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ALGEBRAIC SURFACES OF GENERAL TYPE WITH SMALL c_1^2 IN POSITIVE CHARACTERISTIC

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Abstract. We establish Noether's inequality for surfaces of general type in positive characteristic. Then we extend Enriques' and Horikawa's classification of surfaces on the Noether line, the so-called Horikawa surfaces. We construct examples for all possible numerical invariants and in arbitrary characteristic, where we need foliations and deformation techniques to handle characteristic 2. Finally, we show that Horikawa surfaces lift to characteristic zero.

Introduction

The genus g and the degree of a canonical divisor K_C of a smooth projective curve are related by the well-known formula $\deg K_C = 2g - 2$.

Already in 1875, Max Noether [Noe] has given the following generalisation to surfaces: Given a minimal surface of general type X over the complex numbers with self-intersection K_X^2 of a canonical divisor (playing the role of the degree of a canonical divisor) and with geometric genus p_g then

$$K_X^2 \geq 2p_g - 4.$$

It is natural to classify surfaces for which equality holds, i.e., surfaces on the so-called *Noether line*. Over the complex numbers, this has been sketched in Enriques' book [En, Capitolo VIII.11] and a detailed analysis has been carried out by Horikawa [Hor2]. The result is that surfaces on the Noether line, also called *Horikawa surfaces*, are double covers of rational surfaces via their canonical map. Hence these surfaces may be thought of as a two-dimensional generalisation of hyperelliptic curves.

Another point of view comes from Šafarevič's book [S, Chapter 6.3]: The 3-canonical map of a complex surface of general type is birational as soon as $K_X^2 > 3$. Hence, by Noether's inequality, surfaces with $p_g > 3$ have a birational 3-canonical map. However, Horikawa surfaces with $p_g = 3$

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provide examples of surfaces where the 3-canonical map is not birational. Also this may be thought of as a generalisation of hyperelliptic curves.

In this article, we extend these results to surfaces of general type over algebraically closed fields of arbitrary characteristic. We first show that Noether's inequality still holds. Our contribution here lies in skipping through the literature to find a characteristic-free proof.

THEOREM 2.1. *Let X be a minimal surface of general type. Then*

$$K_X^2 \geq 2p_g - 4.$$

If the canonical system is composed with a pencil and $p_g \geq 3$, then $K_X^2 \geq 2p_g - 2$ holds true.

DEFINITION 2.2. A *Horikawa surface* is a minimal surface X of general type for which the equality $K_X^2 = 2p_g - 4$ holds.

In order to extend the classification of Enriques and Horikawa of these surfaces we need Clifford's theorem on special linear systems for singular curves. With this result we can avoid Bertini's theorem in the classical argumentation and obtain literally the same result, now valid in all characteristics. As usual, there is an extra twist in characteristic 2.

THEOREM 3.3 AND PROPOSITION 4.2. *Let X be a Horikawa surface and $S := \phi_1(X)$ the image of the canonical map, which is a possibly singular surface in $\mathbb{P}^{p_g-1}$.*

Then ϕ_1 is a generically finite morphism of degree 2 and we have the following cases:

(1) *If S is a smooth surface then we have the following possibilities*
   - $S \cong \mathbb{P}^2$ *and* $p_g = 3$.
   - $S \cong \mathbb{P}^2$ *and* $p_g = 6$.
   - $S \cong \mathbb{F}_d$ *and* $p_g \geq \max\{d+4, 2d-2\}$ *and* $p_g - d$ *is even.*

(2) *If S is not smooth then it is the cone over the rational normal curve of degree $d := p_g - 2$. The minimal desingularisation of S is the Hirzebruch surface $\mathbb{F}_d$ and $4 \leq p_g \leq 6$.*

From this description we deduce that Horikawa surfaces are algebraically simply connected and that their Picard schemes are reduced, cf. Proposition 3.7. Another byproduct is Proposition 3.9, which tells us that the 3-canonical map of a Horikawa surface with $p_g = 3$ is not birational onto its image.

Conversely, out of this data one can always construct a Horikawa surface. In characteristic $p \neq 2$ this can be done along the lines of Horikawa's article [Hor2, Section 1], cf. Section 5.

Hence we are interested in Horikawa surfaces in characteristic 2. To obtain such surfaces, we use quotients of minimal rational surfaces by p-closed foliations. The main technical difficulty is that these vector fields necessarily have isolated singularities so that we need to control the singularities of the quotients.

The canonical map of such a surface is a purely inseparable morphism onto a rational surface. In particular, these surfaces are inseparably unirational. This is in contrast to curves, whose canonical maps are always separable.

THEOREM. (Horikawa [Hor2, Section 1], Section 5) *All possible cases of the previous theorem do exist in arbitrary characteristic. In characteristic* 2*, we may even assume the canonical map to be inseparable.*

To get Horikawa surfaces in characteristic 2 with separable canonical map, we use a deformation argument. Morally speaking, it says that surfaces with inseparable canonical map should be at the boundary of the moduli space. In particular, all possible numerical invariants for a Horikawa surface in characteristic 2 do occur with surfaces that have a separable canonical map.

THEOREM 6.3. *In characteristic* 2*, every Horikawa surface with inseparable canonical map can be* (*birationally*) *deformed into a Horikawa surface with a separable canonical map, while fixing* p_g *and the canonical image.*

Since the classification looks the same in every characteristic, it is natural to ask whether Horikawa surfaces over an algebraically closed field of positive characteristic k lift over the Witt ring $W(k)$. I.e., we look for a scheme $\mathcal{X}$, flat over $\operatorname{Spec} W(k)$ and with special fibre the given surface over k.

Theorem 7.1. *The canonical model of a Horikawa surface lifts over $W(k)$. For every Horikawa surface there exists an algebraic space, flat over a possibly ramified extension of $W(k)$, that achieves the lifting.*

As a Horikawa surface is a double cover of a rational surface, the idea of proving Theorem 7.1 is first to lift the rational surface and then the line bundle associated with this double cover, which then defines a lifting of the whole double cover and hence a lifting of the Horikawa surface in question.

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§1. Singular hyperelliptic curves

In order to prove Theorem 2.3 below, we recall some facts about singular hyperelliptic curves. In this section, curves will always be assumed to be reduced and irreducible as well as proper over an algebraically closed field k. We denote by $p_a(C) := 1 - \chi(\mathcal{O}_C)$ the arithmetic genus of C.

Definition 1.1. A reduced and irreducible curve C is called *hyperelliptic* if $p_a(C) \geq 2$ and if there exists a morphism of degree 2 from C onto $\mathbb{P}^1$.

It follows that C is automatically Gorenstein, say with invertible dualising sheaf ω_C. Clearly, a smooth curve is hyperelliptic in the sense of Definition 1.1 if and only if it is hyperelliptic in the classical sense.

An immediate consequence that will be used later on is the following result.

Lemma 1.2. *Let $\mathcal{L}$ be an invertible sheaf on a hyperelliptic curve such that $\mathcal{L}^{\otimes 2} \cong \omega_C$. Then $|\mathcal{L}|$ does not define a birational map.*

Proof. Let $\phi : C \to \mathbb{P}^1$ be a morphism of degree 2. The fibres of ϕ provide us with an infinite number of smooth points x, y (possibly $x = y$) such that $h^0(\mathcal{O}_C(x + y)) \geq 2$.

First, suppose that ϕ is generically étale. Then we may assume $x \neq y$. The long exact sequence in cohomology, Serre duality and the assumption on $\mathcal{L}$ yield

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathcal{L} \otimes \mathcal{O}_C(-x-y)) & \longrightarrow & H^0(\mathcal{L}) & \longrightarrow & \mathcal{L}/(\mathfrak{m}_x \oplus \mathfrak{m}_y) \cdot \mathcal{L} \\ & \stackrel{\delta}{\longrightarrow} & H^0(\mathcal{L} \otimes \mathcal{O}_C(x+y))^\vee & \longrightarrow & H^0(\mathcal{L})^\vee & \longrightarrow & 0\,. \end{array}$$

By the choice of x, y and [Hart, Lemma IV.5.5], we have

$$h^0(\mathcal{L} \otimes \mathcal{O}_C(x+y)) \geq h^0(\mathcal{L}) + h^0(\mathcal{O}_C(x+y)) - 1 \geq h^0(\mathcal{L}) + 1,$$

which implies that the boundary map δ is non-trivial. In particular, $|\mathcal{L}|$ fails to separate an infinite number of distinct points x and y and so the associated map cannot be birational.

If ϕ is not generically étale, then its fibres provide us with an infinite number of points where $|\mathcal{L}|$ fails to be an embedding (similar long exact sequence as before). Again, $|\mathcal{L}|$ cannot be birational. □

It is interesting that Clifford's theorem on special line bundles remains true in the singular case.

THEOREM 1.3. (Clifford's theorem) *Let $\mathcal{L}$ be an invertible sheaf on a reduced Gorenstein curve C. If both, $h^0(C, \mathcal{L})$ and $h^1(C, \mathcal{L})$ are non-zero then*

$$h^0(C, \mathcal{L}) \leq \frac{1}{2} \deg \mathcal{L} + 1.$$

Moreover, if equality holds then $\mathcal{L} \cong \mathcal{O}_C$ or $\mathcal{L} \cong \omega_C$ or C is hyperelliptic.

Proof. [EKS, Theorem A], where even a version for torsion free sheaves is given. □

Although we do not need this result in the sequel we note that being hyperelliptic can be rephrased in terms of the canonical map $|\omega_C|$, just as in the smooth case. This is proved as in the classical case or along the lines of the proof of Lemma 1.2.

PROPOSITION 1.4. *A Gorenstein curve C with $p_a(C) \geq 2$ is hyperelliptic if and only if $|\omega_C|$ is not birational.* □

§2. Noether's inequality and Horikawa surfaces

The results of this section are well-known over the complex numbers. In order to extend them to positive characteristic we run through the classical arguments and have to find new ones whenever Bertini's theorem or vanishing results are used.

We start with Noether's inequality [Noe, Abschnitt 11]. Probably it is known to the experts that it holds in arbitrary characteristic.

Theorem 2.1. (Noether's inequality) *Let X be a minimal surface of general type. Then*

$$K_X^2 \geq 2p_g - 4.$$

If the canonical system is composed with a pencil and $p_g \geq 3$, then $K_X^2 \geq 2p_g - 2$ holds true.

Proof. Since $K_X^2 > 0$ we may assume that $p_g \geq 3$. Thus, the canonical system is non-empty and either is composed with a pencil or has a 2-dimensional image.

If the canonical linear system is composed with a pencil and $p_g \geq 3$, we argue as in the proof of [BHPV, Theorem VII.3.1] to get the inequality $K_X^2 \geq 2p_g - 2$ in this case.

If the canonical map has a 2-dimensional image, [E2, Proposition 0.1.3 (iii)] yields the desired inequality. □

In view of this inequality it is natural to classify those surfaces where K_X^2 attains the minimal value possible given p_g. Over the complex numbers, this classification has been sketched in Enriques' book [En, Capitolo VIII.11] and carried out in detail by Horikawa [Hor2].

Definition 2.2. A *Horikawa surface* is a minimal surface X of general type for which the equality $K_X^2 = 2p_g(X) - 4$ holds.

Sometimes these surfaces are referred to as *even Horikawa surfaces* as K_X^2 is always an even number. Since Horikawa also classified surfaces for which $K_X^2 = 2p_g - 3$ holds and K_X^2 for such a surface is an odd number these latter surfaces are sometimes called *odd Horikawa surfaces*. However, we will only deal with even Horikawa surfaces in this article, and so we will simply refer to them as Horikawa surfaces.

We now establish the structure result about the canonical map of a Horikawa surface in positive characteristic. Although our proof is essentially the same as the original one by Enriques and Horikawa, we have to be a little bit careful applying Bertini's theorem to the canonical linear system. In fact, the Horikawa surfaces in characteristic 2 that we will construct in Section 5 have the property that a generic canonical divisor is a singular rational curve.

Theorem 2.3. (Enriques, Horikawa) *Let X be a Horikawa surface. Then $p_g \geq 3$ and the canonical map ϕ_1 is basepoint-free and without fixed part. More precisely, ϕ_1 is a generically finite morphism of degree 2 onto a possibly singular surface of degree $p_g - 2$ inside $\mathbb{P}^{p_g-1}$.*

Proof. Since Noether's inequality is an equality and since $K_X^2 > 0$ for a minimal surface of general type we have $p_g \geq 3$. In particular, the canonical system is not empty. By Theorem 2.1, the canonical system is not composed with a pencil and hence the image of the canonical map ϕ_1 is a surface.

It follows from [E2, Proposition 0.1.2 (iii)] that ϕ_1 is basepoint-free and either birational or of degree 2 onto a (possibly singular) ruled surface.

Suppose that ϕ_1 is birational. Let D be a canonical divisor, which we can assume to be irreducible by Bertini's theorem [Jou, Théorème I.6.10]. Being birational, ϕ_1 is generically unramified, which implies that D is reduced over an open and dense subset (loc. cit.). We may thus assume that D is a reduced and irreducible curve. Arguing as in [Hor1, Lemma 2] we get $2p_g - 4 \leq D^2 \leq K_X^2$ and by our assumptions we have equality everywhere. Also, it is shown in (loc. cit.) that $\mathcal{L} := \omega_X \otimes \mathcal{O}_D$ is an invertible sheaf on D with non-vanishing h^0 and h^1 for which Clifford's inequality is an equality. By Theorem 1.3, the curve D is hyperelliptic in the sense of Definition 1.1. We see $\mathcal{L}^{\otimes 2} \cong \omega_D$ from the adjunction formula on X and so the map defined by $|\mathcal{L}|$ is not birational by Lemma 1.2. However, this contradicts the birationality of ϕ_1. Hence the canonical map cannot be birational.

Thus, $\deg \phi_1 = 2$ and the image of ϕ_1 is an irreducible but possibly singular surface of degree at most $p_g - 2$ in $\mathbb{P}^{p_g-1}$. However, $p_g - 2$ is the lowest degree possible for a non-degenerate surface in $\mathbb{P}^{p_g-1}$ and so the image of ϕ_1 has degree equal to $p_g - 2$. We write $|K_X| = |M| + F$, where F denotes the fixed part and M denotes the movable part of the canonical system. As K_X is nef, we have $2p_g - 4 = M^2 \leq K_X^2$, and hence equality by our assumptions. From $MF + F^2 = K_X F \geq 0$ and $MF \geq 0$ together with $K_X^2 = M^2$ it is not difficult to deduce $F^2 = 0$ and $K_X F = 0$. Since $K_X^2 > 0$ and $K_X F = F^2 = 0$, the Hodge index theorem implies that F is numerically trivial. On the other hand, F is an effective divisor and being numerically trivial we see that F is the zero divisor. $\square$

Remark 2.4. A surface in $\mathbb{P}^n$ that spans the ambient space has degree at least $n - 1$. Surfaces of minimal degree have been classified by del Pezzo (see [EH] for a modern account) and consist of $\mathbb{P}^2$, Hirzebruch surfaces, as well as cones over rational normal curves.

§3. Classification of Horikawa surfaces

In this section we give a more detailed description of Horikawa surfaces similar to [En, Capitolo VIII.11] and [Hor2, Section 1]. In order for this to work also in presence of inseparable maps and wild ramification, we do not use the language of branch divisors but use the description of flat morphisms of degree 2 in terms of their associated line bundles.

Let $\pi : X \to S$ be a flat double cover, i.e., a finite, flat and surjective morphism of degree 2. Via

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \pi_* \mathcal{O}_X \longrightarrow \mathcal{L}^\vee \longrightarrow 0 \tag{1}$$

we obtain a sheaf $\mathcal{L}^\vee$ on S, which is invertible by our assumptions on π. In this case, we define $\mathcal{L}$ to be its dual.

Definition 3.1. We refer to $\mathcal{L}$ as the *line bundle associated with* π.

Flat double covers of smooth varieties are automatically Gorenstein by [CD, Proposition 0.1.3] and the dualising sheaf ω_X of X is given by

$$\omega_X \cong \pi^*(\omega_S \otimes \mathcal{L}). \tag{2}$$

Using the projection formula and (1), we obtain an extension

$$0 \longrightarrow \omega_S \otimes \mathcal{L} \longrightarrow \pi_* \omega_X \longrightarrow \omega_S \longrightarrow 0. \tag{3}$$

Remark 3.2. If $p_g(S) = 0$, then the canonical map of X factors as π followed by the map from S associated with $\omega_S \otimes \mathcal{L}$.

For a natural number $d \geq 0$, we denote by $\mathbb{F}_d$ the Hirzebruch surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$. This $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ has a section Δ_0 with self-intersection number $\Delta_0^2 = -d$, which is unique if d is positive. We denote by Γ the class of a fibre of this $\mathbb{P}^1$-bundle.

We now state and prove the structure result for Horikawa surfaces.

Theorem 3.3. *Let X be a Horikawa surface and $S := \phi_1(X)$ its canonical image in $\mathbb{P}^{p_g - 1}$.*

(1) *If S is smooth then ϕ_1 factors as*

$$\phi_1 : X \longrightarrow X_{\mathrm{can}} \xrightarrow{\pi} S,$$

where X_{can} denotes the canonical model of X and where π is a finite and flat morphism of degree 2. Let $\mathcal{L}$ be the line bundle associated with π. Then we have the following possibilities:

- *$S \cong \mathbb{P}^2$, $p_g = 3$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(4)$,*
- *$S \cong \mathbb{P}^2$, $p_g = 6$ and $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(5)$,*
- *$S \cong \mathbb{F}_d$, $0 \le d \le p_g - 4$, $p_g - d$ is even and $\mathcal{L} \cong \mathcal{O}_{\mathbb{F}_d}(3\Delta_0 + \frac{1}{2}(p_g + 2 + 3d)\Gamma)$.*

(2) *If S is not smooth then it is the cone over a rational normal curve of degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$. Also $p_g \ge 4$ and the minimal desingularisation $\nu : \tilde{S} \to S$ is isomorphic to $\mathbb{F}_{p_g-2}$. There exists a partial desingularisation X' of the canonical model X_{can} such that ϕ_1 factors as*

$$\phi_1 : X \longrightarrow X' \xrightarrow{\pi} \mathbb{F}_{p_g-2} \xrightarrow{\nu} S,$$

where π is a finite and flat morphism of degree 2. If $\mathcal{L}$ denotes the line bundle associated with π then $\mathcal{L} \cong \mathcal{O}_{\mathbb{F}_d}(3\Delta_0 + \frac{1}{2}(p_g + 2 + 3d)\Gamma)$.

Remark 3.4. We will see in Proposition 4.2 that there are further restrictions on p_g and d.

Proof. By construction, ϕ_1 factors over the canonical model X_{can}. We know from Theorem 2.3 that S is a surface of degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$, i.e., a surface of minimal degree. Such a surface is a smooth rational surface or the cone over the rational normal curves of degree $p_g - 2$ by [EH]. Thus, if S is not smooth its minimal desingularisation is the Hirzebruch surface $\mathbb{F}_{p_g-2}$.

In any case, we denote by $\nu : \tilde{S} \to S$ the minimal desingularisation of S. Copying the proof of [Hor2, Lemma 1.5] we see that ϕ_1 factors over $\tilde{S}$. Let ψ be the induced morphism $X \to \tilde{S}$, which is generically finite of degree 2 by Theorem 2.3. Let X' be the Stein factorisation of ψ and denote by π the induced morphism $\pi : X' \to \tilde{S}$, which is finite of degree 2. Moreover, π is flat since X' is Cohen-Macaulay (being a normal surface) and $\tilde{S}$ is regular. Moreover, since flat double covers of smooth varieties are Gorenstein, it follows that X' is Gorenstein.

The morphism $X \to X'$ is birational. Since ϕ_1 factors over X_{can}, it is not difficult to see that the canonical morphism $X \to X_{\mathrm{can}}$ factors over X'. In particular, X' has at worst rational singularities, which are Gorenstein by what have already proved, i.e., X' has at worst Du Val singularities. As $\varphi : X' \to X_{\mathrm{can}}$ is a birational morphism and X' has only Du Val singularities, X_{can} is also the canonical model of X' and hence φ partially resolves the singularities of X_{can}.

A smooth surface S of minimal degree is either $\mathbb{P}^2$ embedded via $|H|$ or $|2H|$ into projective space or a Hirzebruch surface $\mathbb{F}_d$ embedded into projective n-space via $|\Delta_0 + \frac{1}{2}(n-1+d)\Gamma|$, where $n-d-3$ is an even and non-negative integer (see for example [Hor2, Lemma 1.2]). Thus, if X is a Horikawa surface and $\phi_1(X)$ is a smooth surface S, the canonical system of X factors over $\omega_S \otimes \mathcal{L}$ on S, where $\omega_S \otimes \mathcal{L}$ is one of the linear systems just described. This yields the first list, cf. also [Hor2, Section 1].

If S is a singular surface of minimal degree, then $\tilde{S} \cong \mathbb{F}_{p_g-2}$ and the embedding of S is given by $|\Delta_0 + d\Gamma|$ on $\tilde{S}$. Proceeding as before, we obtain the description of $\mathcal{L}$ in this case. Again, we refer to [Hor2, Section 1] for details. ∎

Before proceeding we need (or recall) a simple vanishing result.

Lemma 3.5. *We have $H^1(\mathbb{P}^2, \mathcal{L}) = 0$ for every line bundle on $\mathbb{P}^2$. On the Hirzebruch surface $\mathbb{F}_d$ we have $H^1(\mathbb{F}_d, \mathcal{O}_{\mathbb{F}_d}(a\Delta_0 + b\Gamma)) = 0$ if*

(1) *$a \geq 0$ and $b \geq 0$ or*
(2) *$a \leq -2$ and $b \leq -(d+2)$.*

Proof. We leave the assertion about line bundles on $\mathbb{P}^2$ to the reader. We consider the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{F}_d}((a-1)\Delta_0) \longrightarrow \mathcal{O}_{\mathbb{F}_d}(a\Delta_0) \longrightarrow \mathcal{O}_{\Delta_0} \longrightarrow 0.$$

Taking cohomology and noting that the statement is clear for $a = b = 0$ we obtain the assertion for $a \geq 0$ and $b = 0$ inductively. Using this result, an induction on b shows the vanishing for $a \geq 0$ and $b \geq 0$. Applying Serre duality we obtain the remaining vanishing result. ∎

The following result is crucial to prove that Horikawa surfaces are simply connected as well as to show that there are further dependencies between p_g and d in Theorem 3.3.

Lemma 3.6. *Let $\mathcal{L}$ be as in Theorem 3.3. Then $h^1(\mathcal{L}) = h^1(\mathcal{L}^\vee) = 0$.*

Proof. This follows immediately from inspecting the list of possible $\mathcal{L}$'s given in Theorem 3.3 and then applying Lemma 3.5. ∎

As an application of Theorem 3.3 we obtain

PROPOSITION 3.7. *A Horikawa surface fulfills $h^{01}(X) := h^1(\mathcal{O}_X) = 0$ and is algebraically simply connected, i.e., has a trivial étale fundamental group. In particular, its Picard scheme is reduced.*

Proof. We use the notations of Theorem 3.3. If the canonical image $\phi_1(X)$ is a smooth surface, the long exact sequence of cohomology applied to (1) together with Lemma 3.6 yields $h^1(\mathcal{O}_{X_{\mathrm{can}}}) = 0$. Since X_{can} has at worst rational singularities, the Grothendieck-Leray spectral sequence associated with the push-forward of the structure sheaf yields $h^1(\mathcal{O}_X) = 0$. The case where the canonical image $\phi_1(X)$ is a singular surface is similar and left to the reader.

From $h^1(\mathcal{O}_X) = 0$ it follows that the Picard scheme of X is reduced.

To prove that X is algebraically simply connected we use the idea of [Bo, Theorem 14]: Let $\hat{X} \to X$ be an étale cover of degree m. Then we compute $\chi(\mathcal{O}_{\hat{X}}) = m\chi(\mathcal{O}_X)$ and $K_{\hat{X}}^2 = mK_X^2$. Using Noether's inequality (Theorem 2.1), we obtain

$$m(1+p_g(X)) = 1 - h^1(\mathcal{O}_{\hat{X}}) + p_g(\hat{X}) \le 1 + p_g(\hat{X}) \le \frac{1}{2}K_{\hat{X}}^2 + 3 = \frac{m}{2}K_X^2 + 3.$$

We assumed X to be a Horikawa surface and so this inequality holds for $m = 1$ only. Thus, every étale cover is trivial and hence X is algebraically simply connected. ∎

Remark 3.8. Over the complex numbers, even the topological fundamental group of a Horikawa surface is trivial [Hor2, Theorem 3.4].

Surfaces with $K^2 = 2$ and $p_g = 3$, i.e., Horikawa surfaces with $p_g = 3$, have also been studied in Šafarevič's book [S, Chapter 6.3]. The emphasis there is on the fact that these complex surfaces are the only surfaces besides those with $K^2 = 1$ and $p_g = 2$ where $|3K_X|$ does not define a birational map.

PROPOSITION 3.9. *The* 3*-canonical map of a minimal surface of general type with $K^2 = 2$ and $p_g = 3$ is a morphism but not birational.*

Proof. From Theorem 3.3, we see that the canonical map of X exhibits the canonical model X_{can} as a flat double cover $\pi : X_{\mathrm{can}} \to \mathbb{P}^2$ with associated line bundle $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^2}(4)$. Using (2), we see that $\omega_{X_{\mathrm{can}}}^{\otimes 3} \cong \pi^*(\mathcal{O}_{\mathbb{P}^2}(3))$. By the projection formula, the pushforward $\pi_*(\omega_{X_{\mathrm{can}}}^{\otimes 3})$ is an extension of

$\mathcal{O}_{\mathbb{P}^2}(-1)$ by $\mathcal{O}_{\mathbb{P}^2}(3)$. Taking global sections, we see that all global sections of $\omega_{X_{\mathrm{can}}}^{\otimes 3}$ are pull-backs of global sections of $\mathcal{O}_{\mathbb{P}^2}(3)$. Hence the 3-canonical map of X is a morphism and factors over π. In particular, it is not birational. $\square$

§4. The canonical double cover

We now take a closer look at the canonical double cover of a Horikawa surface. In particular, we will see in Proposition 4.2 that there are more restrictions on the line bundle $\mathcal{L}$ in Theorem 3.3.

Let $\pi : X \to S$ be a flat double cover where X is a normal and S is a smooth variety. Let $\mathcal{L}$ be the associated line bundle and consider the short exact sequence (1). If the characteristic of the ground field is different from 2 then the extension of function fields $k(X)/k(S)$ is Galois with a cyclic Galois group of order 2. The Galois action extends to an action on $\pi_*\mathcal{O}_X$ and decomposing into the ± 1-eigensheaves we obtain a splitting of (1).

If the characteristic of the ground field is $p = 2$ then there is no reason for (1) to split. Of course, this sequence has to split if $ext^1(\mathcal{L}^\vee, \mathcal{O}_S) = h^1(\mathcal{L}) = 0$. By Lemma 3.6, this condition is fulfilled for Horikawa surfaces.

If π is a flat double cover as above such that (1) splits then there exist global sections $f \in H^0(\mathcal{L})$ and $g \in H^0(\mathcal{L}^{\otimes 2})$ such that the cover $\pi : X \to S$ is *globally* over S given by

$$z^2 + fz + g = 0, \tag{4}$$

where z is a fibre coordinate on $\mathcal{L}$. Moreover, if the characteristic p is different from 2 or if $p = 2$ and π is purely inseparable, we may even assume $f = 0$.

Lemma 4.1. *Let $\pi : X \to S$ be a flat double cover where X is normal and S is smooth. In characteristic $p = 2$ we assume moreover that the associated sequence* (1) *splits.*

(1) *If $p \neq 2$ or $p = 2$ and π is purely inseparable then $\mathcal{L}^{\otimes 2}$ has a reduced section.*
(2) *If $p = 2$ and π is separable then there exist sections f and g of $\mathcal{L}$ and $\mathcal{L}^{\otimes 2}$, respectively such that* $\operatorname{div}(g)$ *has no component of multiplicity ≥ 2 along a component of* $\operatorname{div}(f)$.

Proof. We may assume that π is given by (4).

In the first case we may assume $f = 0$. If h^2 divides g then, by the Jacobian criterion, X is singular along the divisor $\{\pi^*(h) = 0\}$. However, this contradicts the normality of X.

In the second case, if h divides f and h^2 divides g then, again by the Jacobian criterion, X is singular along $\pi^*(h) = 0$. Again, this contradicts the normality of X. ∎

This result imposes further restrictions on the line bundle $\mathcal{L}$ of Theorem 3.3 and we obtain the same result as in [Hor2, Section 1], now valid in arbitrary characteristic.

PROPOSITION 4.2. *Let X be a Horikawa surface and S be the image of its canonical map.*

(1) *If S is the smooth Hirzebruch surface $\mathbb{F}_d$, then $p_g \geq 2d - 2$.*
(2) *If S is singular, then $p_g \leq 6$.*

Proof. Let $\tilde{S}$ be the minimal desingularisation of S and consider the finite and flat double cover $\pi : X' \to \tilde{S}$ induced by ϕ_1 as in Theorem 3.3. Let $\mathcal{L}$ be the line bundle associated with π and L be its class in $\mathrm{NS}(\tilde{S})$. We know from Lemma 3.6 that $H^1(\mathcal{L})$ vanishes and so the short exact sequence (1) associated with π splits.

We may assume that $d > 0$ (else there is nothing to prove), in which case the divisor Δ_0 is unique. We claim that $2L \cdot \Delta_0 \geq -d$. If not then every section of $\mathcal{L}^{\otimes 2}$ vanishes with multiplicity ≥ 2 along Δ_0. Hence every section of $\mathcal{L}$ vanishes along Δ_0. This however, contradicts Lemma 3.6 in all possible cases.

If $S = \mathbb{F}_d$ then the inequality just shown together with Theorem 3.3 yields $p_g \geq 2d - 2$. In case S is the cone over the rational normal curve of degree d we obtain $p_g \leq 6$. ∎

§5. Inseparable canonical maps

Theorem 3.3 and Proposition 4.2 restrict the numerical invariants of a Horikawa surface. Conversely, we now establish the existence of Horikawa surfaces for all remaining invariants. If the characteristic is different from 2, this can be done along the classical arguments of Enriques and Horikawa.

In characteristic 2, we will first treat existence for surfaces with inseparable canonical map. In the next section we get the remaining case by a deformation argument.

A converse to Theorem 3.3 and Proposition 4.2 in characteristic $p \neq 2$ can be proved along the lines of [Hor2, Section 1]. We only sketch the argument here.

THEOREM 5.1. (Horikawa) *Let $\tilde{S}$ be a minimal rational surface in characteristic $p \neq 2$ and $\mathcal{L}$ a line bundle on $\tilde{S}$. Assume moreover, that $\tilde{S}$ and $\mathcal{L}$ are as in Theorem 3.3 and that the additional inequalities of Proposition 4.2 are fulfilled. Then there exists a Horikawa surface X such that $\tilde{S}$ resolves the singularities of $\phi_1(X)$ and such that $\mathcal{L}$ is the line bundle associated with the flat double cover π of Theorem 3.3.*

Proof. Given a line bundle $\mathcal{L}$ on $\tilde{S}$ as above, there exists a reduced section of $\mathcal{L}^{\otimes 2}$ with at worst normal crossing singularities. A double cover of $\tilde{S}$ branched along the divisor of this section yields a surface with at worst Du Val singularities, whose desingularisation is a Horikawa surface with the stated properties, cf. [Hor2, Section 1]. ∎

Henceforth we shall assume that the characteristic of the ground field is $p = 2$. To obtain examples with inseparable canonical maps, we use quotients by foliations as described in [Lie]. We note that this is a feature that does not occur for curves, where the canonical map can never become inseparable. Also, these surfaces of general type are automatically (inseparably) unirational, which cannot happen in dimension 1 by Lüroth's theorem.

Although we only present examples we note that all Horikawa surfaces with inseparable canonical map can be obtained this way. In fact, the canonical morphism is inseparable if and only if the morphism π in Theorem 3.3 is purely inseparable. Using the Frobenius morphism and Hirokado's result [Hir1, Proposition 2.6], that singularities of p-closed vector fields on surfaces can be resolved by blow-ups in characteristic 2, we conclude that Horikawa surfaces with inseparable canonical morphism are quotients of rational surfaces by p-closed vector fields. In particular, these surfaces are all (inseparably) unirational.

THEOREM 5.2. *Given non-negative integers g, d with $g \geq \max\{2d - 2, d + 4\}$ and $g - d$ even, there exists a Horikawa surface with $p_g = g$ such that its canonical morphism maps inseparably onto $\mathbb{F}_d$.*

Proof. We use the coordinates of [Hir2] to express $\mathbb{F}_d$ and its morphism onto $\mathbb{P}^1$ as

$$\big(\operatorname{Proj} k[X_1,Y_1]\times \operatorname{Spec} k[t_1]\big)\cup\big(\operatorname{Proj} k[X_2,Y_2]\times \operatorname{Spec} k[t_2]\big) \longrightarrow \operatorname{Proj} k[T_1,T_2].$$

We set $x_i := X_i/Y_i$ and $y_i := Y_i/X_i$ for $i=1,2$. Then

$$\begin{array}{ll} x_1 = x_2/t_2^d & \partial/\partial x_1 = t_2^d\partial/\partial x_2 \\ y_i = 1/x_i & \partial/\partial x_i = -y_i^2\partial/\partial y_i \\ t_1 = 1/t_2 & \partial/\partial t_1 = -dx_2t_2\partial/\partial x_2 - t_2^2\partial/\partial t_2. \end{array}$$

The elements of $|\Gamma|$ are given by $\{t_i = \mathit{const}\}$ and the section Δ_0 is defined by $\{y_1=0\}\cup\{y_2=0\}$, whereas $\{x_1=0\}\cup\{x_2=0\}$ is linearly equivalent to $\Delta_0 + d\Gamma$.

We will first assume that g and d are even integers. For an arbitrary integer $m \geq 0$ we choose an integer $\ell \geq 0$ such that $0 \leq 2\ell - 2m + 2 \leq 5d$. Then we choose pairwise distinct elements $a_1,\dots,a_\ell, b_1,\dots,b_m$ in the ground field k and define the rational function

$$\psi(t_1) := \prod_{i=1}^{\ell}(t_1-a_i)^{-2}\prod_{j=1}^{m}(t_1-b_j)^2 \in k(t_1).$$

Then we define the vector field

$$\eta := x_1^{-4}\frac{\partial}{\partial x_1} + \psi(t_1)\frac{\partial}{\partial t_1}$$

which is additive in characteristic 2, i.e., $\eta^{[2]} = 0$. Considered as a derivation of $k(x_1,t_1)$ over k it extends to a rational vector field on $\mathbb{F}_d$ with divisor

$$(\eta) \sim -4\Delta_0 - (2m-2+4d)\Gamma.$$

If $5d - 2\ell + 2m - 2 = 0$ then η has $\ell + m$ isolated singular points. More precisely, the quotient $X' := \mathbb{F}_d/\eta$ has ℓ singular points of type D_8 and m singular points of type D_{12}, where we use [Lie, Proposition 2.3] to determine the singularities. Using the results of [Lie, Section 4] we see that X' has an ample canonical sheaf if $g \geq d+4$, i.e., X' is the canonical model of a surface of general type. We find $p_g = 2m-2+2d \geq 2d-2$ and that X' is a Horikawa surface. Using Remark 3.2, we see that the canonical map of X' factors over $\mathbb{F}_d^{(-1)}$, cf. also [Lie, Section 7]. This construction yields all examples of Horikawa surfaces with all possible values of g and d where both are even integers.

If $5d-2\ell+2m-2=4$ then the quotient $X' := \mathbb{F}_d/\eta$ acquires an elliptic singularity of type $(19)_0$ at the point lying below $P := \{x_2 = t_2 = 0\}$ by [Lie, Proposition 2.3]. The other singular points of X' are again Du Val singularities of type D_8 and D_{12}. Desingularising X' we obtain a Horikawa surface X'' with $p_g = 2m-3+2d$. By construction, the canonical image of X'' equals the image of $\mathbb{F}_d$ under the morphism that is defined by imposing a simple base point at P in the linear system $|\Delta_0 + \frac{1}{2}(g-2+d)\Gamma|$. Blowing up P resolves the indeterminacy and since P does not lie on Δ_0, the induced morphism on the blow-up of $\mathbb{F}_d$ factors over $\mathbb{F}_{d-1}$. Hence the canonical system of X'' factors over $\mathbb{F}_{d-1}$. With this construction we obtain examples of Horikawa surfaces with all possible values of g and d where both are odd integers. □

Theorem 5.3. *For every $4 \le g \le 6$ there exists a Horikawa surface with $p_g = g$ such that its canonical morphism maps inseparably onto the cone over a rational normal curve of degree $p_g - 2$ in $\mathbb{P}^{p_g-1}$.*

Proof. To get an example with $p_g = 4$ (resp. $p_g = 6$) we take $d = 2$ (resp. $d = 4$), $m = 1$ (resp. $m = 0$) and $\ell = 4$ (resp. $\ell = 9$) in the series of surfaces constructed in the proof of Theorem 5.2.

To get an example with $p_g = 5$ we take $d = 4$, $m = 0$ and $\ell = 7$. The quotient $\mathbb{F}_4/\eta$ has a singularity of type $(19)_0$ and a desingularisation yields the desired surface. □

Theorem 5.4. *There exist Horikawa surfaces with $p_g = 3$ and $K^2 = 2$ as well as $p_g = 6$ and $K^2 = 8$ such that their canonical morphisms map inseparably onto $\mathbb{P}^2$ and the Veronese surface in $\mathbb{P}^5$, respectively.*

Remark 5.5. By a theorem of Bloch, there are no smooth and inseparable double covers of $\mathbb{P}^2$, cf. [E, Proposition 2.5]. Hence the canonical model of a surface of Theorem 5.4 cannot be smooth.

Proof. We use the Zariski surfaces constructed by Hirokado [Hir2, Example 3.2]. On $\mathbb{F}_1$, the blow-up of $\mathbb{P}^2$ in one point, the vector field Δ_1 with $\ell = 0$ and $n = 3$ (resp. $n = 4$) is multiplicative with isolated singularities of multiplicity 1. The quotient $X := \mathbb{F}_1/\Delta_1$ is a minimal surface of general type with at worst A_1 singularities having the invariants we are looking for, cf. [Hir2, Example 3.2] and [Hir2, Example 3.5].

The canonical map of X factors over $\mathbb{F}_1^{(-1)}$ followed by the map defined by the linear system $|L+E|$ (resp. $|2L+2E|$), where L is the class of line pulled back from $\mathbb{P}^2$ and E is the exceptional (-1)-curve on $\mathbb{F}_1$. Hence the canonical map of X is a purely inseparable morphism onto $\mathbb{P}^2$ (resp. onto the Veronese surface in $\mathbb{P}^5$). ∎

Remark 5.6. Alternatively, we can consider $\delta := (x^2+x^{-4})\frac{\partial}{\partial x}$, which is a rational and additive vector field on $\mathbb{P}^1$. The quotient $X' := (\mathbb{P}^1\times\mathbb{P}^1)/(\delta+\delta)$ has an elliptic singularity of type $(19)_0$ and nine Du Val singularities of type D_4. Resolving these, we obtain another surface with $p_g = 3$ and $K^2 = 2$ such that the canonical map exhibits X_{can} as a purely inseparable double cover of $\mathbb{P}^2$.

§6. Deformations to the separable case

To obtain Horikawa surfaces with separable canonical map in characteristic 2, we show that every Horikawa surface with inseparable canonical map can be deformed into such a surface with a separable canonical map. Morally speaking, the Horikawa surfaces with inseparable canonical map should be boundary components of the moduli space (if it exists in some sense) of all Horikawa surfaces with fixed p_g and fixed image of the canonical map.

First, we establish a result about deformations of proper surface with at worst rational singularities. If resolution of singularities in positive characteristic were known to hold, we could probably argue along the lines of [El] to prove that every deformation of a rational singularity is again rational. For our purposes, it is enough to consider deformations over a 1-dimensional and normal base. We prove the result in wider generality in order to apply it to lifting problems later on as well.

PROPOSITION 6.1. *Let $f : \mathcal{X} \to \operatorname{Spec} R$ be a flat and proper family of surfaces where R is a normal and* 1*-dimensional Nagata ring. Assume that the geometric fibre over some closed point $t \in \operatorname{Spec} R$ is a normal surface with at worst rational* (*resp. Du Val*) *singularities. Then there exists an open set $U \subseteq \operatorname{Spec} R$ containing t such that the geometric generic fibre as well as every geometric fibre above points of U is a normal surface with at worst rational* (*resp. Du Val*) *singularities.*

Proof. By [EGA IV, Thm. 12.2.4], there exists an open and dense subset U of $C := \operatorname{Spec} R$ over which the fibres are geometrically normal. Since U and the fibres of f above U are normal then so is $\mathcal{X}_U$, cf. [EGA IV, Cor. 6.5.4]. Since any localisation of a normal ring remains normal, also the generic fibre $\mathcal{X}_K$ is normal.

The generic fibre $\mathcal{X}_K \to \operatorname{Spec} K$ is a surface over a field K whose singularities can be resolved by a sequence of normalised blow-ups $\mathcal{Y}_K \to \mathcal{X}_K$ at closed points. If $\mathcal{Y}_K$ is regular but not smooth over K we pass to a finite extension L of K where the singularities become visible. Base-changing to this field and resolving the singularities of $\mathcal{Y}_L$, we eventually obtain a regular surface. After possibly extending this field further, base-changing and resolving singularities we finally obtain a regular surface that is also smooth over its ground field. We may thus assume that there exists a finite field extension L of K and a finite sequence of normalised blow-ups $\mathcal{Y}_L \to \mathcal{X}_L$ such that $\mathcal{Y}_L$ is a smooth surface over L.

Let D be the normalisation of C in L. We choose a point $s \in D$ lying above $t \in C$. Then the discrete valuation ring $\mathcal{O}_{D,s}$ dominates $\mathcal{O}_{C,t}$. We denote their respective residue fields by k and ℓ. Let $\mathcal{X}'_D$ be the normalisation of $\mathcal{X} \otimes_{\mathcal{O}_{C,t}} \mathcal{O}_{D,t}$. By the valuative criterion of properness the sequence of normalised blow-ups $\nu_L : \mathcal{Y}_L \to \mathcal{X}_L$ extends to a sequence of normalised blow-ups $\nu'_D : \mathcal{Y}'_D \to \mathcal{X}'_D$. This induces a partial desingularisation of the special fibre $\mathcal{X}_0 \otimes_k \ell$. Since this special fibre has at worst rational singularities, the flat base-change theorem shows that $R^1(\nu'_D)_* \mathcal{O}_{\mathcal{Y}'_D}$ and $R^1(\nu_L)_* \mathcal{O}_{\mathcal{Y}_L}$ have to vanish. In particular, $\mathcal{X}_L$ has at worst rational singularities, which implies that also the geometric generic fibre and $\mathcal{X}_K$ have at worst rational singularities.

There exists an open neighbourhood V of $s \in D$ over which $\mathcal{Y}'_D \to \mathcal{X}'_D \to \operatorname{Spec} \mathcal{O}_{D,s}$ spreads out flatly. We have thus a sequence of normalised blow-ups $\nu_V : \mathcal{Y}_V \to \mathcal{X}_V$ over V which coincides with ν_L after tensorising with L. Since $\mathcal{Y}_L$ is smooth over L, i.e., geometrically regular, there exists a non-empty and open subset W of V over which the fibres of $\mathcal{Y}_V \to V$ are geometrically regular, cf. [EGA IV, Thm. 12.2.4]. Since D is 1-dimensional, the set $W \cup \{s\}$ is still open in D and we replace V by it. There exists a neighbourhood of t such that the geometric fibres of $\mathcal{X}_V \to V$ are normal (same argumentation as beginning of the proof) and we replace V by it. After possibly shrinking even further we may assume that $R^1(\nu_V)_* \mathcal{O}_{\mathcal{Y}_V}$ is zero since this is true generically. This open set V now has the property that every geometric fibre of $\mathcal{X}_V \to V$ is a normal surface with at worst rational singularities.

Since R is a 1-dimensional and normal Nagata ring the map $D \to C$ is a finite morphism between regular schemes and hence flat. Thus, the image W of V in C is open and contains t. We replace U from above by its intersection with W, which yields an open set of C containing t. The geometric fibres of $\mathcal{X} \to C$ over U are normal surfaces with at worst rational singularities.

Now, suppose that the geometric fibre above t has at worst Du Val singularities. These singularities are precisely the rational Gorenstein singularities. The special fibre is Gorenstein and since the maximal ideal of $t \in C$ is generated by a regular sequence, it follows that the generic fibre is Gorenstein and there are points of $\mathcal{X}$ whose local rings are Gorenstein. This argument also shows that the property of a fibre being Gorenstein is stable under generisation. The set $\mathcal{S}$ of points of $\mathcal{X}$ such that $\mathcal{O}_{\mathcal{X},b}$ is not Gorenstein is a proper closed subset of $\mathcal{X}$. The image of $\mathcal{S}$ in C is a constructible set and hence so is its complement, which we have already seen to be stable under generisation. Hence that the set of points in C whose fibres are Gorenstein schemes is open and contains t. Intersecting U with this open set we obtain an open set of C containing t such that every geometric fibre above points of it is a normal surface with at worst rational Gorenstein, i.e., Du Val, singularities. ∎

The only point in the proof where we have used that we deal with surfaces was when we used resolution of singularities of the generic fibre. For example, if the field of fractions of R is of characteristic zero we can apply Hironaka's resolution of singularities and the proof works in arbitrary dimensions.

Hence if we define a *weak rational singularity* of a variety X in positive characteristic (naively) by requiring that $R^i f_* \mathcal{O}_Y = 0$ for every resolution $f : Y \to X$ and every $i \geq 1$ we obtain the

COROLLARY 6.2. *Let k be a perfect field of positive characteristic and $W(k)$ its associated Witt ring. Assume that X is a normal variety over k with at worst weak rational (Gorenstein) singularities and that there exists a flat lifting $f : \mathcal{X} \to \operatorname{Spec} W(k)$ of X. Then the generic fibre of f is a normal variety with at worst rational (Gorenstein) singularities.* ∎

We now apply the previous proposition to deform a given Horikawa surface with inseparable canonical morphism into one with a separable canonical morphism. This is achieved by deforming the canonical map while fixing (the desingularisation of) the canonical image.

THEOREM 6.3. *Let X be a Horikawa surface in characteristic 2 and assume that the canonical map $\phi_1 : X \to S := \phi_1(X)$ is inseparable.*

(1) *If S is smooth then the canonical model of X can be deformed into the canonical model of a Horikawa surface with separable canonical map and with the same canonical image.*
(2) *If S is singular, then the surface X' of Theorem 3.3 can be deformed into the Y' of a Horikawa surface Y with the same canonical image but with a separable canonical map.*

Proof. We do the case where S is a smooth surface and leave the other case to the reader. Let X be a Horikawa surface with inseparable canonical map, S its canonical image and $\mathcal{L}$ be the line bundle associated with $X_{\mathrm{can}} \to S$ as in Theorem 3.3. We know from Lemma 3.6 that $X_{\mathrm{can}} \cong \operatorname{Spec} \mathcal{A}$ with $\mathcal{A} := \mathcal{O}_S \oplus \mathcal{L}$. The algebra structure on $\mathcal{A}$ is given by $z^2 + t = 0$ for some section t of $\mathcal{L}^{\otimes 2}$.

We choose a non-zero global section s of $\mathcal{L}$, which exists by the description of $\mathcal{L}$ in Theorem 3.3. Then we define the $\mathcal{O}_S$-algebra $\mathcal{A}_\lambda$ to be the $\mathcal{O}_S$-module $\mathcal{A}$ with multiplication

$$z^2 + \lambda s z + t = 0.$$

For $\lambda = 0$ we obtain the original algebra $\mathcal{A}$.

We define $X_\lambda := \operatorname{Spec} \mathcal{A}_\lambda$ and consider these surfaces as a flat family of surfaces over the affine line $\mathbb{A}^1_k$ with parameter λ. Then, every surface in this family is a finite and flat double cover $\pi_\lambda : X_\lambda \to S$ with associated line bundle $\mathcal{L}$. The fibre X_0 is the surface X we started with.

There exists an open set $U \subseteq \mathbb{A}^1$ containing $\lambda = 0$ such that X_λ for $\lambda \in U$ is a normal surface with at worst Du Val singularities, cf. Proposition 6.1. Using $\omega_{X_\lambda} \cong \pi_\lambda^*(\omega_S \otimes \mathcal{L})$ (cf. formula (2)) it is not difficult to see that the canonical sheaves of these surfaces are ample, i.e., that all surfaces in this family above U are canonical models of Horikawa surfaces. By Remark 3.2, their canonical morphisms factor as π_λ followed by the morphism of S associated with the complete linear system $|\mathcal{L}|$.

As explained in the beginning of Section 5, the zero set of λs determines the branch divisor of the double cover $\pi_\lambda : X_\lambda \to \tilde{S}$. If $\lambda \neq 0$, then the morphism π_λ, i.e., the canonical morphism, is separable. $\square$

We can summarise the results of Section 5 and Section 6 as follows.

THEOREM 6.4. *Theorem* 5.1 *also holds in characteristic* $p = 2$. *Moreover, these surfaces exist with separable as well as inseparable canonical maps.* ∎

§7. Lifting to characteristic zero

We now prove that Horikawa surfaces lift projectively over $W(k)$. By Theorem 3.3, giving a Horikawa surface is the same thing as giving a rational surface, a line bundle on it and two sections. It is not difficult to see that all this data lifts to characteristic zero from which we obtain a lift of the canonical model of any given Horikawa surface. To achieve a lifting of the desingularisation we use Artin's result on simultaneous resolution of singularities, which provides us with an algebraic space over a possibly ramified extension of $W(k)$ achieving the lift.

Let k be an algebraically closed field of characteristic $p > 0$ and $W(k)$ its associated Witt ring. The following results is probably folklore but worthwhile being stated explicitly.

LEMMA 7.1. *Let S be a smooth rational surface over k and $\mathcal{L}$ a line bundle on S. Then S lifts to a surface over $W(k)$ and $\mathcal{L}$ lifts to a unique line bundle $\widetilde{\mathcal{L}}$ on this lift.*

In case $h^1(\mathcal{L}) = 0$ we have furthermore $h^i(\mathcal{L}) = h^i(\widetilde{\mathcal{L}})$ for all i. In particular, sections of $\mathcal{L}$ lift as well in this case.

Proof. The first assertions follow from Grothendieck's existence theorem and elementary deformation theory. That smooth rational surfaces lift projectively over $W(k)$ is for example explained in [Ill, Section 8.5.26]. From [Ill, Corollary 8.5.6] and $h^1(S, \mathcal{O}_S) = h^2(S, \mathcal{O}_S) = 0$ we get the unique lifting of line bundles.

Let $\widetilde{\mathcal{L}}$ be the unique lift of $\mathcal{L}$ on S and assume that $h^1(\mathcal{L}) = 0$. From the upper semicontinuity theorem we get $h^1(\tilde{\mathcal{L}}) = 0$. Since $\chi(\mathcal{L}) = \chi(\widetilde{\mathcal{L}})$ and h^1 of both line bundles is zero, we conclude $h^i(\mathcal{L}) = h^i(\widetilde{\mathcal{L}})$ for $i = 0, 2$, again using upper semicontinuity. ∎

We now come to the main result of this section.

THEOREM 7.2. *Let X be a Horikawa surface over a field k of positive characteristic. Then the canonical model of X lifts over the Witt ring $W(k)$.*

Also, X can be lifted in the category of algebraic spaces, i.e., there exists an algebraic space, flat over a possibly ramified extension of $W(k)$, with special fibre X.

Proof. We first do the case where the canonical image S of X is a smooth surface.

Let $\pi : X_{\mathrm{can}} \to S$ the canonical flat double cover with associated line bundle $\mathcal{L}$ as in Theorem 3.3. As an $\mathcal{O}_S$-algebra X_{can} is isomorphic to $\mathcal{A} := \mathcal{O}_S \oplus \mathcal{L}$ by Lemma 3.6. The map π is globally given in the form (4) for sections s, t of $\mathcal{L}$, $\mathcal{L}^{\otimes 2}$, respectively.

By Lemma 7.1, we can lift S, $\mathcal{L}$ and $\mathcal{L}^{\otimes 2}$ over $W(k)$. Let $\mathcal{S} \to W(k)$ be the lift of S and $\mathcal{S}_K$ the generic fibre. Using Lemma 3.6 and Lemma 7.1 again, we also lift the sections s and t. Out of this data we construct a flat double $\mathcal{X}' \to \mathcal{S}$ with special fibre $X_{\mathrm{can}} \to S$. By Proposition 6.1 the generic fibre $\mathcal{X}'_K$ of $\mathcal{X}' \to W(k)$ is a normal surface with at worst Du Val singularities. Using the explicit description it follows that $\mathcal{X}_K$ is a possibly singular Horikawa surface with at worst Du Val singularities and with canonical image $\mathcal{S}_K$. Hence, we have lifted the canonical model of X over $W(k)$.

We have to resolve the singularities in the special fibre. However, to achieve a simultaneous resolution of singularities, we may have to base-change to an algebraic space of finite type over $W(k)$, cf. [Ar, Theorem 1]. That this resolution is in fact possible after a finite extension of $W(k)$ follows from the fact that $W(k)$ is Henselian, cf. [Ar, Theorem 2].

In case the canonical image is singular we proceed as before to obtain a lift of X' (in the notation of Theorem 3.3) over $W(k)$ and to get a lift of X over a possibly ramified extension of $W(k)$. The pluri-canonical ring associated with the lift of X' specialises to the canonical model of X and generalises to a normal surface with at worst Du Val singularities. This achieves a lifting of the canonical model of X also in the case of a singular canonical image. $\square$

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