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HECKE'S INTEGRAL FORMULA FOR RELATIVE QUADRATIC EXTENSIONS OF ALGEBRAIC NUMBER FIELDS

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Abstract. Let K/F be a quadratic extension of number fields. After developing a theory of the Eisenstein series over F, we prove a formula which expresses a partial zeta function of K as a certain integral of the Eisenstein series. As an application, we obtain a limit formula of Kronecker's type which relates the 0-th Laurent coefficients at $s = 1$ of zeta functions of K and F.

§1. Introduction

Let $E(z, s)$ be the real analytic Eisenstein series defined by

$$E(z,s) = \frac{1}{2} {\sum_{m,n\in\mathbb{Z}}}' \frac{y^s}{|mz+n|^{2s}} \quad \big(y = \operatorname{Im} z > 0,\ \operatorname{Re}(s) > 1\big),$$

where $\sum'$ means that the sum is taken except for $(m, n) = (0, 0)$. Then it is well-known that, when z is an element of an imaginary quadratic field, $E(z, s)$ represents a zeta function of that field. To be precise, let A be an ideal class of an imaginary quadratic field K, and $\mathfrak{A}$ an element of A^{-1} of the form $\mathfrak{A} = \mathbb{Z}z + \mathbb{Z}$. We fix an embedding of K into $\mathbb{C}$ and assume that $\operatorname{Im}(z) > 0$. Then the partial zeta function

$$\zeta_K(s, A) := \sum_{\substack{\mathfrak{B}\in A \\ \mathfrak{B}\subset O_K}} \mathbf{N}(\mathfrak{B})^{-s}$$

can be written as

$$\zeta_K(s,A) = \frac{2}{w_K}\left(\frac{\sqrt{d_K}}{2}\right)^{-s} E(z,s), \tag{1.0.1}$$

where w_K and d_K denote the number of roots of unity in K and the absolute value of the discriminant of K, respectively.

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Hecke [3] discovered an analogous formula for real quadratic fields. Now let K be real quadratic, and $\mathfrak{A} = \mathbb{Z}z + \mathbb{Z} \in A^{-1}$ as before. Again we fix an embedding $K \hookrightarrow \mathbb{R}$ and denote the conjugate of $x \in K$ over $\mathbb{Q}$ by x'. Assume that $z' > z$. Then we have Hecke's integral formula

$$\zeta_K(s, A) = 2d_K^{-s/2} \frac{\Gamma(s)}{\Gamma(s/2)^2} \int_1^{\varepsilon^2} E(z_t, s) \frac{dt}{t}, \tag{1.0.2}$$

where $\varepsilon > 1$ is the fundamental unit of K and

$$z_t := \frac{t^{1/2} z + t^{-1/2} z' i}{t^{1/2} + t^{-1/2} i}.$$

For further discussions on this formula and related topics, see Meyer [8], Siegel [9] and a recent work of Manin [7, §2].

There are some results related to the formulas (1.0.1) and (1.0.2). For example, when F is a totally real field and K is its CM extension, there is a formula relating the zeta functions of K and the Eisenstein series over F evaluated at certain CM-points, which is a generalization of (1.0.1) (see Yoshida [10]). Another example was given by Konno [6], who found a formula analogous to (1.0.2), which expresses a zeta function of K as an integral of the Eisenstein series over F, when F is imaginary quadratic and $K \supset F$ is absolutely biquadratic. We also note that a recent work of Hiroe and Oda [4] generalizes Hecke's (and Siegel's) formula to the case of $K/\mathbb{Q}$, where K is an arbitrary number field.

In this paper, we consider the case of K/F, an arbitrary quadratic extension of algebraic number fields. Let $T_{K/F}$ be the subgroup of $(K \otimes_{\mathbb{Q}} \mathbb{R})^\times$ consisting of the elements u such that $\mathbf{N}_{K/F}(u) = 1$, and $U_{K/F}$ the intersection of $T_{K/F}$ and the unit group of K. Then our main result is the following:

THEOREM 1.0.1. (= Theorem 3.1.2) *If A is a wide ideal class of K and $\mathfrak{A}$ is an element of A^{-1}, then we have*

$$\xi_K(s, A) = \frac{1}{w_{K/F}} \int_{T_{K/F}/U_{K/F}^2} \widehat{E}\big(\rho(\tilde{u}\mathfrak{A}), s\big)\, d^\times u.$$

Here $\xi_K(s, A)$ denotes the completed zeta function associated with A, and $\widehat{E}$ is the completed Eisenstein series over F (precise definitions are given in Section 2).

For the other notations used in the above theorem, see 3.1.

Remark 1.0.2. The above formula says that if we regard the Eisenstein series as a function on the torus $T_{K/F}/U_{K/F}^2$, $\xi_K(s, A)$ appears as the constant term of its Fourier expansion. If we consider the general Fourier coefficients, we will obtain the zeta functions with Grössencharacters, as Siegel [9] did in the case of real quadratic fields. To be more general, we should also consider the Eisenstein series with Grössencharacters over F. The author would like to thank the referee who pointed out these issues.

The contents of this paper are as follows. We develop a theory of the Eisenstein series over an arbitrary number field F in Section 2. After the definitions (2.1 and 2.2), we prove the functional equation (Theorem 2.3.3), the Fourier expansion (Theorem 2.4.5) and the Kronecker limit formula (Theorem 2.5.1). Note that we consider the Eisenstein series as a function of a lattice in a certain vector space over $\mathbb{R}$, although there is a more traditional notion studied by Asai [1] and Jorgenson-Lang [5]. (In fact, these two formulations are essentially equivalent. See Remark 2.2.3.)

In Section 3, we prove our generalization of Hecke's integral formula (Theorem 3.1.2). We also apply it to the Kronecker limit formula about the constant terms in the Laurent expansions of zeta functions at $s = 1$ (Theorem 3.2.1). The result has a relative nature, in the sense that it compares zeta functions of K and F.

1.1. Notation

As usual, the symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ mean the rings of integers, rationals, real numbers and complex numbers, respectively. We also denote by $\mathbb{H}$ the quaternion division algebra of Hamilton:

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C}j = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k.$$

By $\mathbb{C}_1^\times$, we mean the group of complex numbers of absolute value 1.

For an arbitrary algebraic number field F (of finite degree), we use the following notations:

O_F, U_F, $\mathfrak{d}_F$ and d_F mean the ring of integers, the group of units in F, the different and the absolute value of the discriminant, respectively.

We denote the set of infinite places of F by S_F, and the subset of real (resp. complex) ones by S_F^1 (resp. S_F^2).

For each $v \in S_F$, we denote the corresponding completion by F_v, and the embedding of F into F_v by $x \mapsto x_v$. The same notation is used to indicate the v-th component of an element $x \in F_{\mathbb{R}} := F \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{v \in S_F} F_v$. Moreover, we equip F_v with the Lebesgue measure (resp. twice the Lebesgue measure) when v is real (resp. complex).

We denote the absolute norm (of an ideal, or an element of $F_{\mathbb{R}}$, etc.) by $\mathbf{N}_{F/\mathbb{Q}}$, and often abbreviate it to $\mathbf{N}$. The same rule will be applied to the absolute trace $\mathrm{Tr} = \mathrm{Tr}_{F/\mathbb{Q}}$.

§2. Lattices and the Eisenstein series

In this section, we develop a theory of Eisenstein series over a fixed number field F. We denote the number of real (resp. complex) places of F by r_1 (resp. r_2).

2.1. Lattices in $\mathbb{D}_F$

For each infinite place $v \in S_F$, let $\mathbb{D}_v$ be the quadratic division algebra over F_v. Thus $\mathbb{D}_v$ is isomorphic to $\mathbb{C}$ or $\mathbb{H}$ according to whether v is real or complex. Choosing such an isomorphism, we define $j_v \in \mathbb{D}_v$ to be the element corresponding to $i \in \mathbb{C}$ or $j \in \mathbb{H}$. Then we have that $\mathbb{D}_v = F_v \oplus F_v j_v$ for any v. We also define the Haar measure on $\mathbb{D}_v$ to be the Lebesgue measure (resp. 4 times the Lebesgue measure) for $v \in S_F^1$ (resp. $v \in S_F^2$), so that the above direct sum decomposition preserves the measure.

Next, let us put $\mathbb{D}_F := \prod_{v \in S_F} \mathbb{D}_v$ and $j_F := (j_v)_v \in \mathbb{D}_F$. Then $\mathbb{D}_F = F_{\mathbb{R}} \oplus F_{\mathbb{R}} j_F$ becomes naturally a quadratic algebra over $F_{\mathbb{R}}$. If we write an element z of $\mathbb{D}_F$ as $z = x + y j_F$, we regard x and y as elements of $F_{\mathbb{R}}$.

Moreover, for $z = (z_v)_v \in \mathbb{D}_F$, we set

$$\|z\|_{\mathbb{D}_F} = \prod_{v \in S_F^1} |z_v| \cdot \prod_{v \in S_F^2} |z_v|^2,$$

where $|\cdot|$ is the usual absolute value in $\mathbb{C}$ or $\mathbb{H}$. Note that $\|x\| = \bigl|\mathbf{N}(x)\bigr|$ for $x \in F_{\mathbb{R}}$.

Definition 2.1.1. We call a discrete and cocompact O_F-submodule of $\mathbb{D}_F$ an *O_F-lattice* in $\mathbb{D}_F$. For such a lattice Λ, we denote by $V(\Lambda)$ the volume of the quotient $\mathbb{D}_F/\Lambda$ (with respect to the product Haar measure on $\mathbb{D}_F$).

Lemma 2.1.2. *Let $\Lambda \subset \mathbb{D}_F$ be an O_F-lattice.*

(1) *For any* $z \in \mathbb{D}_F^\times$, *we have* $V(z\Lambda) = V(\Lambda z) = \|z\|^2 V(\Lambda)$.

(2) *There exist elements* $\omega_1, \omega_2 \in \mathbb{D}_F$ *and a fractional ideal* $\mathfrak{a}$ *of* F *such that*

$$\Lambda = \mathfrak{a}\omega_1 + O_F\omega_2.$$

(3) *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be fractional ideals of* F, *and* z *an element of* $\mathbb{D}_F$ *of the form* $z = x + yj_F$ *where* $x \in F_\mathbb{R}$ *and* $y \in F_\mathbb{R}^\times$. *Then* $\Lambda = \mathfrak{a}z + \mathfrak{b}$ *is an* O_F*-lattice, and*

$$V(\Lambda) = d_F \mathbf{N}(\mathfrak{a})\mathbf{N}(\mathfrak{b})\big|\mathbf{N}(y)\big|.$$

Proof. The first assertion is clear from the definition. (2) is a special case of the structure theorem for finitely generated torsion-free modules over Dedekind domains (Bourbaki [2, Chap. 7, §4, Proposition 24]).

For (3), we consider the map

$$F_\mathbb{R} \oplus F_\mathbb{R} \ni (\alpha, \beta) \longmapsto \alpha z + \beta \in \mathbb{D}_F.$$

This is an isomorphism of $\mathbb{R}$-vector spaces, and multiplies the volume by $\big|\mathbf{N}(y)\big|$. On the other hand, we see that the volume of $F_\mathbb{R}/\mathfrak{a}$ is $\mathbf{N}(\mathfrak{a})\sqrt{d_F}$ and similar for $\mathfrak{b}$. Then the claim follows. ∎

2.2. The Eisenstein series

DEFINITION 2.2.1. An O_F-lattice $\Lambda \subset \mathbb{D}_F$ is said to be *non-degenerate* if there is no nonzero element $\lambda \in \Lambda$ satisfying $\|\lambda\| = 0$. For such Λ, we define the *Eisenstein series* $E(\Lambda, s)$ by

$$E(\Lambda, s) := \sideset{}{'}\sum_{\lambda \in \Lambda/U_F} \frac{V(\Lambda)^s}{\|\lambda\|^{2s}} \quad (\mathrm{Re}(s) > 1),$$

where the prime means that the sum is taken for nonzero λ.

$E(\Lambda, s)$ has the 'modularity':

LEMMA 2.2.2. *For* $z \in \mathbb{D}_F^\times$, *we have* $E(z\Lambda, s) = E(\Lambda z, s) = E(\Lambda, s)$.

Proof. This follows from the definition, and Lemma 2.1.2 (1). ∎

Remark 2.2.3. By the above lemma and Lemma 2.1.2 (2), (3), it is sufficient to consider

$$E(z, \mathfrak{a}, \mathfrak{b}, s) = \sideset{}{'}\sum_{(\mu,\nu) \in (\mathfrak{a}\oplus\mathfrak{b})/U_F} \frac{\Big(d_F \mathbf{N}(\mathfrak{a})\mathbf{N}(\mathfrak{b})\big|\mathbf{N}(y)\big|\Big)^s}{\|\mu z + \nu\|^{2s}},$$

for $z = x + yj_F$, $x \in F_{\mathbb{R}}$ and $y \in F_{\mathbb{R}}^{\times}$. Moreover, this series can be regarded as a function of $\big(x_v + |y_v| j_v\big)_v$ (and s), i.e., a function on the product of r_1 copies of the upper half plane and r_2 copies of the hyperbolic 3-space. Hence our notion of Eisenstein series is essentially equivalent to the more traditional one (see Asai [1], Jorgenson-Lang [5] or Yoshida [10]).

To consider the inverse Mellin transform of the Eisenstein series, we need some definitions.

For each infinite place $v \in S_F$, put $n_v = 1$ or 2 according to whether v is real or complex. We also set $T_F := F_{\mathbb{R}}^{\times}$, and choose the Haar measure $d^{\times}t$ on it to be $\prod_v dt_v/|t_v|^{n_v}$. Then we define

$$f(z) := \prod_{v \in S_F} \exp\big(-n_v \pi \, |z_v|^2\big) \quad (z \in \mathbb{D}_F),$$

$$\Gamma_F(s) := \int_{T_F} f(t) \big|\mathbf{N}(t)\big|^s \, d^{\times}t, \quad \widehat{E}(\Lambda, s) := \Gamma_F(2s) E(\Lambda, s).$$

Proposition 2.2.4.

$$\widehat{E}(\Lambda, s) = V(\Lambda)^s \int_{T_F/U_F} \sideset{}{'}\sum_{\lambda \in \Lambda} f(t\lambda) \big|\mathbf{N}(t)\big|^{2s} \, d^{\times}t.$$

Proof. Putting $\lambda = \mu + \nu j_F$, we have

$$\begin{aligned} \Gamma_F(2s) \|\lambda\|^{-2s} &= \prod_{v \in S_F} \int_{F_v^{\times}} \exp\big(-n_v \pi \, |t_v|^2\big) |t_v|^{2n_v s} \big(|\mu_v|^2 + |\nu_v|^2\big)^{-n_v s} \frac{dt_v}{|t_v|^{n_v}} \\ &= \prod_{v \in S_F} \int_{F_v^{\times}} \exp\Big(-n_v \pi \, |t_v|^2 \big(|\mu_v|^2 + |\nu_v|^2\big)\Big) |t_v|^{2n_v s} \frac{dt_v}{|t_v|^{n_v}} \\ &= \int_{T_F} f(t\lambda) \big|\mathbf{N}(t)\big|^{2s} \, d^{\times}t. \end{aligned}$$

Hence, by taking the sum and transforming as

$$\sideset{}{'}\sum_{\lambda \in \Lambda/U_F} \int_{T_F} = \int_{T_F} \sideset{}{'}\sum_{\lambda \in \Lambda/U_F} = \int_{T_F/U_F} \sideset{}{'}\sum_{\lambda \in \Lambda},$$

we obtain the result. ∎

2.3. The functional equation

DEFINITION 2.3.1. Let $\psi_{\mathbb{D}_F} : \mathbb{D}_F \to \mathbb{C}_1^\times$ be the character defined by

$$\psi_{\mathbb{D}_F}(x + yj_F) := \exp\bigl(2\pi i \operatorname{Tr}_{F/\mathbb{Q}}(x)\bigr).$$

Then, for an O_F-lattice $\Lambda \subset \mathbb{D}_F$, we define the *dual lattice* Λ^* by

$$\Lambda^* := \bigl\{\lambda^* \in \mathbb{D}_F \mid \psi_{\mathbb{D}_F}(\lambda\lambda^*) = 1 \ (\forall \lambda \in \Lambda)\bigr\}.$$

PROPOSITION 2.3.2. *For a non-degenerate O_F-lattice $\Lambda \subset \mathbb{D}_F$, set*

$$\Theta(t, \Lambda) := \sum_{\lambda \in \Lambda} f(t\lambda) \quad (t \in T_F).$$

Then we have

$$\Theta(t, \Lambda) = V(\Lambda)^{-1} \bigl|\mathbf{N}(t)\bigr|^{-2} \Theta(t^{-1}, \Lambda^*).$$

In particular, $V(\Lambda^*) = V(\Lambda)^{-1}$ holds.

Proof. If we put $f_t(z) = f(tz)$, its Fourier transform is given by

$$\hat{f}_t(w) := \int_{\mathbb{D}_F} f_t(z) \overline{\psi_{\mathbb{D}_F}(zw)}\, dz = \bigl|\mathbf{N}(t)\bigr|^{-2} f_{t^{-1}}(w).$$

Therefore we can use the Poisson summation formula to get

$$\sum_{\lambda \in \Lambda} f_t(\lambda) = V(\Lambda)^{-1} \bigl|\mathbf{N}(t)\bigr|^{-2} \sum_{\lambda^* \in \Lambda^*} f_{t^{-1}}(\lambda^*)$$

as desired. The last assertion is shown by applying this formula to Λ^*. ∎

THEOREM 2.3.3. *$\widehat{E}(\Lambda, s)$ can be continued meromorphically to the whole s-plane, and satisfies the functional equation*

$$\widehat{E}(\Lambda, s) = \widehat{E}(\Lambda^*, 1 - s).$$

Proof. Proposition 2.2.4 says that

$$\widehat{E}(\Lambda, s) = V(\Lambda)^s \int_{T_F/U_F} \bigl(\Theta(t, \Lambda) - 1\bigr) \bigl|\mathbf{N}(t)\bigr|^{2s}\, d^\times t.$$

We decompose the integral as $\int_{T_F/U_F} = \int_{|\mathbf{N}(t)|\geq 1} + \int_{|\mathbf{N}(t)|\leq 1}$. Then the former integral converges for every $s \in \mathbb{C}$. On the other hand, after inverting the variable t, the latter gives

$$V(\Lambda)^s \int_{|\mathbf{N}(t)|\geq 1} \big(\Theta(t^{-1}, \Lambda) - 1\big)\big|\mathbf{N}(t)\big|^{-2s}\, d^{\times}t$$
$$= V(\Lambda^*)^{1-s} \int_{|\mathbf{N}(t)|\geq 1} \big(\Theta(t, \Lambda^*) - 1\big)\big|\mathbf{N}(t)\big|^{2-2s}\, d^{\times}t + R,$$

where

$$R = V(\Lambda^*)^{1-s} \int_{|\mathbf{N}(t)|\geq 1} \big|\mathbf{N}(t)\big|^{2-2s}\, d^{\times}t - V(\Lambda)^s \int_{|\mathbf{N}(t)|\geq 1} \big|\mathbf{N}(t)\big|^{-2s}\, d^{\times}t.$$

Moreover, we see that

$$\int_{|\mathbf{N}(t)|\geq 1} \big|\mathbf{N}(t)\big|^{-2s}\, d^{\times}t = C \int_1^{\infty} t^{-2s} \frac{dt}{t} = \frac{C}{2s},$$

denoting by C the volume of $\big\{t \in T_F/U_F \mid |\mathbf{N}(t)| = 1\big\}$ with respect to a suitably normalized measure. (In fact, we know the value of $C = C_F$. See 2.5 below.) Hence we obtain the analytic continuation of R, and the functional equation follows immediately. □

2.4. The Fourier expansion

Here we give the 'Fourier expansion' of the Eisenstein series. More precisely, we consider O_F-lattices of the form $\Lambda = \mathfrak{a}z + \mathfrak{b}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals and $z = x + yj_F \in \mathbb{D}_F$. Then $\widehat{E}(\Lambda, s)$ is invariant under the transforms $x \mapsto x + \gamma$ for all $\gamma \in \mathfrak{a}^{-1}\mathfrak{b}$, and hence has the Fourier expansion.

Let us begin with some preparations.

Definition 2.4.1. (1) To a fractional ideal $\mathfrak{a}$ of F, we attach the (*completed*) *zeta function*

$$\zeta_F(s, \mathfrak{a}) := \mathbf{N}(\mathfrak{a})^s \sum_{\alpha\in\mathfrak{a}/U_F}{}' \big|\mathbf{N}(\alpha)\big|^{-s}, \quad \xi_F(s, \mathfrak{a}) := d_F^{s/2}\Gamma_F(s)\zeta_F(s, \mathfrak{a}).$$

(2) For $a, b \in F_{\mathbb{R}}^{\times}$, we define

$$B_F(a, b, s) := \int_{T_F} f_t(a) f_{t^{-1}}(b)\big|\mathbf{N}(t)\big|^{2s}\, d^{\times}t$$
$$= (2\pi)^{r_2}\big|\mathbf{N}(b/a)\big|^s \prod_{v\in S_F} K_{n_v s}\big(n_v\pi|a_v b_v|\big).$$

Here $K_s(x) = \int_0^{\infty} e^{-x(u+u^{-1})} u^{s-1}\, du$ is the K-Bessel function.

Remark 2.4.2. $\zeta_F(s,\mathfrak{a})$ is equal to the partial zeta function

$$\zeta_F(s,A) = \sum_{\mathfrak{b}\in A,\, \mathfrak{b}\subset O_F} \mathbf{N}(\mathfrak{b})^{-s},$$

where A is the wide ideal class containing $\mathfrak{a}^{-1}$.

PROPOSITION 2.4.3. *We have*

$$\xi_F(s,\mathfrak{a}) = V(\mathfrak{a})^s \int_{T_F/U_F} \sum_{\alpha\in\mathfrak{a}}{}' f(t\alpha)\big|\mathbf{N}(t)\big|^s\, d^\times t.$$

Here $V(\mathfrak{a}) = d_F^{1/2}\mathbf{N}(\mathfrak{a})$ *denotes the volume of* $F_\mathbb{R}/\mathfrak{a}$.

Proof. This can be shown in the same way as Proposition 2.2.4. ∎

LEMMA 2.4.4. *For* $z = x + yj_F \in \mathbb{D}_F$, $t \in T_F$ *and a fractional ideal* $\mathfrak{b}$ *of* F, *we have*

$$\sum_{\beta\in\mathfrak{b}} f_t(z+\beta) = V(\mathfrak{b})^{-1}\big|\mathbf{N}(t)\big|^{-1} \sum_{\beta^*\in\mathfrak{b}^*} e^{2\pi i\,\mathrm{Tr}(x\beta^*)} f_t(y) f_{t^{-1}}(\beta^*).$$

Proof. The Fourier transform of the function

$$f_{t,z}(u) := f\big(t(z+u)\big) = f_t(y)f_t(x+u)$$

on $F_\mathbb{R}$ is given by

$$\hat{f}_{t,z}(v) := \int_{F_\mathbb{R}} f_{t,z}(u)e^{-2\pi i\,\mathrm{Tr}(uv)}\,du = e^{2\pi i\,\mathrm{Tr}(xv)}\big|\mathbf{N}(t)\big|^{-1} f_t(y)f_{t^{-1}}(v).$$

Hence the claim is a consequence of the Poisson summation formula. ∎

THEOREM 2.4.5. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be fractional ideals of* F, *and* $z = x + yj_F$ *an element of* $\mathbb{D}_F$ *with* $y \in F_\mathbb{R}^\times$. *Then, for* $\Lambda = \mathfrak{a}z + \mathfrak{b}$, *we have*

$$\begin{aligned}\widehat{E}(\Lambda,s) = &\left(\frac{\mathbf{N}(\mathfrak{a})}{\mathbf{N}(\mathfrak{b})}\big|\mathbf{N}(y)\big|\right)^s \xi_F(2s,\mathfrak{b}) + \left(\frac{\mathbf{N}(\mathfrak{a})}{\mathbf{N}(\mathfrak{b})}\big|\mathbf{N}(y)\big|\right)^{1-s} \xi_F(2s-1,\mathfrak{a})\\ &+ V(\mathfrak{a})^s V(\mathfrak{b})^{s-1}\big|\mathbf{N}(y)\big|^s \sum_{(\alpha,\beta^*)} e^{2\pi i\,\mathrm{Tr}(x\alpha\beta^*)} B_F\left(\alpha y,\beta^*,s-\frac{1}{2}\right).\end{aligned}$$

Here $(\alpha,\beta^*) \in \big(\mathfrak{a}\setminus\{0\}\big)\times\big(\mathfrak{b}^*\setminus\{0\}\big)$ *runs through a system of representatives with respect to the equivalence relation defined by*

$$(\alpha,\beta^*) \sim (\alpha\varepsilon, \beta^*\varepsilon^{-1}) \quad (\forall\varepsilon\in U_F).$$

Proof. First, note that

$$\sum_{\lambda\in\Lambda}{}' f_t(\lambda) = \sum_{\beta\in\mathfrak{b}}{}' f_t(\beta) + \sum_{\alpha\in\mathfrak{a}}{}'\sum_{\beta\in\mathfrak{b}} f_t(\alpha z+\beta).$$

Furthermore, by Lemma 2.4.4, we have

$$\begin{aligned}\sum_{\alpha\in\mathfrak{a}}{}'\sum_{\beta\in\mathfrak{b}} f_t(\alpha z+\beta) &= V(\mathfrak{b})^{-1}\big|\mathbf{N}(t)\big|^{-1}\sum_{\alpha\in\mathfrak{a}}{}'\sum_{\beta^*\in\mathfrak{b}^*} e^{2\pi i\,\mathrm{Tr}(x\alpha\beta^*)} f_t(y\alpha)f_{t^{-1}}(\beta^*)\\ &= V(\mathfrak{b})^{-1}\big|\mathbf{N}(t)\big|^{-1}\sum_{\alpha\in\mathfrak{a}}{}'\sum_{\beta^*\in\mathfrak{b}^*}{}' e^{2\pi i\,\mathrm{Tr}(x\alpha\beta^*)} f_t(y\alpha)f_{t^{-1}}(\beta^*)\\ &\quad + V(\mathfrak{b})^{-1}\big|\mathbf{N}(t)\big|^{-1}\sum_{\alpha\in\mathfrak{a}}{}' f_t(y\alpha).\end{aligned}$$

Therefore, the theorem is deduced from Proposition 2.2.4, Lemma 2.1.2 (3), Proposition 2.4.3, and

$$\begin{aligned}&\int_{T_F/U_F}\sum_{\alpha\in\mathfrak{a}}{}'\sum_{\beta^*\in\mathfrak{b}^*}{}' e^{2\pi i\,\mathrm{Tr}(x\alpha\beta^*)} f_t(y\alpha)f_{t^{-1}}(\beta^*)\big|\mathbf{N}(t)\big|^{2s-1}\,d^\times t\\ &= \sum_{(\alpha,\beta^*)} e^{2\pi i\,\mathrm{Tr}(x\alpha\beta^*)}\int_{T_F} f_t(y\alpha)f_{t^{-1}}(\beta^*)\big|\mathbf{N}(t)\big|^{2s-1}\,d^\times t.\end{aligned}$$

The last follows from the identity

$$f_t(y\alpha\varepsilon)f_{t^{-1}}(\beta^*\varepsilon^{-1}) = f_{t\varepsilon}(y\alpha)f_{(t\varepsilon)^{-1}}(\beta^*)$$

for each $\varepsilon\in U_F$. ∎

2.5. The Kronecker limit formula

For a meromorphic function $\varphi(s)$ around $s=\alpha$, we denote by $\mathrm{CT}_{s=\alpha}\,\varphi(s)$ the constant term in the Laurent expansion at $s=\alpha$, while $\mathrm{Res}_{s=\alpha}\,\varphi(s)$ means the residue. As an application of Theorem 2.4.5, we give a limit formula of Kronecker's type which expresses $\mathrm{CT}_{s=1}\,\widehat{E}(\Lambda,s)$.

Let us denote the regulator of F by R_F, and the number of roots of unity in F by w_F. Then we put

$$C_F := \frac{2^{r_1}(2\pi)^{r_2}R_F}{w_F}.$$

THEOREM 2.5.1. *Let $\Lambda = \mathfrak{a}z + \mathfrak{b}$ be as in Theorem 2.4.5. Then*

$$\mathrm{Res}_{s=1} \widehat{E}(\Lambda, s) = \frac{C_F}{2},$$

$$\mathrm{CT}_{s=1} \widehat{E}(\Lambda, s) = \mathrm{CT}_{s=1} \xi_F(s, \mathfrak{a}) + \frac{C_F}{2}\left(h_F(z, \mathfrak{a}, \mathfrak{b}) - \log\left(\frac{\mathbf{N}(\mathfrak{a})}{\mathbf{N}(\mathfrak{b})}\big|\mathbf{N}(y)\big|\right)\right),$$

where the function h_F is defined by

$$\frac{C_F}{2} h_F(z, \mathfrak{a}, \mathfrak{b}) = \frac{\mathbf{N}(\mathfrak{a})}{\mathbf{N}(\mathfrak{b})}\big|\mathbf{N}(y)\big|\xi_F(2, \mathfrak{b}) + V(\mathfrak{a})\big|\mathbf{N}(y)\big| \sum_{(\alpha,\beta^*)} e^{2\pi i\,\mathrm{Tr}(x\alpha\beta^*)} B_F\left(\alpha y, \beta^*, \frac{1}{2}\right).$$

Proof. It is well-known that

$$\mathrm{Res}_{s=1} \zeta_F(s, \mathfrak{a}) = \frac{C_F}{d_F^{1/2}},$$

or equivalently

$$\mathrm{Res}_{s=1} \xi_F(s, \mathfrak{a}) = C_F.$$

Thus the claim follows from Theorem 2.4.5. ∎

The function $h_F(z, \mathfrak{a}, \mathfrak{b})$ has a modular property.

COROLLARY 2.5.2. *For $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(O_F)$ such that $b \in \mathfrak{a}\mathfrak{b}^{-1}$ and $c \in \mathfrak{a}^{-1}\mathfrak{b}$, we have*

$$h_F\big((az+b)(cz+d)^{-1}, \mathfrak{a}, \mathfrak{b}\big) = h_F(z, \mathfrak{a}, \mathfrak{b}) - 2\log\|cz+d\|.$$

Proof. Put $z' = x' + y' j_F = (az+b)(cz+d)^{-1}$. The conditions on b and c ensure that

$$\mathfrak{a}z + \mathfrak{b} = \mathfrak{a}(az+b) + \mathfrak{b}(cz+d) = (\mathfrak{a}z' + \mathfrak{b})(cz+d),$$

and hence

$$\widehat{E}(\mathfrak{a}z + \mathfrak{b}, s) = \widehat{E}(\mathfrak{a}z' + \mathfrak{b}, s).$$

Then we see from Theorem 2.5.1 that

$$h_F(z, \mathfrak{a}, \mathfrak{b}) - \log\big|\mathbf{N}(y)\big| = h_F(z', \mathfrak{a}, \mathfrak{b}) - \log\big|\mathbf{N}(y')\big|.$$

Hence the claim is reduced to the identity $\mathbf{N}(y') = \mathbf{N}(y)/\|cz+d\|^2$, which is easily shown. ∎

Remark 2.5.3. The function $h_F(z, \mathfrak{a}, \mathfrak{b})$ is a generalization of that studied by Asai [1] and Jorgenson-Lang [5] (up to a constant multiple).

§3. Hecke's integral formula for quadratic extensions

Let K be a quadratic extension of a number field F. The goal in this section is Theorem 3.1.2, which represents a zeta function of K as an integral of the Eisenstein series for F.

In the following, we denote the non-trivial F-automorphism of K by $x \mapsto x'$. We also use the same notation for the induced map on $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$ or on S_K.

3.1. Hecke's integral formula

First, we show an integral formula in a somewhat abstract setting. Let us denote by $d^{\times}u$ the quotient Haar measure on T_K/T_F, i.e. the measure which satisfies

$$\int_{T_K} \phi(t)\, d^{\times}t = \int_{T_K/T_F} \int_{T_F} \phi(t\tilde{u})\, d^{\times}t\, d^{\times}u,$$

where ϕ is any integrable function on T_K and $\tilde{u} \in T_K$ denotes a lift of $u \in T_K/T_F$.

We define

$$g(z) := \prod_{w \in S_K} \exp\bigl(-n_w \pi |z_w|^2\bigr) \quad \bigl(n_w := [K_w : \mathbb{R}]\bigr)$$

for $z = (z_w)_w \in K_{\mathbb{R}}$.

Proposition 3.1.1. *Let $\rho\colon K_{\mathbb{R}} \to \mathbb{D}_F$ be an isomorphism of $F_{\mathbb{R}}$-modules which preserves the Haar measure and satisfies $g(z) = f\bigl(\rho(z)\bigr)$ for any $z \in K_{\mathbb{R}}$. Then, for a fractional ideal $\mathfrak{A}$ of K, we have*

$$\xi_K(s, \mathfrak{A}) = \int_{T_K/T_F U_K} \widehat{E}(\rho(\tilde{u}\mathfrak{A}), s)\, d^{\times}u.$$

Here the lift $\tilde{u} \in T_K$ of $u \in T_K/T_F U_K$ is chosen to satisfy $\bigl|\mathbf{N}_{K/\mathbb{Q}}(\tilde{u})\bigr| = 1$.

Proof. We apply Proposition 2.4.3 and compute as

$$\begin{aligned}
\xi_K(s, \mathfrak{A}) &= V(\mathfrak{A})^s \int_{T_K/U_K} {\sum_{\alpha \in \mathfrak{A}}}' g(t\alpha) \bigl|\mathbf{N}_{K/\mathbb{Q}}(t)\bigr|^s\, d^{\times}t \\
&= V(\mathfrak{A})^s \int_{T_K/T_F U_K} \int_{T_F/T_F \cap U_K} {\sum_{\alpha \in \mathfrak{A}}}' f\bigl(\rho(t\tilde{u}\alpha)\bigr) \bigl|\mathbf{N}_{K/\mathbb{Q}}(t\tilde{u})\bigr|^s\, d^{\times}t\, d^{\times}u \\
&= \int_{T_K/T_F U_K} V\bigl(\rho(\tilde{u}\mathfrak{A})\bigr)^s \int_{T_F/U_F} {\sum_{\alpha \in \mathfrak{A}}}' f\bigl(t\rho(\tilde{u}\alpha)\bigr) \bigl|\mathbf{N}_{F/\mathbb{Q}}(t)\bigr|^{2s}\, d^{\times}t\, d^{\times}u.
\end{aligned}$$

In view of Proposition 2.2.4, this is the desired equation. $\square$

To obtain a more concrete formula, we need to construct ρ, describe the group $T_K/T_F U_K$ with its measure, and choose a lift $\tilde{u}$ for each u.

For each $v \in S_F$, choose and fix a place $w = w_v \in S_K$ above v. Then we define a map $\rho\colon K_\mathbb{R} \to \mathbb{D}_F$ by

$$\bigl(\rho(z)\bigr)_v := z_w + z_{w'} j_v = \begin{cases} z_w + z_{w'} i & (v \in S_F^1,\ w \in S_K^1), \\ (1+i) z_w & (v \in S_F^1,\ w \in S_K^2), \\ z_w + z_{w'} j & (v \in S_F^2,\ w \in S_K^2). \end{cases}$$

It is easy to check that it satisfies the conditions in Proposition 3.1.1.

Next, we set

$$T_{K/F} := \bigl\{ u \in T_K \mid \mathbf{N}_{K/F}(u) = 1 \bigr\}, \quad U_{K/F} := T_{K/F} \cap U_K.$$

Then the map $x \mapsto x/x'$ induces the isomorphisms

$$T_K/T_F \xrightarrow{\cong} T_{K/F}, \quad T_K/T_F U_{K/F} \xrightarrow{\cong} T_{K/F}/U_{K/F}^2,$$

where $U_{K/F}^2$ means $\{u^2 \mid u \in U_{K/F}\}$.

Let us study the structure of $T_{K/F}$ by looking at its v-component

$$(T_{K/F})_v = \begin{cases} \bigl\{ \bigl(z, z^{-1}\bigr) \in K_w^\times \times K_{w'}^\times \mid z \in F_v^\times \bigr\} & (F_v = K_w = K_{w'}), \\ \bigl\{ z \in K_w^\times \mid |z| = 1 \bigr\} & (F_v \subsetneq K_w \cong \mathbb{C}). \end{cases}$$

for each $v \in S_F$. When $F_v = K_w$, we choose the Haar measure on $(T_{K/F})_v$ so that the isomorphism

$$F_v^\times \ni z \longmapsto \bigl(z, z^{-1}\bigr) \in (T_{K/F})_v$$

preserves the measure. Moreover, for $u = \bigl(z, z^{-1}\bigr) \in (T_{K/F})_v$, we define

$$\tilde{u} := \bigl(z|z|^{-1/2}, |z|^{-1/2} \bigr) \in K_w^\times \times K_{w'}^\times.$$

On the other hand, if $F_v \subsetneq K_w$, we equip $(T_{K/F})_v \cong \mathbb{C}_1^\times$ with the measure of total mass 2π, and put $\tilde{u} := \sqrt{u} \in K_w^\times$, any one of the square roots of $u \in (T_{K/F})_v$.

Now, by taking the product, we obtain the Haar measure $d^\times u$ on $T_{K/F}$ and the map

$$T_{K/F} \ni u \longmapsto \tilde{u} \in T_K$$

satisfying $\tilde{u}/\tilde{u}' = u$ and $\bigl|\mathbf{N}(\tilde{u})\bigr| = 1$.

Theorem 3.1.2. *For a fractional ideal* $\mathfrak{A}$ *of* K*, we have*

$$\xi_K(s,\mathfrak{A}) = \frac{1}{w_{K/F}} \int_{T_{K/F}/U_{K/F}^2} \widehat{E}\big(\rho(\tilde{u}\mathfrak{A}), s\big)\, d^\times u,$$

where $w_{K/F}$ *denotes the index* $[U_K : U_F U_{K/F}]$.

Proof. We have only to check the compatibility of Haar measures in the isomorphism $T_K/T_F \cong T_{K/F}$, i.e.

$$\int_{T_K} \phi(t)\, d^\times t = \int_{T_{K/F}} \int_{T_F} \phi(t\tilde{u})\, d^\times t\, d^\times u$$

for integrable functions ϕ. Let us consider componentwise.

If both v and w are real, we have

$$F_v^\times = K_w^\times = K_{w'}^\times \cong \{\pm 1\} \times \mathbb{R}_+^\times.$$

Here $\mathbb{R}_+^\times$, the multiplicative group of positive real numbers, is equiped with the Haar measure dt/t, while $\{\pm 1\}$ has the total mass 2. Therefore we may consider the compatibilities of measures in the isomorphisms

$$\big(\{\pm 1\} \times \{\pm 1\}\big)/\{\pm 1\} \cong \{\pm 1\}, \quad \big(\mathbb{R}_+^\times \times \mathbb{R}_+^\times\big)/\mathbb{R}_+^\times \cong \mathbb{R}_+^\times$$

separately. Then the problem in the former is trivial, while that in the latter is reduced to the elementary formula

$$\int_0^\infty \int_0^\infty \phi(x,y)\, dx\, dy = \int_0^\infty \int_0^\infty \phi\big(tu^{1/2}, tu^{-1/2}\big)\, dt\, du.$$

The case in which v and w are complex can be treated in the same way, using $F_v^\times \cong \mathbb{C}_1^\times \times \mathbb{R}_+^\times$, where $\mathbb{C}_1^\times$ has the total mass 4π (recall that the Haar measure on F_v and K_w are twice the Lebesgue measure). Finally, if v is real and w is complex, we have

$$(T_{K/F})_v = K_w^\times / F_v^\times \cong \big(\mathbb{C}_1^\times \times \mathbb{R}_+^\times\big)/\big(\{\pm 1\} \times \mathbb{R}_+^\times\big) \cong \mathbb{C}_1^\times/\{\pm 1\},$$

and both sides have the total mass 2π. This completes the proof. ∎

Remark 3.1.3. Instead of ρ, we can use the map ρ^* defined by

$$\big(\rho^*(z)\big)_v := z_w - j_v z_{w'} = \begin{cases} z_w - z_{w'} i & (v \in S_F^1,\ w \in S_K^1), \\ (1-i) z_w & (v \in S_F^1,\ w \in S_K^2), \\ z_w - \overline{z_{w'}} j & (v \in S_F^2,\ w \in S_K^2). \end{cases}$$

Then the dual lattice of $\rho(\tilde{u}\mathfrak{A})$ is $\rho^*(\tilde{u}^{-1}\mathfrak{A}^*)$. Hence the functional equation for $\widehat{E}$ (Theorem 2.3.3) leads to the functional equation $\xi_K(s,\mathfrak{A}) = \xi_K(1-s,\mathfrak{A}^*)$.

3.2. Application to the Kronecker limit formula

Combining Theorem 3.1.2 and Theorem 2.5.1, we obtain a 'relative' Kronecker limit formula, which represents a relation between $\mathrm{CT}_{s=1}\,\xi_K(s,\mathfrak{A})$ and $\mathrm{CT}_{s=1}\,\xi_F(s,\mathfrak{a})$.

THEOREM 3.2.1. *Let $\mathfrak{A} \subset K$ be a fractional ideal of the form $\mathfrak{A} = \mathfrak{a}z+\mathfrak{b}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are fractional ideals of F. Then we have*

$$\frac{\mathrm{CT}_{s=1}\,\xi_K(s,\mathfrak{A})}{C_K} = 2\frac{\mathrm{CT}_{s=1}\,\xi_F(s,\mathfrak{a})}{C_F} - \log\frac{\mathbf{N}(\mathfrak{a})}{\mathbf{N}(\mathfrak{b})} + \frac{C_F}{2w_{K/F}C_K}\int_{T_{K/F}/U^2_{K/F}} \Big(h_F(z_{\tilde{u}},\mathfrak{a},\mathfrak{b}) - \log\big|\mathbf{N}(y_{\tilde{u}})\big|\Big)\,d^{\times}u.$$

Here we put

$$z_{\tilde{u}} = x_{\tilde{u}} + y_{\tilde{u}}j_F := \rho(\tilde{u}z)\rho(\tilde{u})^{-1}.$$

Proof. Since

$$\rho(\tilde{u}\mathfrak{A}) = \mathfrak{a}\rho(\tilde{u}z) + \mathfrak{b}\rho(\tilde{u}) = (\mathfrak{a}z_{\tilde{u}} + \mathfrak{b})\rho(\tilde{u}),$$

Theorem 3.1.2 tells us that

$$\xi_K(s,\mathfrak{A}) = \frac{1}{w_{K/F}}\int_{T_{K/F}/U^2_{K/F}} \widehat{E}(\mathfrak{a}z_{\tilde{u}}+\mathfrak{b},s)\,d^{\times}u.$$

By comparing the residues at $s=1$, we obtain

$$C_K = \frac{C_F}{2w_{K/F}}\int_{T_{K/F}/U^2_{K/F}} d^{\times}u.$$

Thus the claimed formula follows from Theorem 2.5.1. ∎

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