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DESCENT FOR l-ADIC POLYLOGARITHMS

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Abstract. Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. Let $z_1, \dots, z_N \in L^* \setminus \{1\}$ and let $m_1, \dots, m_N \in \mathbb{Q}_l$. Let us assume that the linear combination of l-adic polylogarithms $c_n := \sum_{i=1}^N m_i l_n(z_i)_{\gamma_i}$ (constructed in some given way) is a cocycle on G_L and that the formal sum $\sum_{i=1}^N m_i[z_i]$ is G-invariant. Then we show that c_n determines a unique cocycle s_n on G_K. We also prove a weak version of Zagier conjecture for l-adic dilogarithm. Finally we show that if c_2 is "motivic" $(m_1, \dots, m_N \in \mathbb{Q})$ then s_2 is also "motivic".

§0. Introduction

Studying polylogarithms one meets a lot of fascinating identities. Among them there are the following ones

$$p^{n-1}\left(\sum_{k=1}^{p-1} Li_n\left(e^{2\pi ik/p}\right)\right) = (1-p^{n-1})Li_n(1),$$

$$D_3\left(\frac{1+\sqrt{5}}{2}\right) + D_3\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1}{5}Li_3(1)$$

and many more of the similar type (see [14] and [4]). We shall describe common features of all these identities in the following conjecture.

CONJECTURE 0.1. *Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. Let $z_1, \dots, z_N \in L^* \setminus \{1\}$ and let $m_1, \dots, m_N \in \mathbb{Q}$. Let us assume that $\sum_{i=1}^N m_i Li_n(z_i)$ is a regulator of an extension in $Ext^1_{\mathcal{MM}_{Spec\,L}}(\mathbb{Q}(0), \mathbb{Q}(n))$, where $\mathcal{MM}_{Spec\,L}$ is the category of mixed Tate motives over $Spec\,L$. Let us assume further that the formal sum $\sum_{i=1}^N m_i[z_i]$ is G-invariant. Then $\sum_{i=1}^N m_i Li_n(z_i)$ is a regulator of an extension in $Ext^1_{\mathcal{MM}_{Spec\,K}}(\mathbb{Q}(0), \mathbb{Q}(n))$.*

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We recall that in [10] we have defined l-adic polylogarithms. For a given $z \in L$, an l-adic polylogarithm $l_n(z)_\gamma$ is a function from G_L to $\mathbb{Q}_l(n)$, which depends on a choice of a path on $\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}$ from $\overrightarrow{01}$ to z.

The l-adic realization of mixed Tate motives over $\mathit{Spec}\, L$

$$\mathit{real}_l : \mathit{Ext}^1_{\mathcal{MM}_{\mathit{Spec}\, L}}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow H^1(G_L; \mathbb{Q}_l(n))$$

associates to an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ over $\mathit{Spec}\, L$ a cohomology class in $H^1(G_L; \mathbb{Q}_l(n))$. Hence we have the obvious question when a linear combination of l-adic n-th polylogarithms evaluated at elements of L is a cocycle representing a cohomology class $\mathit{real}_l(E)$ for some $E \in \mathit{Ext}^1_{\mathcal{MM}_{\mathit{Spec}\, L}}(\mathbb{Q}(0), \mathbb{Q}(n))$.

In fact in [10] we have also asked a weaker question, when a linear combination of l-adic n-th polylogarithms evaluated at elements of L is a cocycle belonging to $Z^1(G_L; \mathbb{Q}_l(n))$ and we have given an answer to this weaker question (see [10] Theorem 11.0.11).

The l-adic analog of Conjecture 0.1 is the following conjecture.

CONJECTURE 0.2. *Let L be a finite Galois extension of a number field K. Let $G := \mathit{Gal}(L/K)$. Let $z_1, \dots, z_N \in L^* \setminus \{1\}$ and let $m_1, \dots, m_N \in \mathbb{Q}$. Let us assume that the formal sum $\sum_{i=1}^N m_i[z_i]$ is G-invariant. If $\sum_{i=1}^N m_i l_n(z_i)_{\gamma_i}$ is a cocycle representing $\mathit{real}_l(E)$ for some E in $\mathit{Ext}^1_{\mathcal{MM}_{\mathit{Spec}\, L}}(\mathbb{Q}(0), \mathbb{Q}(n))$ then there is $F \in \mathit{Ext}^1_{\mathcal{MM}_{\mathit{Spec}\, K}}(\mathbb{Q}(0), \mathbb{Q}(n))$ such that*

$$\iota^*(\mathit{real}_l(F)) = \sum_{i=1}^N m_i l_n(z_i)_{\gamma_i}$$

in $H^1(G_L; \mathbb{Q}_l(n))$, where $\iota^* : H^1(G_K; \mathbb{Q}_l(n)) \to H^1(G_L; \mathbb{Q}_l(n))$ is induced by the inclusion $\iota : G_L \to G_K$.

Our main result is a weak form of Conjecture 0.2.

THEOREM 0.3. *Let L be a finite Galois extension of a number field K. Let $G := \mathit{Gal}(L/K)$. Let us assume that l does not divide the order of G and that $L \cap K(\mu_{l^\infty}) = K$. Let us assume that the formal sum $\sum_{i=1}^N m_i[z_i]$ is G-invariant. If the linear combination $\sum_{i=1}^N m_i l_n(z_i)_{\gamma_i}$, constructed in the way which will be described below, is a cocycle on G_L then there is a cocycle s_n on G_K such that*

$$\iota^*(s_n) = \sum_{i=1}^N m_i l_n(z_i)_{\gamma_i}$$

in $H^1(G_L;\mathbb{Q}_l(n))$, *where* $\iota : G_L \to G_K$ *is the inclusion.*

Next we are studying Zagier conjecture for l-adic dilogarithm. First we define a $\mathbb{Q}$-vector subspace of "motivic" cocycles of the $\mathbb{Q}_l$-vector space $Z^1(G_L;\mathbb{Q}_l(2))$ of one cocycles on G_L with values in $\mathbb{Q}_l(2)$. Let us set

$$\mathcal{P}_2(L) := \Big\{ c_2 := \sum_{i=1}^{N} m_i l_2(z_i)_{\gamma_i} \ \Big|\ z_1,\dots,z_N \in L^* \setminus \{1\},\ m_1,\dots,m_N \in \mathbb{Q},$$
$$\sum_{i=1}^{N} m_i(z_i)\wedge(1-z_i) = 0 \text{ in } L^*\wedge L^*\otimes\mathbb{Q},\ c_2 \in Z^1(G_L;\mathbb{Q}_l(2)) \Big\}.$$

Let $[\mathcal{P}_2(L)]$ be the image of $\mathcal{P}_2(L)$ in $H^1(G_L;\mathbb{Q}_l(2))$. In [10] we have shown that l-adic dilogarithm satisfies the Abel functional equation. Using this result we show the following theorem.

THEOREM 0.4. *Let* L *be a number field. Then we have*

$$\dim_{\mathbb{Q}}[\mathcal{P}_2(L)] \le r_2(L),$$

where $r_2(L)$ *is a number of complex places of* L.

Observe that if $H^1(G_L;\mathbb{Q}_l(2)) \ne 0$ then it is an infinite dimensional vector space over $\mathbb{Q}$, so the result of the theorem is far from being obvious.

Now we state our last result.

THEOREM 0.5. *Let* L *be a finite Galois extension of a number field* K. *Let* $G := Gal(L/K)$. *Let* $z_1,\dots,z_N \in L^* \setminus \{1\}$ *and let* $m_1,\dots,m_N \in \mathbb{Q}$. *Let* $\sum_{i=1}^N m_i(z_i)\wedge(1-z_i) = 0$ *in* $L^*\wedge L^*\otimes\mathbb{Q}$. *Let us assume that* $c_2 := \sum_{i=1}^N m_i l_2(z_i)_{\gamma_i} \in Z^1(G_L;\mathbb{Q}_l(2))$ *and that formal sum* $\sum_{i=1}^N m_i[z_i]$ *is* G-*invariant. Then there is a cocycle* $s_2 \in \mathcal{P}_2(K)$ *such that* $\iota^*(s_2) = c_2$ *in* $H^1(G_L;\mathbb{Q}_l(2))$, *where* $\iota : G_L \to G_K$ *is the inclusion of Galois groups.*

The present note has it source in the work of the second author (see [11]). In [11] the descent problem for Galois representations arising from actions of Galois groups on torsors of paths is studied.

Finally the second author would like to thank very much Herbert Gangl who pointed to him that the fact that l-adic dilogarithm satisfies 5-term Abel equation implies some form of Zagier conjecture.

§1. Action of a finite Galois group

Let l be a fixed prime number. Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. Let $z \in L$. We denote by $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ the maximal pro-l quotient of the étale fundamental group of $\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}$ based at $\overrightarrow{01}$ and by $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ the $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$-torsor of l-adic paths from $\overrightarrow{01}$ to z. The Galois group G_L acts on $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ and on $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$.

Definition 1.1. (see [11] Definitions 2.0 and 2.2, [12] Definitions 17.4 and 17.5) Let $z \in L \setminus \{0,1\}$. (resp. Let $x \in \{0,1\}$ and let $z = \overrightarrow{xy}$ be a tangential point on $\mathbb{P}^1_L \setminus \{0,1,\infty\}$ defined over L.) We say that a triple $(\mathbb{P}^1_L \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ has good reduction at a prime ideal $\mathfrak{p}$ of $\mathcal{O}_L$ if $\mathbf{v}_\mathfrak{p}(z) = 0$ and $\mathbf{v}_\mathfrak{p}(1-z) = 0$ (resp. $\mathbf{v}_\mathfrak{p}(y-x) = 0$), where $\mathbf{v}_\mathfrak{p} : L^* \to \mathbb{Z}$ is the valuation associated with the prime ideal $\mathfrak{p}$.

Let S be a finite set of prime ideals of $\mathcal{O}_L$ containing all prime ideals lying over l such that the triplet $(\mathbb{P}^1_L \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ has good reduction outside S.

We have the following important result.

Proposition 1.2. (see [11] Proposition 2.3 or [12] Theorem 17.7) *The action of G_L on the torsor of paths $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ is unramified outside S.*

Proof. One repeats arguments of the Ihara proof of Theorem 1 in [3], that the action of $G_\mathbb{Q}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ is unramified outside l (see [11] or [12] for more details). ∎

Let us assume also that S is G-invariant.

Let $M(L)_S$ be a maximal, pro-l, unramified outside S extension of $L(\mu_{l^\infty})$. It follows from [3] Theorem 1 and from Proposition 1.2 that the action of G_L on $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ and on $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ factors through $Gal(M(L)_S/L)$.

Throughout all this paper we shall assume that

a) l does not divide the order of G;

b) $L \cap K(\mu_{l^\infty}) = K$.

It follows immediately from a) and b) that

$$Gal(L(\mu_{l^\infty})/K(\mu_{l^\infty})) = Gal(L/K). \tag{1.3}$$

Let us consider the following three exact sequences and morphisms between them:

$$\begin{array}{ccccccccc}
1 \longrightarrow & Gal(M(L)_S/L(\mu_{l^\infty})) & \longrightarrow & Gal(M(L)_S/K(\mu_{l^\infty})) & \overset{s'}{\underset{\longrightarrow}{\longleftarrow}} & Gal(L(\mu_{l^\infty})/K(\mu_{l^\infty})) & \longrightarrow 1 \\
 & \downarrow & & \downarrow & & = \downarrow & \\
1 \longrightarrow & Gal(M(L)_S/L) & \longrightarrow & Gal(M(L)_S/K) & \overset{s}{\underset{\longrightarrow}{\longleftarrow}} & Gal(L/K) & \longrightarrow 1 \\
 & \uparrow & & \uparrow & & = \uparrow & \\
1 \longrightarrow & G_L & \longrightarrow & G_K & \longrightarrow & Gal(L/K) & \longrightarrow 1.
\end{array}$$

The assumption a) that l does not divide the order of G implies that the upper exact sequence has a section

$$s' : G \longrightarrow Gal(M(L)_S/K(\mu_{l^\infty})).$$

Therefore the midle exact sequence has also a section

$$s : G \longrightarrow Gal(M(L)_S/K),$$

which is the composition of the isomorphism

$$Gal(L/K) \approx Gal(L(\mu_{l^\infty})/K(\mu_{l^\infty}))$$

and s'.

Let $\tilde{g}$ be a lifting of $s(g)$ to G_K. Then $\tilde{g}$ induces an isomorphism

$$\tilde{g}_* : \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \longrightarrow \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; \overrightarrow{01})$$

and a bijection

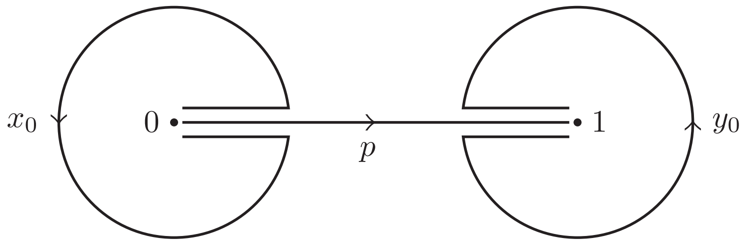
$$\tilde{g}_\diamond : \pi(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; z, \overrightarrow{01}) \longrightarrow \pi(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; g(z), \overrightarrow{01}).$$

Let us fix a path γ from $\overrightarrow{01}$ to z. Let us consider the following diagram

$$\begin{array}{ccc}
\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01}) & \xrightarrow{\tilde{g}_\diamond} & \pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; g(z), \overrightarrow{01}) \\
{\scriptstyle t_\gamma}\downarrow & & {\scriptstyle t_{\tilde{g}_\diamond(\gamma)}}\downarrow \\
\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01}) & \xrightarrow{\tilde{g}_*} & \pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \\
{\scriptstyle k}\downarrow & & {\scriptstyle k_g}\downarrow \\
\mathbb{Q}_l\{\{X,Y\}\}^* & \xrightarrow{id} & \mathbb{Q}_l\{\{X,Y\}\}^*
\end{array}$$

Diagram 1

We recall that $t_\gamma(q) = \gamma^{-1} \cdot q$ (see [9], Section 1). We choose standard generators of $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$. Let $x := x_0$ and let $y := p^{-1} \cdot y_0 \cdot p$ (see Picture 1).



Picture 1

Then k (resp. k_g) is a continous multiplicative homomorphism given by $k(x) = e^X$ and $k(y) = e^Y$ (resp. $k_g(\tilde{g}_*(x)) = e^X$ and $k_g(\tilde{g}_*(y)) = e^Y$). Observe that with these choices of k and k_g all squares of Diagram 1 commute.

Lemma 1.4. *Let x_0 (resp. y_0) be a small loop around* 0 (*resp.* 1) *based at* $\overrightarrow{01}$ (*resp.* $\overrightarrow{10}$) (*see Picture* 1). *Then we have*

$$\tilde{g}_*(x_0) = x_0 \quad and \quad \tilde{g}_*(y_0) = y_0.$$

Proof. The assumptions a) and b) imply that $\tilde{g}$ acts as the identity on $K(\mu_{l^\infty})$. The action of G_K on $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1\}; v)$ factors through $Gal(K(\mu_{l^\infty})/K)$. Hence $\tilde{g}$ acts as the identity on $\pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1\}; v)$. □

For a number field K we denote by $\{K/l\}$ the set of prime ideals of $\mathcal{O}_K$ lying over l.

LEMMA 1.5. *Let p be a path from $\overrightarrow{01}$ to $\overrightarrow{10}$, an interval* $[0,1]$ (*see Picture* 1). *Then we have*

$$\tilde{g}_\diamond(p) = p.$$

Proof. We recall that the section

$$s : G \longrightarrow Gal(M(L)_S/K)$$

is the composition of the isomorphism $Gal(L/K) \approx Gal(L(\mu_{l^\infty})/K(\mu_{l^\infty}))$ and the section $s' : Gal(L(\mu_{l^\infty})/K(\mu_{l^\infty})) \to Gal(M(L)_S/K(\mu_{l^\infty}))$. Therefore the restriction of $s(g)$ to $K(\mu_{l^\infty})$ is the identity. Hence it follows that the image of $s(g)$ in $Gal(M(K)_{\{K/l\}}/K)$ by the epimorphism

$$Gal(M(L)_S/K) \longrightarrow Gal(M(K)_{\{K/l\}}/K)$$

is trivial because $Gal(M(K)_{\{K/l\}}/K(\mu_{l^\infty}))$ is a pro-l group and l does not divide the order of G.

The action of G_K on $\pi(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; \overrightarrow{10}, \overrightarrow{01})$ factors through $Gal(M(K)_{\{K/l\}}/K)$ because the triple $(\mathbb{P}^1_K \setminus \{0,1,\infty\}; \overrightarrow{10}, \overrightarrow{01})$ has good reduction at every prime ideal of $\mathcal{O}_K$. This implies that $\tilde{g} \in G_K$ acts as $s(g)$ on $\pi(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; \overrightarrow{10}, \overrightarrow{01})$ and therefore it acts as the identity. Hence we have $\tilde{g}_\diamond(p) = p$. ∎

COROLLARY 1.6. *The isomorphism*

$$\tilde{g}_* : \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \longrightarrow \pi_1(\mathbb{P}^1_{\bar{K}} \setminus \{0,1,\infty\}; \overrightarrow{01})$$

is the identity. Consequently the embedding $k_g = k$. ∎

§2. l-adic polylogarithms

We start by recalling briefly the definition of l-adic polylogarithms. First however we introduce the following notation.

Let a, $b \in \mathbb{Q}_l\{\{X, Y\}\}$. The Lie bracket is defined by $[a, b] := a \cdot b - b \cdot a$. We set $[Y, X^0] := Y$, $[Y, X^1] := [Y, X]$ and $[Y, X^n] := [[Y, X^{n-1}], X]$ for $n \geq 1$. We use the similar notation for successive commutators of elements of a group.

We denote by

$$\chi : G_L \longrightarrow \mathbb{Z}_l^*$$

the cyclotomic character.

We denote by $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \otimes \mathbb{Q}$ the rationalization of the pro-l, pronilpotent group $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$ and by $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01}) \otimes \mathbb{Q}$ the $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \otimes \mathbb{Q}$-torsor deduced from the $\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01})$-torsor of paths $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ (see [10], 10.3). Elements of $\pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z, \overrightarrow{01}) \otimes \mathbb{Q}$ we shall call $\mathbb{Q}_l$-paths on $\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}$ from z to $\overrightarrow{01}$.

Definition 2.0. (see [10] Definition 11.0.1) Let z be an L-point of $\mathbb{P}^1_L \setminus \{0,1,\infty\}$ or a tangential point defined over L. Let γ be a $\mathbb{Q}_l$-path on $\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}$ from $\overrightarrow{01}$ to z. Let $\sigma \in G_L$. Then we set

$$\Lambda_\gamma(\sigma) := k(\gamma^{-1} \cdot \sigma(\gamma))$$

and we define l-adic polylogarithms by the congruence

$$\log \Lambda_\gamma(\sigma) \equiv l(z)_\gamma(\sigma) X + \sum_{n=1}^{\infty} l_n(z)_\gamma(\sigma)[Y, X^{n-1}] \mod I_2,$$

where I_2 is an ideal of $\mathbb{Q}_l\{\{X,Y\}\}$ generated topologically by monomials with two or more Y's.

The l-adic polylogarithms depend on a choice of a path γ. We have the following result which we shall need later.

Lemma 2.1. *Let $S \in \pi_1(V_{\bar{L}}; \overrightarrow{01}) \otimes \mathbb{Q}$, where $V = \mathbb{P}^1 \setminus \{0,1,\infty\}$.*

i) *Let $S \equiv x^a \cdot y^b \mod \Gamma^2 \pi_1(V_{\bar{L}}; \overrightarrow{01}) \otimes \mathbb{Q}$. Then*

$$l(z)_{\gamma \cdot S} = l(z)_\gamma + a(\chi - 1) \quad \textit{and} \quad l_1(z)_{\gamma \cdot S} = l_1(z)_\gamma + b(\chi - 1).$$

ii) *Let $n > 1$ and let $S \equiv (y, x^{n-1})^a \mod \Gamma^{n+1} \pi_1(V_{\bar{L}}; \overrightarrow{01}) \otimes \mathbb{Q}$. Then*

$$l_k(z)_{\gamma \cdot S} = l_k(z)_\gamma \quad \textit{for } k < n \quad \textit{and} \quad l_n(z)_{\gamma \cdot S} = l_n(z)_\gamma + a(\chi^n - 1).$$

Proof. The point i) of the lemma is already proved in [10] (see [10] Lemma 11.0.10), hence we shall prove only the second part of the lemma.

Let $\sigma \in G_L$. Then

$$\Lambda_{\gamma \cdot S}(\sigma) = k((\gamma \cdot S)^{-1} \cdot \sigma(\gamma \cdot S)) = k(S)^{-1} \cdot \Lambda_\gamma(\sigma) \cdot k(\sigma(S)).$$

Let $I := ker(\mathbb{Q}_l\{\{X,Y\}\} \to \mathbb{Q}_l)$ be the augmentation ideal. One shows easily by recurrence on n that $k((y,x^{n-1})) \equiv 1 + [Y, X^{n-1}] \mod I^{n+1}$ and $k(\sigma(y,x^{n-1})) \equiv 1 + \chi(\sigma)^n [Y, X^{n-1}] \mod I^{n+1}$. Hence we get that

$$\Lambda_{\gamma \cdot S}(\sigma) \equiv (1 - a[Y, X^{n-1}]) \cdot \Lambda_\gamma(\sigma) \cdot (1 + a\chi(\sigma)^n [Y, X^{n-1}]) \mod I^{n+1}.$$

After taking logarithm we get

$$\log \Lambda_{\gamma \cdot S}(\sigma) \equiv \log \Lambda_\gamma(\sigma) + a(\chi(\sigma)^n - 1)[Y, X^{n-1}] \mod I^{n+1}.$$

Hence it follows that $l_k(z)_{\gamma \cdot S} = l_k(z)_\gamma$ for $k < n$ and $l_n(z)_{\gamma \cdot S} = l_n(z)_\gamma + a(\chi^n - 1)$. ∎

In [1], assuming the existence of the category of mixed Tate motives over *Spec* L, one gives a condition when a linear combination of n-th polylogarithms evaluated at elements of L is a regulator of an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ over *Spec* L. In [9] we are presenting an l-adic version of the problem. As in a case of classical complex polylogarithms, one wants to know, when a linear combination of l-adic polylogarithms is an l-adic regulator (we should rather say "an l-adic realisation") of an extension of $\mathbb{Q}(0)$ by $\mathbb{Q}(n)$ over *Spec* L. The l-adic realization induces a map

$$real_l : Ext^1_{\mathcal{MM}_{Spec\, L}}(\mathbb{Q}(0), \mathbb{Q}(n)) \longrightarrow H^1(G_L; \mathbb{Q}_l(n)).$$

Hence we can ask a weaker question, when is a linear combination of l-adic polylogarithms evaluated at elements of L a cocycle? We recall here a result from [10] .

PROPOSITION 2.2. (see [10] Theorem 11.0.11) *Let $n \geq 2$. Let $z_1, \dots, z_N \in L^* \setminus \{1\}$ and let $m_1, \dots, m_N \in \mathbb{Q}_l$. Let $\mathcal{Z}$ be a subgroup of L^* generated by $z_1, \dots, z_N$ and $1 - z_1, \dots, 1 - z_N$. Let us assume that there are $\mathbb{Q}_l$-paths γ_i from $\overrightarrow{01}$ to z_i $(i = 1, \dots, N)$ such that*

i) *the map*
$$z_i \longrightarrow l(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$
and
$$1 - z_i \longrightarrow l_1(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$
is well defined on the set $\{z_1, \dots, z_N,\ 1 - z_1, \dots, 1 - z_N\}$ and it defines a homomorphism from $\mathcal{Z}$ to $Z^1(G_L; \mathbb{Q}_l(1))$;

ii) *if* $n > 2$ *then* $\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i)(z_i) \wedge (1-z_i) = 0$ *in* $L^* \wedge L^* \otimes \mathbb{Q}_l$ *for any homomorphisms* $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ $(j = 1, \dots, n-2)$ *and if* $n = 2$ *then* $\sum_{i=1}^{N} m_i (z_i) \wedge (1 - z_i) = 0$ *in* $L^* \wedge L^* \otimes \mathbb{Q}_l$;

iii) *if* $n > 2$ *then* $\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\gamma_i} = 0$ *for* $2 \le k \le n-1$ *and any homomorphisms* $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ $(j = 1, \dots, n-2)$.

Then the map $\sum_{i=1}^{N} m_i l_n(z_i)_{\gamma_i}$ *from* G_L *to* $\mathbb{Q}_l(n)$ *is a cocycle.* □

We explain how this result is related to motivic considerations in [1] and also in [9]. In fact we shall repeat in l-adic situation considerations from [1].

Let L be a number field. In [1], and also in [9] in l-adic case one defines in an inductive way some groups $\mathcal{L}_k$ $(k = 1, 2, \dots)$, operators $d_k : \mathcal{L}_k \to \bigwedge^2(\bigoplus_{i=1}^{k-1} \mathcal{L}_i)$ $(k = 1, 2, \dots)$ and homomorphisms $\varphi_k : ker\, d_k \to \mathrm{Ext}^1_{\mathcal{MM}_{Spec\, L}}(\mathbb{Q}(0), \mathbb{Q}(n))$ with the following conjectural properties.

Let $\sum_{i=1}^{N} m_i \{z_i\}_n \in \mathcal{L}_n$ be such that $d_n(\sum_{i=1}^{N} m_i \{z_i\}_n) = 0$. Then the cohomology class $\mathrm{real}_l(\varphi_n(\sum_{i=1}^{N} m_i \{z_i\}_n))$ in $H^1(G_L; \mathbb{Q}_l(n))$ restricted to the subgroup $\bigcap_{i=1}^{N} H_n(\mathbb{P}^1_L \setminus \{0, 1, \infty\}; z_i, \overrightarrow{01})$ of G_L is equal to the function $\sum_{i=1}^{N} m_i l_n(z_i)$ restricted to the same subgroup of G_L. (The l-adic polylogarithms restricted to this subgroup of G_L do not depend on choices of paths, so we drop out subscripts indicating paths along which one calculates them. Subgroups $H_n(\mathbb{P}^1_L \setminus \{0, 1, \infty\}; z_i, \overrightarrow{01})$ of G_L are defined in [9], Section 3.)

Assuming motivic formalism as in [1] we shall show the following result (compare it with Proposition 4.6 in [1]).

Theorem 2.3. *Let* $\sum_{i=1}^{N} m_i \{z_i\}_n \in \mathcal{L}_n$ *be such that* $d_n(\sum_{i=1}^{N} m_i \{z_i\}_n) = 0$. *Then there are* $\mathbb{Q}_l$*-paths* δ_i *from* $\overrightarrow{01}$ *to* z_i *such that* $\sum_{i=1}^{N} m_i l_n(z_i)_{\delta_i}$ *is a cocycle representing the cohomology class* $\mathrm{real}_l(\varphi_n(\sum_{i=1}^{N} m_i \{z_i\}_n))$.

The next two lemmas are technical. We need these lemmas in order to show Theorem 2.3. In the first lemma we show that we can always choose $\mathbb{Q}_l$-paths in such a way that the assumption i) of Proposition 2.2 is satisfied. The second lemma shows that in the motivic situation described above the assumption iii) of Proposition 2.2 will be satisfied.

Lemma 2.4. *Let* $z_1, \dots, z_N \in L^* \setminus \{1\}$ *and let* $\mathcal{Z}$ *be a subgroup of* L^* *generated by* $z_1, \dots, z_N$ *and* $1 - z_1, \dots, 1 - z_N$. *Then there are* $\mathbb{Q}_l$*-paths* γ_i

from $\overrightarrow{01}$ *to* z_i $(i = 1, \ldots, N)$ *such that the map*

$$z_i \longrightarrow l(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$

and

$$1 - z_i \longrightarrow l_1(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$

is well defined on the set $\{z_1, \ldots, z_N, 1 - z_1, \ldots, 1 - z_N\}$ *and it defines a homomorphism from* $\mathcal{Z}$ *to* $Z^1(G_L; \mathbb{Q}_l(1))$.

Proof. The subgroup $\mathcal{Z}$ of L^* is a finitely generated abelian group, hence

$$\mathcal{Z} = \mathcal{Z}_0 \oplus torsion(\mathcal{Z}),$$

where $\mathcal{Z}_0$ is a finitely generated free abelian group and $torsion(\mathcal{Z}) \subset \mu(L)$.

Let $x_1, \ldots, x_r \in L^*$ be free generators of $\mathcal{Z}_0$. The group $torsion(\mathcal{Z})$ is cyclic. We present it as a product of two cyclic subgroups A and B such that the order of A is prime to l and the order of B is a power of l. Let ξ be a generator of A. For ξ we choose a compatible family of l^n-th roots $(\xi^{1/l^n})_{n\in\mathbb{N}}$ such that $\xi^{1/l^n} \in L$ for all $n \in \mathbb{N}$. Then the corresponding Kummer character is zero.

Let $\zeta = \exp(2\pi i/l^{n_0})$ be a generator of B. Then the Kummer character corresponding to the family $(\exp(2\pi i/l^{n_0+n}))_{n\in\mathbb{N}}$ of l^n-th roots of ζ is $\frac{1}{l^{n_0}}(\chi - 1)$ and it is calculated along a certain path γ from $\overrightarrow{01}$ to ζ. Hence we have $l(\zeta)_\gamma = \frac{1}{l^{n_0}}(\chi - 1)$. Calculating the l-adic logarithm $l(\zeta)$ along the $\mathbb{Q}_l$-path $\gamma \cdot x^{-1/l^{n_0}}$ we get

$$l(\zeta)_{\gamma \cdot x^{-1/l^{n_0}}} = l(\zeta)_\gamma + \Big(-\frac{1}{l^{n_0}}\Big) \cdot (\chi - 1) = 0$$

by [10] Lemma 11.0.10.

We define the compatible family of l^n-th roots of ζ to be the constant family $(1)_{n\in\mathbb{N}}$ and this family corresponds to the path $\gamma \cdot x^{-1/l^{n_0}}$.

We also choose a compatible family of l^n-th roots $(x_i^{1/l^n})_{n\in\mathbb{N}}$ for any x_i $(i = 1, \ldots, r)$. If z_j (resp. $1 - z_j$) is equal $\zeta^h \cdot \xi^k \cdot \prod_{i=1}^r x_i^{k_i}$ then we choose compatible family of l^n-th roots of z_j (resp. $1 - z_j$) by taking $\big((\xi^{1/l^n})^k \cdot \prod_{i=1}^r (x_i^{1/l^n})^{k_i}\big)_{n\in\mathbb{N}}$.

The choice of a compatible family of l^n-th roots of z_j and $1 - z_j$ determines a $\mathbb{Q}_l$-path γ_{z_j} from $\overrightarrow{01}$ to z_j modulo $\Gamma^2(\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; \overrightarrow{01}) \otimes \mathbb{Q})$. The l-adic logarithm $l(z_j)_{\gamma_j}$ (resp. $l_1(z_j)_{\gamma_j}$) is the Kummer character corresponding to given above compatible family of l^n-roots of z_j (resp. $1 - z_j$). □

LEMMA 2.5. *Let $z_1, \dots, z_N \in L^* \setminus \{1\}$ and let $m_1, \dots, m_N \in \mathbb{Q}_l$. Let $n > r$ be two natural numbers greater than 2. Let γ_i be $\mathbb{Q}_l$-paths on $\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to z_i for $i = 1, \dots, N$ such that*

i) *Lemma 2.4 holds for paths γ_i;*
ii) *$\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\gamma_i} = 0$ for $2 \le k \le r-1$ and for any homomorphisms $\nu_s : \mathcal{Z} \to \mathbb{Q}_l$ $(s = 1, \dots, n-2)$;*
iii) *$\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-r}(z_i) l_r(z_i)_{\gamma_i} \in B^1(G_L; \mathbb{Q}_l(r))$ for any homomorphisms $\nu_s : \mathcal{Z} \to \mathbb{Q}_l$ $(s = 1, \dots, n-r)$.*

Then there are $\mathbb{Q}_l$-paths $\bar{\gamma}_i$ from $\overrightarrow{01}$ to z_i $(i = 1, \dots, N)$ such that

1) *$\gamma_i \equiv \bar{\gamma}_i \mod \Gamma^r(\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; \overrightarrow{01}) \otimes \mathbb{Q})$ for $i = 1, \dots, N$;*
2) *$\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\bar{\gamma}_i} = 0$ for $2 \le k \le r$ and for any homomorphisms $\nu_s : \mathcal{Z} \to \mathbb{Q}_l$ $(s = 1, \dots, n-2)$.*

Proof. In a $\mathbb{Q}_l$-vector space $Maps(G_L; \mathbb{Q}_l)$ we consider a subspace $\mathcal{A}_r$ generated by $l_r(z_i)_{\gamma_i}$ $(i = 1, \dots, N)$ and by $\chi^r - 1$. Let $\{x_1, \dots, x_p\} \subset \{z_1, \dots, z_N\}$ be such that $l_r(x_j)_{\gamma_j}$ $(j = 1, \dots, p)$ and $\chi^r - 1$ is a base of $\mathcal{A}_r$. Then for any $1 \le i \le N$ we have

$$l_r(z_i)_{\gamma_i} = \sum_{j=1}^{p} a_j^i l_r(x_j)_{\gamma_j} + a_i(\chi^r - 1) \tag{2.5.1}$$

for some a_j^i $(j = 1, \dots, p)$ and a_i in $\mathbb{Q}_l$.

Let $\bar{\gamma}_i := \gamma_i \cdot (y, x^{r-1})^{-a_i}$. Then it is clear that

$$\bar{\gamma}_i \equiv \gamma_i \mod \Gamma^r(\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; \overrightarrow{01}) \otimes \mathbb{Q}).$$

It follows from Lemma 2.1 that

$$l_k(z_i)_{\gamma_i} = l_k(z_i)_{\bar{\gamma}_i}$$

for $k < r$ and $i = 1, \dots, N$. This implies that 2) holds for $2 \le k < r$.

Let us observe that

$$l_r(z_i)_{\bar{\gamma}_i} = l_r(z_i)_{\gamma_i} - a_i(\chi^r - 1)$$

by Lemma 2.1. Hence it follows from (2.5.1) that the point 2) also holds for $k = r$. ∎

Proof of Theorem 2.3. Let $\sum_{i=1}^N m_i\{z_i\}_n \in \mathcal{L}_n$ be such that $d_n\big(\sum_{i=1}^N m_i\{z_i\}_n\big) = 0$. By the very definition of d_n we get

$$(*_{n-1}) \qquad d_n\Big(\sum_{i=1}^N m_i\{z_i\}_n\Big) = \sum_{i=1}^N m_i(z_i) \otimes \{z_i\}_{n-1} = 0.$$

in $\mathcal{L}_1 \otimes \mathcal{L}_{n-1}$.

Let $1 \le k < n-2$. Let us suppose that

$$(*_{n-k}) \qquad \sum_{i=1}^N m_i(z_i)^{\otimes k} \otimes \{z_i\}_{n-k} = 0$$

in $\big(\bigotimes_{i=1}^k \mathcal{L}_1\big) \otimes \mathcal{L}_{n-k}$. Let d be the prolongation of d_k on tensor products. Then

$$0 = d\Big(\sum_{i=1}^N m_i(z_i)^{\otimes k} \otimes \{z_i\}_{n-k}\Big) = \sum_{i=1}^N m_i(z_i)^{\otimes(k+1)} \otimes \{z_i\}_{n-k-1}$$

in $\big(\bigotimes_{i=1}^{k+1} \mathcal{L}_1\big) \otimes \mathcal{L}_{n-k-1}$. Hence we have got the equality $(*_{n-k-1})$. Therefore the equality $(*_{n-k})$ holds for $1 \le k \le n-2$.

Let $k = n-2$. Then we get

$$0 = d\Big(\sum_{i=1}^N m_i(z_i)^{\otimes(n-2)} \otimes \{z_i\}_2\Big) = \sum_{i=1}^N m_i(z_i)^{\otimes(n-2)} \otimes ((z_i) \wedge (1-z_i)) = 0$$

in $\big(\bigotimes_{i=1}^{n-2} \mathcal{L}_1\big) \otimes (\mathcal{L}_1 \wedge \mathcal{L}_1)$. Hence we have

$$(*_1) \qquad \sum_{i=1}^N m_i(z_i)^{\otimes(n-2)} \otimes ((z_i) \wedge (1-z_i)) = 0.$$

The equalities $(*_{n-k})$ $(1 \le k \le n-2)$ and $(*_1)$ are equivalent to the equalities

$$(**_{n-k}) \qquad \sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_k(z_i)\{z_i\}_{n-k} = 0$$

for $1 \le k \le n-2$ and

$$(**_1) \qquad \sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i)(z_i) \wedge (1-z_i) = 0$$

for any homomorphisms $\nu_1, \dots, \nu_{n-2}$ from $\mathcal{Z}$ to $\mathbb{Q}_l$.

We chose $\mathbb{Q}_l$-paths γ_i $(i = 1, \dots, N)$ from $\overrightarrow{01}$ to z_i such as in Lemma 2.4. It follows from Proposition 2.2 ii) and the equality $(**_1)$ that

$$\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i) l_2(z_i)_{\gamma_i}$$

is a cocycle. In $\mathcal{L}_2$ we have the equality $(**_2)$, i.e. the equality

$$\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i) \{z_i\}_2 = 0.$$

Applying real_l to the element $\varphi_2(\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i)\{z_i\}_2)$ we get 0 in $H^1(G_L; \mathbb{Q}_l(2))$. Hence the cocycle $\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i) l_2(z_i)_{\gamma_i}$ is a coboundary.

Observe that the conditions i), ii) (trivially) and iii) of Lemma 2.5 are satisfied for $r = 2$. Hence there are $\mathbb{Q}_l$-paths $\gamma_i^{(2)}$ such that

1) $\gamma_i \equiv \gamma_i^{(2)} \mod \Gamma^2(\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; \overrightarrow{01}) \otimes \mathbb{Q})$ for $i = 1, 2, \dots, N$;
2) $\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i) l_2(z_i)_{\gamma_i^{(2)}} = 0$ for any homomorphisms $\nu_1, \dots, \nu_{n-2}$ from $\mathcal{Z}$ to $\mathbb{Q}_l$.

Let $r \le n - 1$ be such that

$$\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\gamma_i^{(r-1)}} = 0$$

for $2 \le k < r \le n - 1$ and for any homomorphisms $\nu_1, \dots, \nu_{n-2}$ from $\mathcal{Z}$ to $\mathbb{Q}$. Then Proposition 2.2 implies that

$$\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-r}(z_i) l_r(z_i)_{\gamma_i^{(r-1)}}$$

is a cocycle for any homomorphisms $\nu_1, \dots, \nu_{n-2}$ from $\mathcal{Z}$ to $\mathbb{Q}$. Applying real_l to the element $\varphi_r(\sum_{i=1}^{N} m_i \nu_1(z_i) \cdot \nu_2(z_i) \cdots \nu_{n-r}(z_i)\{z_i\}_r)$ we get that the cocycle $\sum_{i=1}^{N} m_i \nu_1(z_i) \cdots \nu_{n-r}(z_i) l_r(z_i)_{\gamma_i^{(r-1)}}$ is a coboundary because of the equality $(**_{n-r})$. Lemma 2.5 implies that there are $\mathbb{Q}_l$-paths $\gamma_i^{(r)}$ $(i = 1, \dots, N)$ such that

$1)_r$ $\gamma_i^{(r)} \equiv \gamma_i^{(r-1)} \mod \Gamma^r(\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; \overrightarrow{01}) \otimes \mathbb{Q})$ for $i = 1, 2, \dots, N$;

$2)_{r,k}$ $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\gamma_i^{(r)}} = 0$ for $2 \leq k \leq r$ for any homomorphisms $\nu_1, \dots, \nu_{n-2}$ from $\mathcal{Z}$ to $\mathbb{Q}_l$.

Hence after a finite number of steps we get equations $2)_{n-1,k}$ for all $2 \leq k \leq n-1$. Then Proposition 2.2 implies that

$$\sum_{i=1}^N m_i l_n(z_i)_{\gamma_i^{(n-1)}}$$

is a cocycle. We recall that the map

$$H^1(G_L; \mathbb{Q}_l(n)) \longrightarrow \mathrm{Hom}\Big(\bigcap_{i=1}^N H_n(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; z_i, \overrightarrow{01}); \mathbb{Q}_l(n)\Big)$$

is injective. Hence it follows from the motivic formalism developped in [9], Section 7 that the cohomology class $\mathrm{real}_l(\varphi_n(\sum_{i=1}^N m_i\{z_i\}_n))$ is represented by the cocycle $\sum_{i=1}^N m_i l_n(z_i)_{\gamma_i^{(n-1)}}$. $\square$

§3. Action of G and l-adic polylogarithms

Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. We recall that we assume that

a) l does not divide the order of G;

b) $L \cap K(\mu_{l^\infty}) = K$.

In this section we shall study a G-equivariant version of results from Section 2. Let $g \in G$. We start with a relation between l-adic polylogarithms evaluated at z and $g(z)$. We denote by $\tilde{g}$ a lifting of $g \in G$ to G_K constructed in Section 1.

Proposition 3.1. *Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. Let z be an L-point of $\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}$ and let γ be a path from $\overrightarrow{01}$ to z. Then we have*

$$l_n(z)_\gamma(\tilde{g}^{-1} \cdot \tau \cdot \tilde{g}) = l_n(g(z))_{\tilde{g}(\gamma)}(\tau)$$

for any $\tau \in G_L$ and any $g \in G$.

Proof. Let $\sigma \in G_L$ and let $g \in G$. Then it follows from Corollary 1.6 that $\Lambda_\gamma(\sigma) = k(\gamma^{-1} \cdot \sigma(\gamma)) = k(\tilde{g}(\gamma^{-1} \cdot \sigma(\gamma))) = k(\tilde{g}(\gamma)^{-1} \cdot \tilde{g}(\sigma(\gamma))) = k(\tilde{g}(\gamma)^{-1} \cdot (\tilde{g} \cdot \sigma \cdot \tilde{g}^{-1})(\tilde{g}(\gamma))) = \Lambda_{\tilde{g}(\gamma)}(\tilde{g} \cdot \sigma \cdot \tilde{g}^{-1})$. For $\sigma = \tilde{g}^{-1} \cdot \tau \cdot \tilde{g}$ we get $\Lambda_\gamma(\tilde{g}^{-1} \cdot \tau \cdot \tilde{g}) = \Lambda_{\tilde{g}(\gamma)}(\tau)$. The proposition follows immediately from the definition of l-adic polylogarithms. ∎

Let $z_1, \ldots, z_N \in L^* \setminus \{1\}$ and let $m_1, \ldots, m_N \in \mathbb{Q}_l$. We shall assume that the set $\{z_1, \ldots, z_N\}$ and the formal sum $\sum_{i=1}^N m_i[z_i]$ are G-invariant. We shall show that the family of paths $\{\gamma_i \mid 1 \le i \le N\}$ from Lemmas 2.4 and 2.5 and from Proposition 2.2 can be chosen G-invariant.

Let S be a finite set of prime ideals of $\mathcal{O}_L$ containing all prime ideals lying over l and such that for any $1 \le i \le N$ a triplet $(\mathbb{P}^1_L \setminus \{0, 1, \infty\}; z_i, \overrightarrow{01})$ has good reduction for any prime ideal of $\mathcal{O}_L$ not belonging to S. We assume also that S is G-invariant.

Observe that then the action of G_L on a disjoint union of torsors of paths $\coprod_{i=1}^N \pi(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; z_i, \overrightarrow{01})$ factors through $Gal(M(L)_S/L)$.

Lemma 3.2. *Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. We assume that*

a) *l does not divide the order of G,*

b) *$K(\mu_{l^\infty}) \cap L = K$.*

Let $z_1, \ldots, z_N \in L^* \setminus \{1\}$. We assume that the set $\{z_1, \ldots, z_N\}$ is G-invariant. Then there are paths γ_{z_i} from $\overrightarrow{01}$ to z_i for $i = 1, \ldots, N$ and liftings $\tilde{g}$ of $g \in G$ to G_K such that

$$\tilde{g}(\gamma_{z_i}) = \gamma_{g(z_i)}$$

for $1 \le i \le N$ and for any $g \in G$.

Proof. The action of G_K on the disjoint union of torsors of paths $\coprod_{i=1}^N \pi(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; z_i, \overrightarrow{01})$ factors through $Gal(M(L)_S/K)$. The assumptions a) and b) imply that the exact sequence

$$1 \longrightarrow Gal(M(L)_S/L) \longrightarrow Gal(M(L)_S/K) \longrightarrow G \longrightarrow 1$$

has a section $s : G \to Gal(M(L)_S/K)$. Let $\tilde{g}$ be a lifting of $s(g)$ to G_K. Then in $Aut(\coprod_{i=1}^N \pi(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; z_i, \overrightarrow{01}))$ we have $\tilde{g}_1 \cdot \tilde{g}_2 = \widetilde{g_1 g_2}$ for any $g_1, g_2 \in G$.

Let $z \in L$ and let $K(Gz)$ be a subfield of L generated by elements $g(z)$ for $g \in G$. It is clear that $K(Gz)$ is a Galois extension of K.

Let us fix a path γ_{z_1} from $\overrightarrow{01}$ to z_1. If $K(Gz_1) = L$ then we take $\gamma_{g(z_1)} := \tilde{g}(\gamma_{z_1})$ for $g \in G$. So let us assume that $K(Gz_1) = L' \neq L$. Observe that

i) in the commutative diagram

$$\begin{array}{ccc}
\mathrm{Gal}(M(L)_S/K) & \xrightarrow{\overset{s}{\leftarrow}} & G = \mathrm{Gal}(L/K) \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle pr}\downarrow \\
\mathrm{Gal}(M(L')_S/K) & \longrightarrow & G' := \mathrm{Gal}(L'/K)
\end{array}$$

we have $ker(\pi \circ s) = ker\, pr$;

ii) the action of G_K on $\coprod_{g\in G} \pi(\mathbb{P}^1_{\bar{L}} \setminus \{0,1,\infty\}; g(z_1), \overrightarrow{01})$ factors through $\mathrm{Gal}(M(L')_S/K)$.

Hence we set $\gamma_{g(z_1)} := \pi(s(g))(\gamma_{z_1})$ for $g \in G$.

Now we take an element with the smallest index in

$$\{z_1, \dots, z_N\} \setminus \{g(z_1) \mid g \in G\}$$

and we repeat the construction. It is clear that the constructed family of paths is G-invariant. ∎

From now on we denote γ_{z_i} by γ_i and $\gamma_{g(z_i)}$ by $\gamma_{g(i)}$.

LEMMA 3.3. *Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. We assume that*

a) *l does not divide the order of G;*

b) *$K(\mu_{l^\infty}) \cap L = K$.*

Let $z_1, \dots, z_N \in L^* \setminus \{1\}$. Let us assume that the set $\{z_1, \dots, z_N\}$ is G-invariant. Let $\mathcal{Z}$ be a subgroup of L^* generated by $z_1, \dots, z_N$ and $1 - z_1, \dots, 1 - z_N$. Then there are $\mathbb{Q}_l$-paths γ_{z_i} from $\overrightarrow{01}$ to z_i such that

i) *the map*

$$z_i \longrightarrow l(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1)),$$
$$1 - z_i \longrightarrow l_1(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$

is well defined on the set $\{z_1, \dots, z_N, 1 - z_1, \dots, 1 - z_N\}$ and it defines a homomorphism from $\mathcal{Z}$ to $Z^1(G_L; \mathbb{Q}_l(1))$;

ii) *for any* $1 \leq i \leq N$ *and any* $g \in G$ *we have*

$$\tilde{g}(\gamma_{z_i}) = \gamma_{g(z_i)}.$$

Proof. It follows from Lemma 3.2 that there are paths γ_i from $\overrightarrow{01}$ to z_i such that

$$\tilde{g}(\gamma_i) = \gamma_{g(i)}$$

for $1 \leq i \leq N$ and $g \in G$. We shall modify these paths in such a way that the condition i) will be satisfied.

Let $V := \langle l(z_i)_{\gamma_i},\, l_1(z_i)_{\gamma_i},\, \chi - 1 \mid 1 \leq i \leq N\rangle$ be a $\mathbb{Q}_l$-vector subspace of $Maps(G_L, \mathbb{Q}_l)$ generated by $l(z_i)_{\gamma_i}$, $l_1(z_i)_{\gamma_i}$ $(1 \leq i \leq N)$ and by $\chi - 1$. It follows from Proposition 3.1 that V is also a $\mathbb{Q}_l[G]$-module. Let $I(G)$ be the augmentation ideal of $\mathbb{Q}_l[G]$. Then

$$V = I(G)V \oplus V^G$$

is a decomposition into a direct sum of two $\mathbb{Q}_l[G]$-modules. Let $v_1, \ldots, v_s$, $\chi - 1$ be a base of V^G. Then for any $1 \leq i \leq N$ we have

$$l(z_i)_{\gamma_i} = w_i^0 + u_i^0 + a_i(\chi - 1)$$

and

$$l_1(z_i)_{\gamma_i} = w_i^1 + u_i^1 + b_i(\chi - 1),$$

where $w_i^0,\, w_i^1 \in I(G)V$ and $u_i^0,\ u_i^1$ belong to a subspace generated by $v_1, \ldots, v_s$. Observe that

$$a_{g(i)} = a_i \quad \text{and} \quad b_{g(i)} = b_i \tag{3.3.1}$$

for any $1 \leq i \leq N$ and $g \in G$.

If $\prod_{i=1}^N z_i^{p_i}(1 - z_i)^{q_i} = 1$ is a multiplicative relation between $z_1, \ldots, z_N$, $1 - z_1, \ldots, 1 - z_N$ then

$$\sum_{i=1}^{N} (p_i l(z_i)_{\gamma_i} + q_i l_1(z_i)_{\gamma_i}) = \alpha(\chi - 1)$$

for some $\alpha \in \mathbb{Q}_l$. Hence we get $\sum_{i=1}^N (p_i a_i + q_i b_i) = \alpha$. Let us set $\bar{\gamma}_i := \gamma_i \cdot y^{-b_i} \cdot x^{-a_i}$ for $1 \leq i \leq N$. It follows from Lemma 2.1 that $l(z_i)_{\bar{\gamma}_i} = l(z_i)_{\gamma_i} - a_i(\chi - 1)$ and $l_1(z_i)_{\bar{\gamma}_i} = l_1(z_i)_{\gamma_i} - b_i(\chi - 1)$. Hence we get that $\sum_{i=1}^N (p_i l(z_i)_{\bar{\gamma}_i} + q_i l_1(z_i)_{\bar{\gamma}_i}) = 0$ for any multiplicative relation $\prod_{i=1}^N z_i^{p_i} \cdot (1 - z_i)^{q_i} = 1$. It follows from Corollary 1.6 and from (3.3.1) that $\tilde{g}(\bar{\gamma}_i) = \bar{\gamma}_{g(i)}$ for $1 \leq i \leq N$. □

LEMMA 3.4. *Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. Let $z_1, \dots, z_N \in L^* \setminus \{1\}$ and let $m_1, \dots, m_N \in \mathbb{Q}_l$. Let us assume that the formal sum $\sum_{i=1}^N m_i[z_i]$ is G-invariant. Let $n > r$ be two natural numbers greater than 2. Let γ_i be $\mathbb{Q}_l$-paths on $\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}$ from $\overrightarrow{01}$ to z_i for $i = 1, \dots, N$ such that*

i) *the family of paths $\{\gamma_i \mid 1 \le i \le N\}$ is G-invariant;*
ii) *Lemma 2.4 holds for the family of paths $\{\gamma_i \mid 1 \le i \le N\}$;*
iii) $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\gamma_i} = 0$ *for $2 \le k \le r-1$ and for any homomorphisms $\nu_s : \mathcal{Z} \to \mathbb{Q}_l$ $(s = 1, \dots, n-2)$;*
iv) $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-r}(z_i) l_r(z_i)_{\gamma_i} \in B^1(G_L; \mathbb{Q}_l(r))$ *for any homomorphisms $\nu_s : \mathcal{Z} \to \mathbb{Q}_l$ $(s = 1, \dots, n-r)$.*

Then there are $\mathbb{Q}_l$-paths $\bar{\gamma}_i$ from $\overrightarrow{01}$ to z_i $(i = 1, \dots, N)$ such that

1) $\gamma_i \equiv \bar{\gamma}_i \mod \Gamma^r(\pi_1(\mathbb{P}^1_{\bar{L}} \setminus \{0, 1, \infty\}; \overrightarrow{01}) \otimes \mathbb{Q})$ *for $i = 1, \dots, N$;*
2) $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\bar{\gamma}_i} = 0$ *for $2 \le k \le r$ and for any homomorphisms $\nu_s : \mathcal{Z} \to \mathbb{Q}_l$ $(s = 1, \dots, n-2)$;*
3) *the family of paths $\{\bar{\gamma}_i \mid 1 \le i \le N\}$ is G-invariant.*

Proof. Let $\mathcal{A}_r$ be a $\mathbb{Q}_l$-vector subspace of $Maps(G_L; \mathbb{Q}_l)$ generated by $l_r(z_i)_{\gamma_i}$ for $1 \le i \le N$ and by $\chi^r - 1$. It follows from Proposition 3.1 that $\mathcal{A}_r$ is G-invariant. Let us decompose $\mathcal{A}_r$ into a direct sum

$$\mathcal{A}_r = I(G)\mathcal{A}_r \oplus (V \oplus \langle \chi^r - 1 \rangle),$$

where $\mathcal{A}_r^G = V \oplus \langle \chi^r - 1 \rangle$. Hence we can write

$$l_r(z_i)_{\gamma_i} = I_{z_i} + v_i + \alpha_i(\chi^r - 1),$$

where $I_{z_i} \in I(G)\mathcal{A}_r$ and $v_i \in V$. It follows from Proposition 3.1 that

$$l_r(g(z_i))_{\tilde{g}(\gamma_i)} = I_{g(z_i)} + v_i + \alpha_i(\chi^r - 1) \tag{3.4.1}$$

for $g \in G$.

Let us set

$$\bar{\gamma}_i := \gamma_i \cdot (y, x^{r-1})^{-\alpha_i}$$

for $1 \le i \le N$. It follows from Corollary 1.6 that the family of paths $\{\bar{\gamma}_i \mid 1 \le i \le N\}$ is G-invariant. It follows from Lemma 2.1 that

$$l_r(z_i)_{\bar{\gamma}_i} = I_{z_i} + v_i \quad \text{and} \quad l_r(g(z_i))_{\tilde{g}(\bar{\gamma}_i)} = I_{g(z_i)} + v_i.$$

Hence it follows immediately from (3.4.1) that conditions 1 and 2 of the lemma are satisfied. ∎

Now we shall formulate our main result.

THEOREM 3.5. *Let $n \geq 2$. Let $z_1, \ldots, z_N \in L^* \backslash \{1\}$ and let $m_1, \ldots, m_N \in \mathbb{Q}_l$. Let $\mathcal{Z}$ be a subgroup of L^* generated by $z_1, \ldots, z_N$ and $1 - z_1, \ldots, 1 - z_N$. Let us assume that there are $\mathbb{Q}_l$-paths γ_i from $\overrightarrow{01}$ to z_i $(i = 1, \ldots, N)$ such that*

i) *the map*
$$z_i \longrightarrow l(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$
and
$$1 - z_i \longrightarrow l_1(z_i)_{\gamma_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$
is well defined on the set $\{z_1, \ldots, z_N, 1 - z_1, \ldots, 1 - z_N\}$ and it defines a homomorphism from $\mathcal{Z}$ to $Z^1(G_L; \mathbb{Q}_l(1))$;

ii) *if $n > 2$ then $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i)(z_i) \wedge (1 - z_i) = 0$ in $L^* \wedge L^* \otimes \mathbb{Q}_l$ for any homomorphisms $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ $(j = 1, \ldots, n-2)$ and if $n = 2$ then $\sum_{i=1}^N m_i(z_i) \wedge (1 - z_i) = 0$ in $(L^* \wedge L^*) \otimes \mathbb{Q}_l$;*

iii) *if $n > 2$ then $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\gamma_i} = 0$ for $2 \leq k \leq n - 1$ and any homomorphisms $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ $(j = 1, \ldots, n-2)$.*

Let us assume further that the set $\{z_1, z_2, \ldots, z_N\}$ and the formal sum $\sum_{i=1}^N m_i[z_i]$ are G-invariant. Then there are $\mathbb{Q}_l$-paths $\bar{\gamma}_i$ from $\overrightarrow{01}$ to z_i $(i = 1, \ldots, N)$ such that

1) *the family of paths $\{\bar{\gamma}_i \mid 1 \leq i \leq N\}$ is G-invariant;*

2) *the map*
$$z_i \longrightarrow l(z_i)_{\bar{\gamma}_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$
and
$$1 - z_i \longrightarrow l_1(z_i)_{\bar{\gamma}_i} \in Z^1(G_L; \mathbb{Q}_l(1))$$
is well defined on the set $\{z_1, \ldots, z_N, 1 - z_1, \ldots, 1 - z_N\}$ and it defines a homomorphism from $\mathcal{Z}$ to $Z^1(G_L; \mathbb{Q}_l(1))$;

3) *if $n > 2$ then $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-k}(z_i) l_k(z_i)_{\bar{\gamma}_i} = 0$ for $2 \leq k \leq n - 1$ and any homomorphisms $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ $(j = 1, \ldots, n-2)$.*

For the family of paths $\{\bar{\gamma}_i \mid 1 \leq i \leq N\}$, $c_n := \sum_{i=1}^N m_i l_n(z_i)_{\bar{\gamma}_i}$ *is a cocycle on* G_L *satisfying* $c_n(\tilde{g}^{-1} \cdot \tau \cdot \tilde{g}) = c_n(\tau)$ *for any* $\tau \in G_L$ *and any* $g \in G$. *Consequently there exists a cocycle* $s_n : G_K \to \mathbb{Q}_l(n)$ *such that*

$$s_n \circ \iota = c_n,$$

in $H^1(G_L; \mathbb{Q}_l(n))$, *where* $\iota : G_L \to G_K$ *is the inclusion.*

Proof. It follows from Lemmas 3.2 and 3.3 that there exists a family of paths $\{\delta_i \mid 1 \leq i \leq N\}$ satisfying conditions 1) and 2).

The assumption ii) and Proposition 2.2 imply that $\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i) l_2(z_i)_{\delta_i}$ is a cocycle for any homomorphisms $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ ($j = 1, \ldots, n-2$). The assumption iii) implies that this cocycle is a coboundary. It follows from Lemma 3.4 that there is a family of paths $\{\delta_i^{(2)} \mid 1 \leq i \leq N\}$ satisfying conditions 1) and 2) and such that

$$\sum_{i=1}^N m_i \nu_1(z_i) \cdots \nu_{n-2}(z_i) l_2(z_i)_{\delta_i^{(2)}} = 0$$

for any homomorphisms $\nu_j : \mathcal{Z} \to \mathbb{Q}_l$ ($j = 1, \ldots, n-2$). Repeating this reasoning for $3, 4, \ldots, n-1$ we finally get a family of paths $\{\delta_i^{(n-1)} \mid 1 \leq i \leq N\}$ satisfying conditions 1), 2) and 3).

We set $\bar{\gamma}_i := \delta_i^{(n-1)}$ for $1 \leq i \leq N$. It follows from Proposition 2.2 that

$$c_n := \sum_{i=1}^N m_i l_n(z_i)_{\bar{\gamma}_i}$$

is a cocycle on G_L. It follows from Proposition 3.1 that for any $\tau \in G_L$ and any $g \in G$ we have $c_n(\tilde{g}^{-1} \cdot \tau \cdot \tilde{g}) = \sum_{i=1}^N m_i l_n(z_i)_{\bar{\gamma}_i}(\tilde{g}^{-1} \cdot \tau \cdot \tilde{g}) = \sum_{i=1}^N m_i l_n(g(z_i))_{\tilde{g}(\bar{\gamma}_i)}(\tau) = \sum_{i=1}^N m_i l_n(z_i)_{\bar{\gamma}_i}(\tau) = c_n(\tau)$, i.e.,

$$c_n(\tilde{g}^{-1} \cdot \tau \cdot \tilde{g}) = c_n(\tau),$$

because the family of paths $\{\bar{\gamma}_i \mid 1 \leq i \leq N\}$ is G-invariant.

Therefore the cohomology class $[c_n]$ belongs to $H^1(G_L; \mathbb{Q}_l(n))^G$. The inclusion of Galois groups $\iota : G_L \to G_K$ induces a homomorphism of Galois cohomology $\iota^* : H^1(G_K; \mathbb{Q}_l(n)) \to H^1(G_L; \mathbb{Q}_l(n))$. It follows from the Lyndon spectral sequence of the exact sequence of Galois groups

$$1 \longrightarrow G_L \longrightarrow G_K \longrightarrow G \longrightarrow 1$$

that

$$H^1(G_K; \mathbb{Q}_l(n)) \simeq H^1(G_L; \mathbb{Q}_l(n))^G.$$

Hence there exists a cocycle $s_n \in Z^1(G_K; \mathbb{Q}_l(n))$ such that

$$s_n \circ \iota = c_n$$

in $H^1(G_L; \mathbb{Q}_l(n))$. □

§4. Zagier conjecture for l-adic dilogarithm

In this section we shall prove a part of Zagier conjecture for l-adic dilogarithm. We shall use F instead of L to denote a number field.

Definition 4.0. Let n be a natural number greater than 1. Let F be a number field. We set

$$P_n(F) := \left\{ c_n := \sum_{i=1}^{N} m_i l_n(z_i)_{\gamma_i} \;\middle|\; z_1, \ldots, z_N \in F^*, \right. \\ \left. m_1, \ldots, m_N \in \mathbb{Q},\, c_n \in Z^1(G_F; \mathbb{Q}_l(n)) \right\}.$$

We define $[P_n(F)]$ to be the image of $P_n(F)$ in $H^1(G_F; \mathbb{Q}_l(n))$.

It is clear from the definition that $[P_n(F)]$ is a $\mathbb{Q}$-vector subspace of $H^1(G_F; \mathbb{Q}_l(n))$. Observe that if $H^1(G_F; \mathbb{Q}_l(n))$ is non null then it is an infinite dimensional vector space over $\mathbb{Q}$, because already $\mathbb{Q}_l$ is infinite dimensional over $\mathbb{Q}$. Hence $[P_n(F)]$ considered as a vector space over $\mathbb{Q}$ can have a priori an infinite dimension over $\mathbb{Q}$. Below we state a conjecture that $[P_n(F)]$ is a finite dimensional vector space over $\mathbb{Q}$. We view this conjecture as an analogue of Zagier conjecture for polylogarithms (see [14] and [1]).

We denote by $r_1(F)$ (resp. $r_2(F)$) the number of real (resp. complex) places of F.

Conjecture 4.1. *Let F be a number field. Then*

$$\dim_{\mathbb{Q}}[P_n(F)] \le r_2(F) \quad \textit{if } n \textit{ is even}$$

and

$$\dim_{\mathbb{Q}}[P_n(F)] \le r_1(F) + r_2(F) \quad \textit{if } n \textit{ is odd and } n > 1.$$

DEFINITION 4.2. We define $\mathcal{P}_n(F)$ to be the set of $c_n := \sum_{i=1}^{N} m_i l_n(z_i)_{\gamma_i}$, where $z_1, \ldots, z_N \in F^*$, $m_1, \ldots, m_N \in \mathbb{Q}$ and c_n is a cocycle obtained by the way of Proposition 2.2. We define $[\mathcal{P}_n(F)]$ to be the image of $\mathcal{P}_n(F)$ in $H^1(G_F; \mathbb{Q}_l(n))$.

Observe that for $n = 2$ we have

$$\mathcal{P}_2(F) := \left\{ \sum_{i=1}^{N} m_i l_2(z_i)_{\gamma_i} \;\middle|\; z_1, \ldots, z_N \in F^*,\ m_1, \ldots, m_N \in \mathbb{Q}, \right.$$
$$\sum_{i=1}^{N} m_i (z_i) \wedge (1 - z_i) = 0 \text{ in } F^* \wedge F^* \otimes \mathbb{Q} \text{ and the paths}$$
$$\gamma_i \ (i = 1, \ldots, N) \text{ are such that } l(z_i)_{\gamma_i} \text{ and } l_1(z_i)_{\gamma_i} \text{ respect all}$$
$$\left. \text{multiplicative relations between } z_1, \ldots, z_N,\ 1 - z_1, \ldots, 1 - z_N \right\}.$$

Observe that Conjecture 4.1 implies the following conjecture.

CONJECTURE 4.3. *Let F be a number field. Then*

$$\dim_{\mathbb{Q}} [\mathcal{P}_n(F)] \leq r_2(F) \quad \textit{if } n \textit{ is even}$$

and

$$\dim_{\mathbb{Q}} [\mathcal{P}_n(F)] \leq r_1(F) + r_2(F) \quad \textit{if } n \textit{ is odd and } n > 1.$$

We recall here the definition of the Bloch group (see [6]). Let $D(F)$ be a $\mathbb{Q}$-vector space with basis $[x]$ ($x \in F^* \setminus \{1\}$). Let

$$\lambda : D(F) \longrightarrow (F^* \wedge F^*) \otimes \mathbb{Q}$$

be the homomorphism $[x] \to (x \wedge (1 - x)) \otimes 1$. Let $R(F)$ be the $\mathbb{Q}$-vector subspace of $D(F)$ generated by elements of the form

$$[x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - x}{1 - y}\right]$$

($x \neq y$ belong to $F^* \setminus \{1\}$). We define the Bloch group by setting

$$B(F) := \mathit{ker}\, \lambda / R(F).$$

Remark 4.3.1. The group $B(F)$ defined here is equal to the corresponding group in [6] tensored by $\mathbb{Q}$.

We define a tower of fields

$$F \subset F_1 := F(\mu_{l^\infty}) \subset F_2 := F_1(x^{1/l^\infty} \mid x \in F)$$

and a corresponding tower of Galois groups

$$G_{F_2} \subset G_{F_1} \subset G_F.$$

Let us set

$$\Gamma := \mathrm{Gal}(F(\mu_{l^\infty})/F).$$

PROPOSITION 4.4. *The restriction homomorphism*

$$H^1(G_F; \mathbb{Q}_l(2)) \longrightarrow Hom_\Gamma(G_{F_2}; \mathbb{Q}_l(2))$$

is injective.

Proof. We have an isomorphism

$$H^1(G_F; \mathbb{Q}_l(2)) \longrightarrow Hom_\Gamma(G_{F(\mu_{l^\infty})}; \mathbb{Q}_l(2))$$

induced by the inclusion $G_{F(\mu_{l^\infty})} \subset G_F$. Hence it is enough to show that the restriction map

$$Hom_\Gamma(G_{F(\mu_{l^\infty})}; \mathbb{Q}_l(2)) \longrightarrow Hom_\Gamma(G_{F_2}; \mathbb{Q}_l(2))$$

is injective.

Let $c_2 : G_{F(\mu_{l^\infty})} \to \mathbb{Q}_l(2)$ be a Γ-homomorphism such that the composition

$$G_{F_2} \longrightarrow G_{F(\mu_{l^\infty})} \longrightarrow \mathbb{Q}_l(2)$$

is the zero map. Passing to the quotient group $G_{F(\mu_{l^\infty})}/G_{F_2} = \mathrm{Gal}(F_2/F_1)$ we get a Γ-homomorphism $\bar{c}_2 : \mathrm{Gal}(F_2/F_1) \to \mathbb{Q}_l(2)$. Let $x \in F^*$ be not a root of 1. Observe that then we have an isomorphism of Γ-modules $\mathrm{Gal}(F(\mu_{l^\infty})(x^{1/l^\infty})/F(\mu_{l^\infty})) \simeq \mathbb{Z}_l(1)$. Therefore the map $\bar{c}_2$ must be zero because weights are incompatible. Hence c_2 is also the zero map. □

For any $z \in F$ we denote by $\mathcal{L}_2(z)$ the restrictin of $l_2(z)_\gamma$ to the subgroup G_{F_2} of G_F. Observe that $\mathcal{L}_2(z)$ does not depend on a choice of a path γ and that $\mathcal{L}_2(z)$ is a Γ-homomorphism from G_{F_2} to $\mathbb{Q}_l(2)$ (see [9] Lemma 3.2.1).

We define a homomorphism

$$\mathcal{D} : D(F) \longrightarrow Hom_\Gamma(G_{F_2}; \mathbb{Q}_l(2))$$

by setting

$$\mathcal{D}\Big(\sum_{i=1}^{N} m_i[z_i]\Big) := \sum_{i=1}^{N} m_i \mathcal{L}_2(z_i).$$

LEMMA 4.5. *The subgroup $R(F)$ of $D(F)$ is contained in $ker\,\mathcal{D}$.*

Proof. We must show that the l-adic polylogarithm $\mathcal{L}_2(z)$ satisfies the functional equation

$$\mathcal{L}_2(x) - \mathcal{L}_2(y) + \mathcal{L}_2\Big(\frac{y}{x}\Big) - \mathcal{L}_2\Big(\frac{1-x^{-1}}{1-y^{-1}}\Big) + \mathcal{L}_2\Big(\frac{1-x}{1-y}\Big) = 0. \tag{4.5.1}$$

This is of course one of the forms of the Abel functional equation.

Let $f : \mathbb{P}^1 \setminus \{0,1,\infty\} \to \mathbb{P}^1 \setminus \{0,1,\infty\}$ be given by $f(z) = \frac{z}{z-1}$. Then one has $id_* + f_* = 0$ in $\mathrm{Hom}\big(\Gamma^2\pi/\Gamma^3\pi; \Gamma^2\pi/\Gamma^3\pi\big)$, where $\pi := \pi_1(\mathbb{P}^1_{\bar{F}} \setminus \{0,1,\infty\}; v)$. Hence it follows from [10] Theorem 11.2.1 that

$$\mathcal{L}_2(z) + \mathcal{L}_2\Big(\frac{z}{z-1}\Big) = 0.$$

In [10] we have shown that the l-adic dilogarithm satisfies the functional equation

$$\mathcal{L}_2\Big(\frac{(1-t)x}{x-1}\Big) - \mathcal{L}_2(tx) + \mathcal{L}_2\Big(\frac{(x-1)t}{1-t}\Big) - \mathcal{L}_2\Big(\frac{t}{t-1}\Big) + \mathcal{L}_2(x) = 0.$$

Substituting t by y^{-1} and using functional equations $\mathcal{L}_2(z) + \mathcal{L}_2(z^{-1}) = 0$ (see [10] Corollary 11.2.6) and $\mathcal{L}_2(z) + \mathcal{L}_2\big(\frac{z}{z-1}\big) = 0$ we get the equation (4.5.1). □

THEOREM 4.6. *Let F be a number field. Then we have*

$$\dim_{\mathbb{Q}}[\mathcal{P}_2(F)] \leq r_2(F).$$

Proof. It follows from Lemma 4.5 that $\mathcal{D} : D(F) \to Hom_\Gamma(G_{F_2}; \mathbb{Q}_l(2))$ induces a homomorphism

$$\bar{\mathcal{D}} : ker\,\lambda/R(F) = B(F) \longrightarrow Hom_\Gamma(G_{F_2}; \mathbb{Q}_l(2)).$$

It is well known that $B(F) \simeq K_3(F) \otimes \mathbb{Q}$ (see [7] Theorem 5.2). It follows from the Borel theorem that $\dim_{\mathbb{Q}}(K_3(F) \otimes \mathbb{Q}) = r_2(F)$ (see [2] Proposition 12.2). Therefore we get that

$$\dim_{\mathbb{Q}}(Image\,\bar{\mathcal{D}}) \leq r_2(F). \tag{4.6.1}$$

It follows from Proposition 2.2 that to any element x of $ker\,\lambda$ we can associate a cocycle in $Z^1(G_F;\mathbb{Q}_l(2))$. It follows from Proposition 4.4 that the corresponding cohomology class in $H^1(G_F;\mathbb{Q}_l(2))$ is uniquely determined by the class of x in $B(F)$. Let us denote by $cl_F(x)$ this cohomology class. Observe that we have a commutative diagram

$$\begin{array}{ccc} B(F) & \xrightarrow{cl_F} & H^1(G_F;\mathbb{Q}_l(2)) \\ = \Big\downarrow & & \text{restriction} \Big\downarrow \\ B(F) & \xrightarrow{\bar{\mathcal{D}}} & Hom_\Gamma(G_{F_2};\mathbb{Q}_l(2)). \end{array}$$

It follows from Proposition 4.4 and the already proved inequality (4.6.1) that $\dim_{\mathbb{Q}}[\mathcal{P}_2(F)] \leq r_2(F)$.

Conjecture 4.7. *For a number field F we have*

$$\dim_{\mathbb{Q}}[\mathcal{P}_2(F)] = r_2(F).$$

Remark 4.8. To show Conjecture 4.7 we only need to show that the following diagram commmutes (perhaps up to a multiplication by a constant in $\mathbb{Q}_l^*$)

$$\begin{array}{ccc} B(F) & \xrightarrow{cl_F} & H^1(G_F;\mathbb{Q}_l(2)) \\ = \Big\uparrow & & c_{2,1} \Big\uparrow \\ B(F) & \longleftarrow & H_3(SL(2,F);\mathbb{Z}) \end{array}$$

where the map from $H_3(SL(2,F);\mathbb{Z})$ to $B(F)$ is induced by sending the homogenous 3-simplex (g_0, g_1, g_2, g_3), $g_i \in SL(2,F)$ onto the cross-ratio $[g_0(\infty) : g_1(\infty) : g_2(\infty) : g_3(\infty)]$ of four points of $\mathbb{P}^1(F)$ and $c_{2,1}$ is deduced from the morphisms $c_{2,1} : H_3(SL(2,F);\mathbb{Z}/l^n) \to H^1(G_F;\mathbb{Z}/l^n(2))$ (see [5] p. 258).

The Bloch-Wigner function $D(z)$ defines a continous cocycle on $SL(2,\mathbb{C})$ which is the universal class from which one gets all generators of $H^3(SL(2,F);\mathbb{Q})$ for any number field F.

We shall define a cohomology class

$$\mathcal{D}(F) \in H^3(SL(2,F); Hom_\Gamma(G_{F_2};\mathbb{Q}_l(2))$$

by setting

$$\mathcal{D}(F)(g_0, g_1, g_2, g_3) := \mathcal{L}_2([g_0(\infty) : g_1(\infty) : g_2(\infty) : g_3(\infty)])$$

on the homogenous 3-simplex. $\mathcal{D}(F)$ is a cocycle because $\mathcal{L}_2$ satisfies 5-term functional equation (see [10] Theorem 11.1.14). Evaluating the cocycle $\mathcal{D}(F)$ on $H_3(SL(2,F))$ we get a homomorphism

$$H_3(SL(2,F);\mathbb{Z}) \longrightarrow Hom_\Gamma(G_{F_2};\mathbb{Q}_l(2))$$

which factors through

$$\mathcal{D}(F) : H_3(SL(2,F);\mathbb{Z}) \longrightarrow H^1(G_F;\mathbb{Q}_l(2)).$$

We do not know if this map is $c_{2,1}$.

Let F' be a finite extension of F. We have two maps

$$(j_{F'/F})_* : H_3(SL(2,F);\mathbb{Z}) \longrightarrow H_3(SL(2,F');\mathbb{Z})$$

and

$$(\iota_{F'/F})^* : H^1(G_F;\mathbb{Q}_l(2)) \longrightarrow H^1(G_{F'};\mathbb{Q}_l(2))$$

induced by the inclusion $F \subset F'$. One easily sees that

$$(\iota_{F'/F})^* \circ \mathcal{D}(F) = \mathcal{D}(F') \circ (j_{F'/F})_*.$$

§5. Action of G and Zagier conjecture for l-adic polylogarithms

We return to our study of descent properties of l-adic polylogarithms.

Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$.

Definition 5.1. We define $\mathcal{P}_n(L/K)$ to be the set of $c_n := \sum_{i=1}^N m_i \, l_n(z_i)_{\gamma_i}$, where $z_1, \dots, z_N \in L^* \setminus \{1\}$, $m_1, \dots, m_N \in \mathbb{Q}$, the set $\{z_1, \dots, z_N\}$ and the formal sum $\sum_{i=1}^N m_i[z_i]$ are G-invariant and c_n is a cocycle obtained by the way of Proposition 2.2. We define $[\mathcal{P}_n(L/K)]$ to be the image of $\mathcal{P}_n(L/K)$ in $H^1(G_L;\mathbb{Q}_l(n))$.

It follows from Theorem 3.5 that

$$[\mathcal{P}_n(L/K)] \subset H^1(G_L;\mathbb{Q}_l(n))^G. \tag{5.1.1}$$

It follows from the Lyndon spectral sequence that

$$H^1(G_L;\mathbb{Q}_l(n))^G \simeq H^1(G_K;\mathbb{Q}_l(n)). \tag{5.1.2}$$

Hence we get that

$$[\mathcal{P}_n(L/K)] \subset H^1(G_K;\mathbb{Q}_l(n)). \tag{5.1.3}$$

DEFINITION 5.2. Let $[\mathcal{P}_n(\bar{K}/K)]$ be the $\mathbb{Q}$-vector subspace of $H^1(G_K;\mathbb{Q}_l(n))$ generated by subspaces $[\mathcal{P}_n(L/K)]$ of $H^1(G_K;\mathbb{Q}_l(n))$ for all finite Galois extensions L of K.

CONJECTURE 5.3. *Let K be a number field. Then*

$$\dim_{\mathbb{Q}}[\mathcal{P}_n(\bar{K}/K)] \leq r_2(K) \quad \textit{if } n \textit{ is even}$$

and

$$\dim_{\mathbb{Q}}[\mathcal{P}_n(\bar{K}/K)] \leq r_1(K) + r_2(K) \quad \textit{if } n \textit{ is odd and } n > 1.$$

THEOREM 5.4. *Let K be a number field. Then we have*

$$\dim_{\mathbb{Q}}[\mathcal{P}_2(\bar{K}/K)] \leq r_2(K).$$

Proof. Let L be a finite Galois extension of a number field K. Let $G := Gal(L/K)$. We have the following commutative diagram

$$\begin{array}{ccc} B(L) & \longleftarrow & K_3(L)\otimes\mathbb{Q} \\ \uparrow & & \uparrow \\ B(K) & \longleftarrow & K_3(K)\otimes\mathbb{Q} \end{array}$$

where the vertical maps are induced by the inclusion $K \subset L$ and the horizontal maps are isomorphisms (see [7] Theorem 5.2). The group G acts on $B(L)$ and on $K_3(L)\otimes\mathbb{Q}$ and the upper horizontal map is G-invariant. Hence we get the following diagram

$$\begin{array}{ccc} B(L)^G & \longleftarrow & (K_3(L)\otimes\mathbb{Q})^G \\ \uparrow & & \uparrow \\ B(K) & \longleftarrow & K_3(K)\otimes\mathbb{Q} \end{array}$$

For the K-theory of fields we have $K_3(K)\otimes\mathbb{Q}\simeq(K_3(L)\otimes\mathbb{Q})^G$. Hence we get

$$B(K)\simeq B(L)^G. \tag{5.4.1}$$

Let $c\in\mathcal{P}_2(L/K)$. Then $c=\sum_{i=1}^N m_i l_2(z_i)_{\gamma_i}$ for some $\sum_{i=1}^N m_i[z_i]\in B(L)$, such that the set $\{z_1,\dots,z_N\}$ and the formal sum $\sum_{i=1}^N m_i[z_i]$ are G-invariant. Therefore there exists $\sum_{j=1}^M \mu_j[\zeta_j]\in B(K)$ corresponding to $\sum_{i=1}^N m_i[z_i]$ via isomorphism (5.4.1). The cocycles c and $s=\sum_{j=1}^M \mu_j l_2(\zeta_j)_{\delta_j}$ represent the same cohomology class in $H^1(G_L;\mathbb{Q}_l(2))$. The cohomology class of s belongs to $H^1(G_K;\mathbb{Q}_l(2))$, hence $[\mathcal{P}_2(L/K)]\subset[\mathcal{P}_2(K)]$. Now the theorem follows from Theorem 4.6. ∎

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