive number. We denote by $A_{\rho,j}$ the space of formal analytic symbols $h(z, y, \lambda) =$

$\sum_{k\geq 0}\lambda^{-k}h_k(z,y,\lambda)$ such that

$$\sup_{(z,y)\in\Omega_t^j}|h_k(z,y)|\leq f_{k,j}(h)k^k t^{-k},\quad 0<t\leq r_j, \tag{5.15}$$

where $f_{k,j}(h)$ is the best constant and the series $\sum_{k=0}^{\infty} f_{k,j}(h)\rho^k$ is convergent. This definition shows that

$$\begin{aligned}\|h\|_{\rho,j} &= \sum_{k=0}^{\infty} f_{k,j}(h)\rho^k\\ &= \sum_{k=0}^{\infty}\frac{\rho^k}{k^k}\sup_{0<t\leq r_j}\left\{t^k\sup_{(z,y)\in\Omega_t^j}|h_k(z,y)|\right\},\end{aligned} \tag{5.16}$$

is a norm on $A_{\rho,j}$. The solution of (5.9) given by (5.11) gives rise to the formal symbol $h(z,y,\lambda)=\sum_{k\geq 0}\lambda^{-k}h_k(z,y)$ which is a solution of the Cauchy problem

$$\begin{cases}\dfrac{\partial h}{\partial z_n}-\displaystyle\sum_{l=1}^{m-1}\lambda^{-l}W_l h=0, & \operatorname{Re} z_n\geq s_j,\\ h|_{z_n=s_j}=h(z',s_j,y). & \end{cases} \tag{5.17}$$

We denote by h^j the value of the solution in Ω_0^j. Then h^j satisfies

$$\begin{cases}\dfrac{\partial h^j}{\partial z_n}-\displaystyle\sum_{l=1}^{m-1}\lambda^{-l}W_l h^j=0,\\ h^j|_{z_n=s_j}=h^{j-1}|_{z_n=s_j},\end{cases}$$

where we define $h^{-1}|_{z_n=0}=1$. We set $\beta^j=h^j-h^{j-1}|_{z_n=s_j}$. The system can be written

$$\begin{cases}(\mathrm{Id}-B)\beta^j=B\left(h^{j-1}|_{z_n=s_j}\right),\\ \beta^j|_{z_n=s_j}=0,\end{cases} \tag{5.18}$$

where

$$B=\partial_{z_n}^{-1}\sum_{l=1}^{m-1}\lambda^{-l}W_l(z,y,D_y)=\lambda\partial_{z_n}^{-1}\sum_{l=1}^{m-1}\lambda^{-(l+1)}W_l(z,y,D_y), \tag{5.19}$$

and $\partial_{z_n}^{-1}v=\int_{s_j}^{z_n}v(z',\tau,y)\,d\tau$.

As proved in [23] we have the following lemma.

Lemma 5.1. *There exists a positive constant C such that for $j\geq 0$ we have*

$$\left\|\lambda\partial_{z_n}^{-1}\gamma\right\|_{\rho,j}\leq\frac{C}{\rho}\frac{R_j}{r_j}\|\gamma\|_{\rho,j},\quad \textit{for}\quad \gamma=\sum_{k=1}^{\infty}\lambda^{-(k+1)}\gamma_{k+1}\in A_{\rho,j}, \tag{5.20}$$

$$\left\| \frac{1}{\lambda} \frac{\partial}{\partial y_s} \gamma \right\|_{\rho,j} \le \rho \|\gamma\|_{\rho,j}, \quad \textit{for} \quad \gamma \in A_{\rho,j}, \quad 1 \le s \le n, \tag{5.21}$$

$$\left\| \frac{1}{\lambda} \gamma \right\|_{\rho,j} \le C\rho \|\gamma\|_{\rho,j}, \quad \textit{for} \quad \gamma \in A_{\rho,j}. \tag{5.22}$$

It follows from Lemma 5.1 and (5.10) that we have

$$\begin{aligned} \|B\beta\|_{\rho,j} &= \left\| \lambda \partial_{z_n}^{-1} \sum_{l=1}^{m-1} \lambda^{-(l+1)} W_l(z, y, D_y)\beta \right\|_{\rho,j} \\ &\le \frac{C}{\rho} \frac{R_j}{r_j} \left\| \sum_{l=1}^{m-1} \lambda^{-(l+1)} W_l(z, y, D_y)\beta \right\|_{\rho,j} \\ &\le \frac{C}{\rho} \frac{R_j}{r_j} \left\| \sum_{l=1}^{m-1} \sum_{|\alpha| \le l+1} w_\alpha^l(z, y) \lambda^{-(l+1)} D_y^\alpha \beta \right\|_{\rho,j} \\ &\le \frac{C}{\rho} \frac{R_j}{r_j} \sup_{(z,y)\in\Omega_0^j} \left| w_\alpha^l(z, y) \right| \left\| \sum_{l=1}^{m-1} \sum_{|\alpha| \le l+1} \lambda^{-(l+1)} D_y^\alpha \beta \right\|_{\rho,j} \\ &\le \frac{C}{\rho} \frac{R_j}{r_j} \left(\sup_{(z,y)\in\Omega_0^j, l, \alpha} \left| w_\alpha^l(z, y) \right| \right) C\rho^2 \|\beta\|_{\rho,j}, \end{aligned} \tag{5.23}$$

when $\rho >$ is small enough. For $j \ge 1$ we have

$$\sup_{(z,y)\in\Omega_0^j, l, \alpha} \left| w_\alpha^l(z, y) \right| \le \frac{C}{(1 + |s_j|)^{1+\sigma_0}} \left(\log(1 + |s_j|) \right)^{l+1-|\alpha|},$$

and

$$\sup_{(z,y)\in\Omega_0^0, l, \alpha} \left| w_\alpha^l(z, y) \right| \le C_0.$$

We define the sequence $\{K_j\}$ as

$$K_j = C \frac{(\log(1 + s_j))^m}{(1 + s_j)^{1+\sigma_0}} \frac{R_j}{r_j} \rho, \quad \text{for} \quad j \ge 1,$$

and $K_0 = C R_0 r_0^{-1} C_0 \rho$. If we choose a small enough ρ, we have

$$\begin{cases} K_j \le \dfrac{C\rho}{r_\infty} < 1, \\ K_j \le C 2^{-(1/3m)\sigma_0 j}, \end{cases}$$

for $j \geq 1$ and $K_0 < 1$. From (5.18) we have

$$\|\beta^j\|_{\rho,j} \leq K_j \|\beta^j\|_{\rho,j} + K_j \left\| \left(h^{j-1}\big|_{x=s_j} \right) \right\|_{\rho,j},$$

so

$$\|\beta^j\|_{\rho,j} \leq \frac{K_j}{1-K_j} \left\| \left(h^{j-1}\big|_{x=s_j} \right) \right\|_{\rho,j} \quad \text{for} \quad j \geq 0. \tag{5.24}$$

Since $(z', s_j, y) \in \Omega_t^j$ implies $(z', s_j, y) \in \Omega_t^{j-1}$, we have

$$\left| h_k^{j-1}(z', s_j, y) \right| \leq \sup_{(z,y)\in\Omega_t^{j-1}} \left| h_k^{j-1}(z, y) \right| \leq f_{k,j-1}\left(h^{j-1}\right) k^k t^{-k},$$

for $0 < t \leq r_j$ $(< r_{j-1})$, and

$$\begin{aligned}
\left\| \left(h^{j-1}\big|_{z_n=s_j} \right) \right\|_{\rho,j} &= \sum_{k=0}^{\infty} \frac{\rho^k}{k^k} \sup_{0<t\leq r_j} \left\{ t^k \sup_{(z,y)\in\Omega_t^j} \left| h_k^{j-1}(z', s_j, y) \right| \right\} \\
&\leq \sum_{k=0}^{\infty} \frac{\rho^k}{k^k} \sup_{0<t\leq r_j} \left\{ t^k f_{k,j-1}\left(h^{j-1}\right) k^k t^{-k} \right\} \\
&= \sum_{k=0}^{\infty} \rho^k f_{k,j-1}(h^{j-1}) \\
&= \left\| h^{j-1} \right\|_{\rho,j-1}.
\end{aligned}$$

Since $\beta^j = h^j - h^{j-1}|_{z_n=s_j}$ and $h|_{z_n=0}^{-1} = 0$, we obtain

$$\|h^j\|_{\rho,j} \leq \frac{1}{1-K_j} \left\| h^{j-1} \right\|_{\rho,j-1},$$

and

$$\|h^j\|_{\rho,j} \leq \left(\prod_{l=1}^{j} \frac{1}{1-K_l} \right) \left\| h^0 \right\|_{\rho,0} \leq C \left\| h^0 \right\|_{\rho,0} \leq C. \tag{5.25}$$

Since h_k^j is the restriction of the solution h_k to Ω_t^j, we obtain

$$|h_k(z, y)| \leq \sup_{\Omega^j_{(1/2)r_\infty}} |h_k| \leq C\rho^{-k} k^k \left(\frac{r_\infty}{2} \right)^{-k}.$$

This shows that the system (5.9) has a solution $h(z, y, \lambda) = \sum_{k\geq 0} \lambda^{-k} h_k(z, y)$ such that for $k \geq 0$ we have

$$|h_k(z, y)| \leq C^k k^k, \tag{5.26}$$

in the set $\mathcal{O}_2 = \{(z, y) \in \mathbb{C}^n \times \mathbb{C}^n \,;\, |z' - z_0'| + |y - y_0| + |\mathrm{Im}\, z_n| \le (1/2) r_\infty, \mathrm{Re}\, z_n \ge 0\}$.

From the definitions of $A(z, y), h_k(z, y)$ and the estimates above, the symbol $g(z, y, \lambda) = \sum_{k \ge 0} \lambda^{-k} g_k(z, y)$ satisfies

$$|g_k(z, y)| \le C^k k^k (1 + |z_n|)^{N_1}, \quad (z, y) \in \mathcal{O}_2, \tag{5.27}$$

where N_1 is a fixed integer. Let us take $g = \sum_{k=0}^{K} g_k(z, y)$ and $K \in \mathbb{N}$ is chosen later. It follows from (5.7), (5.27) and Cauchy's formula that we have

$$\begin{aligned} \left| i\lambda e^{-i\lambda\varphi} F \circ \Psi \right| &\le \frac{1}{\lambda^{K+1}} C^K K^K (1 + |z_n|)^{N_1} \\ &\le \frac{1}{\lambda} \left(\frac{CK}{\lambda} \right)^K (1 + |z_n|)^{N_1}. \end{aligned} \tag{5.28}$$

We define k_0 and λ_0 with $0 < k_0 \le 1/(2C)$ and $\lambda_0 \ge 2/k_0$ respectively. Choosing the integer $K = K(\lambda)$ with $k_0\lambda_0 - 1 < K = [k_0\lambda] \le k_0\lambda$ for $\lambda \ge \lambda_0$, there exists a positive constant μ_1 such that we have

$$\log \left(\frac{CK}{\lambda} \right) \le -\mu_1 < 0.$$

This shows

$$\begin{aligned} \left| i\lambda e^{-i\lambda\varphi} F \circ \Psi \right| &\le \frac{1}{\lambda} e^{K \log(CK/\lambda)} (1 + |z_n|)^{N_1} \\ &\le \frac{1}{\lambda} e^{-\mu_1 [k_0 \lambda]} (1 + |z_n|)^{N_1} \\ &\le \frac{e^{\mu_1}}{\lambda} e^{-\mu_1 k_0 \lambda} (1 + |z_n|)^{N_1}. \end{aligned} \tag{5.29}$$

The proof of Theorem 5.1 is completed by setting a positive number $\mu_0 = \mu_1 k_0$ and making λ_0 large enough. □

6. Proof of the main theorem

Let $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ given in Theorem 1.1. Thanks to Lemma 2.1 and Lemma 2.2, for $\rho_1 = (y_1, \eta_1) \in T^*\mathbb{R}^n \setminus 0$ which is constructed in Lemma 2.2, the property $\rho_1 = (y_1, \eta_1) \notin \widetilde{WF}_A[u(t_0, \cdot)]$ shows our conclusion. Instead of ρ_1, we use the same notation $\rho_0 = (y_0, \eta_0) \in T^*\mathbb{R}^n \setminus 0$ with $|y_0| > 2R$, $\langle y_0, \nabla_\xi \widetilde{p}_m(\eta_0) \rangle \ge 0$ and $\det \nabla_\xi^2 \widetilde{p}_m(\eta_0)$. Let $\varphi = \varphi(z, x)$ be the phase function which is constructed in Theorem 4.1 and $f = f(z, x, \lambda)$ be the amplitude function given in Theorem 5.1. For a small enough constant $\varepsilon_0 > 0$, they are defined in the set

(6.1)

$$\begin{aligned} E_0 = \Big\{ & \big(z', z_n, x', x_n \big) \in \mathbb{C}^{n-1} \times \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C} \,; \\ & \left| z' - z_0' \right| < \varepsilon_0, \ \ \mathrm{Re}\, z_n > -\varepsilon_0, \ \ |\mathrm{Im}\, z_n| < \varepsilon_1, \ \ |x - Y(z_n; y_0, \eta_0)| < \varepsilon_0 (1 + |z_n|) \Big\}. \end{aligned}$$

and we have $e^{\delta_0|x|^{1/(m-1)}}u_0 \in L^2\left(\gamma^+_{\varepsilon_0,\rho_0}\right)$.

Let $\chi \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq \chi \leq 1$ and

$$\chi(t) = \begin{cases} 1, & |t| \leq \dfrac{1}{2}\varepsilon_0, \\ 0, & |t| \geq \varepsilon_0. \end{cases}$$

In Section 4 we defined the operator $Su(t, z, \lambda)$. Now we choose χ_1 in (4.1) as $\chi_1(z, x) = \chi((x - Y(\operatorname{Re} z_n; y_0, \eta_0))/(1 + |z_n|))$, that is,

$$Su(t, z, \lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(z,x)} f(z, x, \lambda)\chi\left(\frac{x - Y(\operatorname{Re} z_n; y_0, \eta_0)}{1 + |z_n|}\right) u(t, x)\, dx, \tag{6.2}$$

where $\varphi(z, x)$ and $f(z, x, \lambda)$ are given in Theorem 4.1 and Theorem 5.1, respectively. Let $u(t, \cdot)$ be the solution to the initial value problem

$$\begin{cases} D_t u + P(x, D_x)u = 0, \quad t < 0, \quad x \in \mathbb{R}^n, \\ u|_{t=0} = u_0(x). \end{cases}$$

Taking the integral transform S, we obtain

$$\begin{cases} SD_t u + SP(x, D_x)u = 0, \\ Su|_{t=0} = Su_0. \end{cases}$$

Using the notation in Section 4, we have

$$\begin{cases} \left(\dfrac{\partial}{\partial t} + \lambda^{m-1}\dfrac{\partial}{\partial z_n}\right) Su(t, z, \lambda) = i\lambda^m I(t, z, \lambda), \\ Su(t, z, \lambda)|_{t=0} = Su_0(z, \lambda). \end{cases}$$

This equation is solved as

$$S(t, z, \lambda) = Su_0\left(z', z_n - \lambda^{m-1}t, \lambda\right) + i\lambda^m \int_0^t I\left(\sigma, \left(z', z_n + \lambda^{m-1}(\sigma - t), \lambda\right)\right) d\sigma. \tag{6.3}$$

for $t < 0$. We have

$$\begin{aligned} &Su_0\left(z', z_n - \lambda^{m-1}t, \lambda\right) \\ &= \int_{\mathbb{R}^n} e^{i\lambda\varphi\left(x, \left(z', z_n - \lambda^{m-1}t\right)\right)} f\left(x, \left(z', z_n - \lambda^{m-1}t\right), \lambda\right) \\ &\qquad \times \chi\left(\frac{x - Y(\operatorname{Re} z_n - \lambda^{m-1}t; y_0, \eta_0)}{1 + |z_n - \lambda^{m-1}t|}\right) e^{-(\delta_0/4)|x|^{1/(m-1)}} e^{(\delta_0/4)|x|^{1/(m-1)}} u_0(x)\, dz. \end{aligned}$$

On the support of χ, there exists $\mu > 0$ such that

$$\left|x - Y\left(z_n - \lambda^{m-1}t; y_0, \eta_0\right)\right| \leq \varepsilon_0\left(1 + \left|z_n - \lambda^{m-1}t\right|\right) \leq C\varepsilon_0\left(1 + \lambda^{m-1}t_0\right),$$

for $z \in \mathbb{C}^n$ with $\left|z' - z_0'\right| \leq \mu,\ |z_n| \leq \mu$, and $|t - t_0| < \mu$. On the other hand (3.11) shows

$$\left|Y\left(\operatorname{Re} z_n - \lambda^{m-1}t; y_0, \eta_0\right)\right| \geq R + \left|\operatorname{Re} z_n - \lambda^{m-1}t\right| \left|\nabla_\xi \widetilde{p}_m(\eta_0)\right| \geq \frac{R}{2} + C\lambda^{m-1}|\eta_0|^{m-1}|t_0|.$$

By making $\mu > 0$ small enough and R large enough, we have $|x| \geq \delta_0|\eta_0|^{m-1}|t_0|\lambda^{m-1}$. We note that (4.60) shows $\Phi(z', z_n - \lambda^{m-1}t) = \Phi(z)$. Theorem 5.1 implies that $|f(z', z - \lambda^{m-1}t, x, \lambda)|$ is bounded. It follows from these properties that we have

$$\begin{aligned}
&\left|Su_0\left(z', z_n - \lambda^{m-1}t, \lambda\right)\right| \\
&\leq Ce^{\lambda\Phi(z) - (1/16)|\eta_0|\,|t_0|^{1/(m-1)}\delta_0\lambda} \int_{\mathbb{R}^n} \left|\chi(\cdots)e^{(\delta_0/4)|x|^{1/(m-1)}}u_0(x)\right| dz \qquad (6.4)\\
&\leq Ce^{\lambda\Phi(z) - (1/16)|\eta_0|\,|t_0|^{1/(m-1)}\delta_0\lambda} \left\|e^{\delta_0|x|^{1/(m-1)}}u_0\right\|_{L^2\left(\Gamma^+_{\rho_0,\varepsilon_0}\right)}.
\end{aligned}$$

We have shown the next lemma:

Lemma 6.1. *For $u_0 \in L^2\left(\Gamma^+_{\rho_0,\varepsilon_0}\right)$ there exist positive constants μ_1, λ_0 and C such that for $\lambda > \lambda_0$ and $|t - t_0| < \mu_1$*

$$\left|Su_0\left(z', z_n - \lambda^{m-1}t, \lambda\right)\right| \leq Ce^{\lambda\Phi(z) - \mu_1\lambda} \left\|e^{\delta_0|x|^{1/(m-1)}}u_0\right\|_{L^2\left(\Gamma^+_{\rho_0,\varepsilon_0}\right)}. \qquad (6.5)$$

On the other hand we write

$$I(t, z, \lambda) = I_1(t, z, \lambda) + I_2(t, z, \lambda),$$

where

$$I_k(t, z, \lambda) = \int_{\mathbb{R}^n} J_k(z, x, \lambda)u(t, x)\,dx, \quad (k = 1, 2),$$

and

$$J_1(z, x, \lambda) = \left\{\left(\frac{1}{\lambda}D_{z_n} - \frac{1}{\lambda^m}{}^tP(x, D_x)\right)\left(e^{i\lambda\varphi}f\right)\right\}\chi,$$

$$J_2(z, x, \lambda) = \left[\left(\frac{1}{\lambda}D_{z_n} - \frac{1}{\lambda^m}{}^tP(x, D_x)\right), \chi\right]\left(e^{i\lambda\varphi}f\right),$$

where $[P, Q] = PQ - QP$. To estimate I_1 we use Theorem 5.1,

$$\begin{aligned}
&|I_1(t, z, \lambda)| \\
&\leq Ce^{\lambda\Phi(z) - \mu_0\lambda}(1 + |z_n|)^{N_0} \int \chi\left(\frac{x - Y(\operatorname{Re} z_n; y_0, \eta_0)}{1 + |z_n|}\right) |u(t, x)|\,dx \\
&\leq Ce^{\lambda\Phi(z) - \mu_0\lambda}(1 + |z_n|)^{N_0 + (1/2)n}\,\|u(t, \cdot)\|_{L^2\left(\Gamma^+_{\rho_0,\varepsilon_0}\right)}.
\end{aligned}$$

We estimate the term I_2. On the support of a derivative of $\chi_1(z,x)$, we have

$$|x - Y(z_n; y_0, \eta_0)| \geq \frac{1}{2}\varepsilon_0(1 + |z_n|).$$

For $z \in \mathbb{C}^n$ and $x \in \mathbb{R}^n$ we have the Taylor's expansion with respect to x at $x(z) \in \mathbb{R}^n$

$$\begin{aligned}\operatorname{Re}(i\varphi(z,x)) = &- \operatorname{Im}\varphi(z, x(z)) - \left\langle \operatorname{Im}\frac{\partial\varphi}{\partial x}(z, x(z)), x - x(z)\right\rangle \\ &- 2\sum_{|\gamma|=2}\frac{1}{\gamma!}(x - x(z))^\gamma \int_0^1 (1-\theta)\operatorname{Im}\frac{\partial^\gamma\varphi}{\partial x^\gamma}(z, x(z) + \theta(x - x(z)))\,d\theta,\end{aligned}$$

where $\partial/\partial x_j$ is the real differentiation and $x(z) = (x_1(z), \ldots, x_n(z))$ is the real valued functions given in Section 4. On the support of χ_1 Lemma 4.2 or Lemma 4.3 implies

$$\operatorname{Re}(i\varphi(z,x)) \leq \Phi(z) - \frac{C}{(1+|z_n|)^2}|x - x(z)|^2. \tag{6.6}$$

Since we have

$$|x - x(z)| \geq \frac{1}{4}\varepsilon_0(1 + |z_n|),$$

on the support of a derivative of χ_1, there exist $\delta > 0$ and $\mu_1 > 0$ such that for $|t - t_0| < \delta$ we have

$$\left|\int_0^t I_2(\tau, z', z_n + \lambda^{m-1}(\tau - t), \lambda)\,d\tau\right| \leq Ce^{\lambda\Phi(z) - \mu_1\lambda}\sup_{|t-t_0|<\delta}\|u(t,\,\cdot\,)\|_{L^2\left(\Gamma^+_{\rho_0,\varepsilon_0}\right)}. \tag{6.7}$$

By (6.3), (6.5) and (6.7) we obtain the next theorem:

Theorem 6.1. *Let $t_0 < 0$. Assume $u_0 \in X^+_{\rho_0}$. Then there exist $C > 0$, $\lambda_0 > 0$, $\varepsilon > 0$, $\delta > 0$ and $\mu > 0$ such that for $|z' - z'_0| + |z_n| < \varepsilon$, $|t - t_0| < \delta$ and $\lambda > \lambda_0$, we have*

$$|Su(t,z,\lambda)| \leq Ce^{\lambda\Phi(x) - \mu\lambda}\left(\left\|e^{\delta_0|x|^{1/(m-1)}}u_0\right\|_{L^2\left(\Gamma^+_{\rho_0,\varepsilon_0}\right)} + \sup_{|t-t_0|<\delta}\|u(t,\,\cdot\,)\|_{L^2(\mathbb{R}^n)}\right). \tag{6.8}$$

Let us set

$$Tu(t,z,\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(z,x)}f(z,x,\lambda)\chi(x - y_0)u(t,x)\,dx. \tag{6.9}$$

We also obtain the next theorem in view of Proposition 5.3 in [21]:

Theorem 6.2. *Let $t_0 < 0$. Assume $u_0 \in X^+_{\rho_0}$. Then there exist $C > 0$, $\varepsilon > 0$, $\delta > 0$ and $\mu > 0$ such that for $\left|z' - z'_0\right| + |z_n| < \varepsilon$ and $|t - t_0| < \delta$, we have*

$$|Tu(t,z,\lambda) - Su(t,z,\lambda)| \leq Ce^{\lambda\Phi(x) - \mu\lambda}. \tag{6.10}$$

Theorem 6.1 and Theorem 6.2 show that we have

$$|Tu(t, z, \lambda)| \le Ce^{\lambda\Phi(z)-\mu_1\lambda}, \tag{6.11}$$

which implies $\rho_0 \notin \widetilde{WF}_A[u(t, \cdot)]$. We obtain the desired conclusion.

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