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HARTOGS TYPE THEOREMS FOR CR L^2 FUNCTIONS ON COVERINGS OF STRONGLY PSEUDOCONVEX MANIFOLDS

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Abstract. We prove an analog of the classical Hartogs extension theorem for CR L^2 functions defined on boundaries of certain (possibly unbounded) domains on coverings of strongly pseudoconvex manifolds. Our result is related to a question formulated in the paper of Gromov, Henkin and Shubin [GHS] on holomorphic L^2 functions on coverings of pseudoconvex manifolds.

§1. Introduction

1.1. In this paper, following our previous work [Br4], we continue to study holomorphic L^2 functions on coverings of strongly pseudoconvex manifolds. The subject was originally motivated by the paper [GHS] of Gromov, Henkin and Shubin. In [GHS] the von Neumann dimension was used to measure the space of holomorphic L^2 functions on *regular* (i.e., *Galois*) coverings of a strongly pseudoconvex manifold M. In particular, it was shown that the space of such functions is infinite-dimensional. It was also asked whether the regularity of the covering is relevant for the existence of many holomorphic L^2 functions on M' or it is just an artifact of the chosen method of the proof which requires a use of von Neumann algebras.

In an earlier paper [Br4] we proved that actually the regularity of M' is irrelevant for the existence of many holomorphic L^2 functions on M'. Moreover, we obtained an extension of some of the main results of [GHS]. The method of the proof used in [Br4] is completely different and (probably) easier than that used in [GHS] and is based on L^2 cohomology techniques, as well as, on the geometric properties of M. Also, in [Br1]–[Br3] the case of coverings of pseudoconvex domains in Stein manifolds was considered. Using the methods of the theory of coherent Banach sheaves together with

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Cartan's vanishing cohomology theorems, we proved some results on holomorphic L^p functions, $1 \leq p \leq \infty$, defined on such coverings.

1.2. The present paper is related to one of the open problems posed in [GHS], a Hartogs type theorem for coverings of strongly pseudoconvex manifolds. Let us recall that for a bounded open set $D \subset \mathbb{C}^n$ $(n > 1)$ with a connected smooth boundary bD the classical Hartogs theorem states that any holomorphic function in some neighbourhood of bD can be extended to a holomorphic function on a neighbourhood of the closure $\overline{D}$. In [Bo] Bochner proved a similar extension result for functions defined on the bD only. In modern language his result says that for a smooth function defined on the bD and satisfying the tangential Cauchy-Riemann equations there is an extension to a holomorphic function in D which is smooth on $\overline{D}$. In fact, this statement follows from Bochner's proof (under some smoothness conditions). However at that time there was not yet the notion of a CR-function. Over the years significant contributions to the area of Hartogs theorem were made by many prominent mathematicians, see the history and the references in the paper of Harvey and Lawson [HL, Section 5]. A general Hartogs-Bochner type theorem for bounded domains D in Stein manifolds was proved by Harvey and Lawson [HL, Theorem 5.1]. The proof relies heavily upon the fact that for $n \geq 2$ any $\overline{\partial}$-equation with compact support on an n-dimensional Stein manifold has a compactly supported solution. In [Br2] and [Br3] we proved some extensions of the theorem of Harvey and Lawson for certain (possibly unbounded) domains on coverings of Stein manifolds. In the present paper we prove an analogous result for CR L^2 functions defined on boundaries of certain domains on coverings of strongly pseudoconvex manifolds. More general Hartogs type theorems for CR-functions of slow growth on boundaries of such domains will be presented in a forthcoming paper.

1.3. Let $M \subset\subset N$ be a domain with smooth boundary bM in an n-dimensional complex manifold N, specifically,

$$M = \{z \in N : \rho(z) < 0\} \tag{1.1}$$

where ρ is a real-valued function of class $C^2(\Omega)$ in a neighbourhood Ω of the compact set $\overline{M} := M \cup bM$ such that

$$d\rho(z) \neq 0 \quad \text{for all } z \in bM. \tag{1.2}$$

Let $z_1, \dots, z_n$ be complex local coordinates in N near $z \in bM$. Then the tangent space T_zN at z is identified with $\mathbb{C}^n$. By $T_z^c(bM) \subset T_zN$ we denote the complex tangent space to bM at z, i.e.,

$$T_z^c(bM) = \left\{ w = (w_1, \dots, w_n) \in T_z(N) : \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z) w_j = 0 \right\}. \tag{1.3}$$

The *Levi form* of ρ at $z \in bM$ is a hermitian form on $T_z^c(bM)$ defined in local coordinates by the formula

$$L_z(w, \overline{w}) = \sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z) w_j \overline{w}_k. \tag{1.4}$$

The manifold M is called *pseudoconvex* if $L_z(w, \overline{w}) \geq 0$ for all $z \in bM$ and $w \in T_z^c(bM)$. It is called *strongly pseudoconvex* if $L_z(w, \overline{w}) > 0$ for all $z \in bM$ and all $w \neq 0$, $w \in T_z^c(bM)$.

Equivalently, strongly pseudoconvex manifolds can be described as the ones which locally, in a neighbourhood of any boundary point, can be presented as strictly convex domains in $\mathbb{C}^n$. It is also known (see [C], [R]) that any strongly pseudoconvex manifold admits a proper holomorphic map with connected fibres onto a normal Stein space. In particular, if M is a strongly pseudoconvex non-Stein manifold of complex dimension $n \geq 2$, then the union C_M of all compact complex subvarieties of M of complex dimension ≥ 1 is a compact complex subvariety of M.

Let $r : M' \to M$ be an unbranched covering of M. Assume that N is equipped with a Riemannian metric g_N. By d we denote the path metric on M' induced by the pullback of g_N. Consider a domain $\widetilde{D} \subset\subset M$ with a connected C^1 smooth boundary $b\widetilde{D}$ such that

$$b\widetilde{D} \cap C_M = \emptyset. \tag{1.5}$$

Let D be a connected component of $r^{-1}(\widetilde{D})$. By bD we denote the boundary of D and by $\overline{D} \subset M'$ the closure of D. Also, by $\mathcal{O}(D)$ we denote the space of holomorphic functions on D. Now, recall that a continuous function f on bD is called CR if for every smooth $(n, n-2)$-form ω on M' with compact support one has

$$\int_{bD} f \cdot \overline{\partial} \omega = 0.$$

If f is smooth this is equivalent to f being a solution of the tangential CR-equations: $\overline{\partial}_b f = 0$ (see, e.g., [KR]).

Let $dV_{M'}$ and dV_{bD} be the Riemannian volume forms on M' and bD obtained by the pullback of the Riemannian metric g_N on N. By $H^2(D)$ we denote the Hilbert space of holomorphic functions g on D with norm

$$\left(\int_{z\in D} |g(z)|^2\, dV_{M'}(z)\right)^{1/2}.$$

Also, $L^2(bD)$ stands for the Hilbert space of functions g on bD with norm

$$\left(\int_{z\in bD} |g(z)|^2\, dV_{bD}(z)\right)^{1/2}.$$

The following question was asked in [GHS, Section 4]:

Suppose that D is a regular covering of a strongly pseudoconvex manifold $\widetilde{D} \subset\subset M$. Is it true that for every CR-function $f \in L^2(bD)\cap C(\overline{D})$ there exists $F \in H^2(D)\cap C(\overline{D})$ such that $F|_{bD} = f$?

In the present paper we give a particular answer to this question. To formulate our results we require the following definitions.

For every x from the closure of $\widetilde{D}$ we introduce the Hilbert space $l_{2,x}(D)$ of functions g on $r^{-1}(x)\cap\overline{D}$ with norm

$$|g|_x := \left(\sum_{y\in r^{-1}(x)\cap\overline{D}} |g(y)|^2\right)^{1/2}. \tag{1.6}$$

Next, we introduce the Banach space $\mathcal{H}_2(D)$ of holomorphic on D functions f with norm

$$|f|_D := \sup_{x\in\widetilde{D}} |f|_x.$$

Similarly, we introduce the Banach space $\mathcal{L}_2(bD)$ of continuous on bD functions g with norm

$$|g|_{bD} := \sup_{x\in b\widetilde{D}} |f|_x.$$

Let $\mathcal{U} = (U_i)_{i\in I}$ be a finite open cover of $b\widetilde{D}$ by open simply connected sets $U_i \subset\subset M$. Then $r^{-1}(U_i)\cap bD$ is homeomorphic to $(U_i\cap b\widetilde{D})\times Q$ where Q is the fibre of the covering $r : D \to \widetilde{D}$. In what follows we identify $r^{-1}(U_i)\cap bD$ with $(U_i\cap b\widetilde{D})\times Q$.

Suppose that $f \in C(bD)$ is a CR-function satisfying the following conditions

(1) $f \in \mathcal{L}_2(bD)$;
(2) for any $i \in I$ and any $z_1, z_2 \in b\widetilde{D} \cap U_i$ there is a constant L_i such that

$$\left(\sum_{q\in Q}\left|\frac{f(z_1,q)-f(z_2,q)}{d((z_1,q),(z_2,q))}\right|^2\right)^{1/2} \le L_i.$$

(It is easy to show that condition (2) is independent of the choice of the cover.)

THEOREM 1.1. *For any CR-function f on bD satisfying conditions* (1) *and* (2) *there exists $\hat{f} \in \mathcal{H}_2(D) \cap C(\overline{D})$ such that*

$$\hat{f}|_{bD} = f \quad \textit{and} \quad |\hat{f}|_D = |f|_{bD}.$$

Remark 1.2. (A) If, in addition, bD is smooth of class C^k, $1 \le k \le \infty$, and $f \in C^s(bD)$, $1 \le s \le k$, then the extension $\hat{f}$ belongs to $\mathcal{O}(D) \cap C^s(\overline{D})$. This follows from [HL, Theorem 5.1].

(B) From the Cauchy integral formula it follows that the hypotheses of the theorem are true if f is the restriction to bD of a holomorphic function from $\mathcal{H}_2(W)$ where $\widetilde{W} := r(W) \subset\subset M$ is a neighbourhood of $b\widetilde{D}$ and W is a connected component of $r^{-1}(\widetilde{W})$ containing bD (see [Br1, Proposition 2.4] for similar arguments).

(C) It was shown in [Br4, Theorem 1.1] that holomorphic functions from $\mathcal{H}_2(M')$ separate points on $M' \setminus C'_M$ where $C'_M := r^{-1}(C_M)$. Thus there are sufficiently many CR-functions f on bD satisfying conditions (1) and (2).

As before by $\mathcal{L}_2(M')$ we denote the Banach space of continuous functions f on M' with norm

$$|f|_{M'} := \sup_{x\in M} |f|_x.$$

where $|\cdot|_x$, $x \in M$, is defined as in (1.6) with M' substituted for $\overline{D}$. Also, for a measurable locally bounded $(0,1)$-differential form η on M' by $|\eta|_z$, $z \in M'$, we denote the norm of η at z defined by the natural hermitian metric on the fibres of the cotangent bundle T^*M' on M'. We say that such η belongs to the space $\mathcal{E}_2(M')$ if

$$|\eta|_{M'} := \sup_{x\in M}\left(\sum_{z\in r^{-1}(x)} |\eta|_z^2\right)^{1/2} < \infty. \tag{1.7}$$

(Note that this definition does not depend on the choice of the Riemannian metric on N, and that the expression in the brackets is correctly defined for almost all $x \in M$.)
By $\operatorname{supp}\eta$ we denote support of η, i.e., the minimal closed set $K \subset M'$ such that η equals zero almost everywhere on $M' \setminus K$.

As mentioned above, the proof of the classical Hartogs theorem is based on the fact that for $n \geq 2$ any $\overline{\partial}$-equation with compact support on an n-dimensional Stein manifold has a compactly supported solution. Similarly our proof of Theorem 1.1 is based on the following result.

THEOREM 1.3. *Let $O \subset\subset M \setminus C_M$. Assume that a $(0,1)$-form η on M' belongs to $\mathcal{E}_2(M')$, is $\overline{\partial}$-closed (in the distributional sense) and*

$$r(\operatorname{supp}\eta) \subset O.$$

Then there are a function $F \in \mathcal{L}_2(M')$ and a neighborhood $U \subset M$ of bM such that $\overline{\partial}F = \eta$ (in the distributional sense) and $F|_{r^{-1}(U)} = 0$.

(Since M' can be thought of as a subset of a covering L' of a neighbourhood L of $\overline{M}$, the boundary bM' of M' is correctly defined.)

Remark 1.4. (A) Condition (2) in the formulation of Theorem 1.1 means that f is a Lipschitz section of a Hilbert vector bundle on $b\widetilde{D}$ with fibre $l_2(Q)$ associated with the natural action of the fundamental group $\pi_1(b\widetilde{D})$ of $b\widetilde{D}$ on $l_2(Q)$ (see [Br1, Example 2.2(b)] for a similar construction). This condition is required by the method of the proof. It would be interesting to know to what extent it is necessary.

(B) Another interesting question is whether a general extension theorem for CR-functions on bD without growth condition might hold.

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§2. Proof of Theorem 1.1

In this section we prove Theorem 1.1 modulo Theorem 1.3. Then in the next section we prove Theorem 1.3.

Since $b\widetilde{D}$ is a compact C^1 smooth manifold, there are a neighbourhood $O \subset\subset M \setminus C_M$ of $b\widetilde{D}$ and a C^1 retraction $p : O \to b\widetilde{D}$. (As such O one can take, e.g., a neighbourhood of the zero section of the normal vector bundle on $b\widetilde{D}$.) Without loss of generality we may assume also that fundamental groups $\pi_1(O)$ and $\pi_1(b\widetilde{D})$ are isomorphic. Let O' be a connected component of $r^{-1}(O) \subset M'$ containing bD. Then by the covering homotopy theorem there is a C^1 retraction $p' : O' \to bD$ such that $r \circ p' = p \circ r$.

Let ρ, $0 \le \rho \le 1$, be a C^∞ function on M equals 1 in a neighbourhood of $b\widetilde{D}$ with $\operatorname{supp} \rho \subset\subset O$. Consider the C^∞ function $\rho' := \rho \circ r$ on M'.

Let $\mathcal{V} = (V_j)_{j \in J}$ be a finite open cover of $\widetilde{D} \cup b\widetilde{D}$ by simply connected coordinate charts $V_j \subset\subset M$. We naturally identify $r^{-1}(V_j)$ with $V_j \times S$ where S is the fibre of $r : M' \to M$. Then in these local coordinates on M' we have

$$p'(z,s) = (p(z),s), \;\; \rho'(z,s) = \rho(z), \;\; (z,s) \in O' \cap r^{-1}(V_j), \;\; j \in J. \tag{2.1}$$

Next, for a CR-function f satisfying the assumptions of the theorem we define

$$f_1(z) := \rho'(z) \cdot f(p'(z)), \quad z \in \overline{D}. \tag{2.2}$$

LEMMA 2.1. *In the above local coordinates on M' one has*

$$\left(\sum_{s \in S} \left| \frac{f_1(z_1,s) - f_1(z_2,s)}{d((z_1,s),(z_2,s))} \right|^2 \right)^{1/2} \le C_j, \;\; (z_1,s), (z_2,s) \in \overline{D} \cap r^{-1}(V_j), \;\; j \in J,$$

for some numerical constants C_j.

Proof. By d_N we denote the path metric on N determined by the Riemannian metric g_N. Since the path metric d on M' is obtained by the pullback of g_N, we have $d((z_1,s),(z_2,s)) = d_N(z_1,z_2)$. Also, by the definition of p' and ρ' we clearly have for some $C > 0$,

$$d(p'(z_1,s),p'(z_2,s)) \le C d_N(z_1,z_2) \quad \text{for all } z_1,z_2 \in \operatorname{supp} \rho, \text{ and}$$
$$|\rho'(z_1,s) - \rho'(z_2,s)| \le C d_N(z_1,z_2) \quad \text{for all } z_1,z_2 \in M.$$

Using these inequalities, condition (2) of the theorem and the triangle inequality for l_2 norms we obtain that there is $A > 0$ such that for $z_1, z_2 \in$

$\operatorname{supp}\rho$

$$\begin{aligned}&\left(\sum_{s\in S}\left|\frac{f_1(z_1,s)-f_1(z_2,s)}{d_N(z_1,z_2)}\right|^2\right)^{1/2}\\&\le\left(\sum_{s\in S}\left\{\left|\frac{\rho(z_1)-\rho(z_2)}{d_N(z_1,z_2)}\right|\cdot|f(p(z_1),s)|\right.\right.\\&\qquad\qquad\left.\left.+|\rho(z_2)|\cdot\left|\frac{f(p(z_1),s)-f(p(z_2),s)}{d_N(z_1,z_2)}\right|\right\}^2\right)^{1/2}\\&\le C\left\{\left(\sum_{s\in S}|f(p(z_1),s)|^2\right)^{1/2}+\left(\sum_{s\in S}\left|\frac{f(p(z_1),s)-f(p(z_2),s)}{d((p(z_1),s),(p(z_2),s))}\right|^2\right)^{1/2}\right\}\\&\le A.\end{aligned}$$

Suppose now that, e.g., $z_1\in\operatorname{supp}\rho$ and $z_2\notin\operatorname{supp}\rho$. Then the term with $|\rho(z_2)|$ in the second line of the above inequalities disappears and again we get the require estimate. Finally, the case $z_1,z_2\notin\operatorname{supp}\rho$ is obvious. □

This lemma in particular implies that f_1 is a bounded Lipschitz function on $\overline{D}$. Now, using the McShane extension theorem [M] we extend f_1 to a Lipschitz function $\widetilde{f}$ on M'.

Further, since locally the metric d is equivalent to the Euclidean metric and since $\widetilde{f}$ is Lipschitz on M', by the Rademacher theorem, see, e.g., [Fe, Section 3.1.6], $\widetilde{f}$ is differentiable almost everywhere. In particular, $\overline{\partial}\widetilde{f}$ is a $(0,1)$-form on M' whose coefficients in its local coordinate representations are L^∞-functions. Let χ_D be the characteristic function of D. Consider the $(0,1)$-form on M' defined by

$$\omega:=\chi_D\cdot\overline{\partial}\widetilde{f}.$$

Then repeating word-for-word the arguments of [Br3, Lemma 3.3] we get

Lemma 2.2. *ω is $\overline{\partial}$-closed in the distributional sense.* □

Also, the inequality of Lemma 2.1 implies immediately that $\omega\in\mathcal{E}_2(M')$, see (1.7). Moreover, by our construction $r(\operatorname{supp}\omega)\subset\subset M\setminus C_M$. Thus according to Theorem 1.3 there is a continuous function $F\in\mathcal{L}_2(M')$ such that $\overline{\partial}F=\omega$ and $F|_{r^{-1}(U)}=0$ for a neighbourhood $U\subset M$ of bM. Since

$D \subset M'$ is a domain with a connected boundary, and F is holomorphic outside $\overline{D}$ (by the definition of ω), the latter implies that $F|_{bD} = 0$.

We set

$$\hat{f}(z) := f_1(z) - F(z), \quad z \in \overline{D}.$$

Using the above properties of f_1 and F one obtains easily that

$$\hat{f} \in \mathcal{O}(D) \cap C(\overline{D}) \quad \text{and} \quad \hat{f}|_{bD} = f.$$

Since f_1 and $F|_{\overline{D}}$ belong to $\mathcal{L}_2(\overline{D})$, $\hat{f} \in \mathcal{H}_2(D)$. Now, the identity $|\hat{f}|_D = |f|_{bD}$ follows from the fact that the function $z \mapsto |f|_z$, $z \in \widetilde{D} \cup b\widetilde{D}$, see (1.6), is continuous and plurisubharmonic on $\widetilde{D}$.

This completes the proof of the theorem. □

§3. Proof of Theorem 1.3

3.1. In Sections 3.1–3.6 we collect some auxiliary results required in the proof. Then in Section 3.7 we prove the theorem.

Let X be a complete Kähler manifold of dimension n with a Kähler form ω and E be a hermitian holomorphic vector bundle on X with curvature Θ. Let $L_2^{p,q}(X,E)$ be the space of L^2 E-valued (p,q)-forms on X with the L^2 norm, and let $W_2^{p,q}(X,E)$ be the subspace of forms such that $\overline{\partial}\eta$ is L^2. (The forms η may be taken to be either smooth or just measurable, in which case $\overline{\partial}\eta$ is understood in the distributional sense.) The cohomology of the resulting L^2 Dolbeault complex $(W_2^{\cdot,\cdot}, \overline{\partial})$ is the L^2 cohomology

$$H_{(2)}^{p,q}(X,E) = Z_2^{p,q}(X,E)/B_2^{p,q}(X,E),$$

where $Z_2^{p,q}(X,E)$ and $B_2^{p,q}(X,E)$ are the spaces of $\overline{\partial}$-closed and $\overline{\partial}$-exact forms in $L_2^{p,q}(X,E)$, respectively.

If $\Theta \geq \epsilon\omega$ for some $\epsilon > 0$ in the sense of Nakano, then the L^2 Kodaira-Nakano vanishing theorem, see [D], [O], states that

$$H_{(2)}^{n,r}(X,E) = 0 \quad \text{for } r > 0. \tag{3.1}$$

Assume now that $\Theta \leq -\epsilon\omega$ for some $\epsilon > 0$ in the sense of Nakano. Then using a duality argument and the Kodaira-Nakano vanishing theorem (3.1) one obtains, see [L, Corollary 2.4],

$$H_{(2)}^{0,r}(X,E) = 0 \quad \text{for } r < n. \tag{3.2}$$

3.2. Let $M \subset\subset N$ be a strongly pseudoconvex manifold. Without loss of generality we will assume that $\pi_1(M) = \pi_1(N)$ and N is strongly pseudoconvex, as well. Then there exist a normal Stein space X_N, a proper holomorphic surjective map $p : N \to X_N$ with connected fibres and points $x_1, \dots, x_l \in X_N$ such that

$$p : N \setminus \bigcup_{1 \le i \le l} p^{-1}(x_i) \longrightarrow X_N \setminus \bigcup_{1 \le i \le l} \{x_i\}$$

is biholomorphic, see [C], [R]. By definition, the domain $X_M := p(M) \subset X_N$ is strongly pseudoconvex, and so it is Stein. Without loss of generality we may assume that $x_1, \dots, x_l \in X_M$. Thus $\bigcup_{1 \le i \le l} p^{-1}(x_i) = C_M$.

Next, we introduce a complete Kähler metric on the complex manifold $M \setminus C_M$ as follows.

First, according to [N] there is a proper one-to-one map $i : X_M \hookrightarrow \mathbb{C}^{2n+1}$, $n = \dim_{\mathbb{C}} X_M$, which is an embedding in regular points of X_M. Then $i(X_M)$ is a complex subvariety of $\mathbb{C}^{2n+1}$. By ω_e we denote the $(1,1)$-form on M obtained as the pullback by $i \circ p$ of the Euclidean Kähler form on $\mathbb{C}^{2n+1}$. Clearly, ω_e is d-closed and positive outside C_M.

Similarly we can embed X_N into $\mathbb{C}^{2n+1}$ as a closed complex subvariety. Let $j : X_N \hookrightarrow \mathbb{C}^{2n+1}$ be an embedding such that $j(X_M)$ belongs to the open Euclidean ball B of radius $1/4$ centered at $0 \in \mathbb{C}^{2n+1}$. Set $z_i := j(x_i)$, $1 \le i \le l$. By ω_i we denote the restriction to $M \setminus C_M$ of the pullback with respect to $j \circ p$ of the form $-\sqrt{-1} \cdot \partial\overline{\partial} \log(\log \|z - z_i\|^2)^2$ on $\mathbb{C}^{2n+1} \setminus \{z_i\}$. (Here $\|\cdot\|$ stands for the Euclidean norm on $\mathbb{C}^{2n+1}$.) Since $j(X_M) \subset B$, the form ω_i is Kähler. Its positivity follows from the fact that the function $-\log(\log \|z\|^2)^2$ is strictly plurisubharmonic for $\|z\| < 1$. Also, ω_i is extended to a smooth form on $M \setminus p^{-1}(x_i)$. Now, let us introduce a Kähler form ω_M on $M \setminus C_M$ by the formula

$$\omega_M := \omega_e + \sum_{1 \le i \le l} \omega_i. \tag{3.3}$$

Proposition 3.1. *The path metric d on $M \setminus C_M$ induced by ω_M is complete.*

Proof. Assume, on the contrary, that there is a sequence $\{w_j\}$ convergent either to C_M or to the boundary bM of M such that the sequence $\{d(o, w_j)\}$ is bounded (for a fixed point $o \in M \setminus C_M$). Then, since $\omega_L \ge \omega_e$, the sequence $\{i(p(w_j))\} \subset \mathbb{C}^{2n+1}$ is bounded. This implies that $\{w_j\}$ converges to C_M. But since $\omega_L \geq \sum \omega_i$, the latter is impossible. One can check this using single blow-ups of $\mathbb{C}^{2n+1}$ at points z_i and rewriting the pullbacks to the resulting manifold of $(1,1)$-forms $-\sqrt{-1} \cdot \partial\overline{\partial} \log(\log \|z - z_i\|^2)^2$ in local coordinates near exceptional divisors, see, e.g., [GM] for similar arguments. □

Similarly one obtains complete Kähler metrics on unbranched coverings of $M \setminus C_M$ induced by pullbacks to these coverings of the Kähler form ω_M on $M \setminus C_M$.

3.3. We retain the notation of the previous section.

Let $r : N' \to N$ be an unbranched covering. Consider the corresponding covering $(M \setminus C_M)' := r^{-1}(M \setminus C_M)$ of $M \setminus C_M$. We equip $(M \setminus C_M)'$ with the complete Kähler metric induced by the form $\omega'_M := r^*\omega_M$. Next we consider the function $f := \sum_{0 \leq s \leq l} f_s$ on $(M \setminus C_M)'$ such that f_0 is the pullback by $i \circ p \circ r$ of the function $\|z\|^2$ on $\mathbb{C}^{2n+1}$ and f_s is the pullback by $j \circ p \circ r$ of the function $-\log(\log \|z - z_s\|^2)^2$ on $\mathbb{C}^{2n+1} \setminus \{z_s\}$, $1 \leq s \leq l$. Clearly we have

$$\omega'_M := \sqrt{-1} \cdot \partial\overline{\partial} f. \tag{3.4}$$

Let $E := (M \setminus C_M)' \times \mathbb{C}$ be the trivial holomorphic line bundle on $(M \setminus C_M)'$. Let g be the pullback to $(M \setminus C_M)'$ of a smooth plurisubharmonic function on M. We equip E with the hermitian metric e^{f+g} (i.e., for $z \times v \in E$ the square of its norm in this metric equals $e^{f(z)+g(z)}|v|^2$ where $|v|$ is the modulus of $v \in \mathbb{C}$). Then the curvature Θ_E of E satisfies

$$\Theta_E := -\sqrt{-1} \cdot \partial\overline{\partial} \log(e^{f+g}) = -\omega'_M - \sqrt{-1} \cdot \partial\overline{\partial} g \leq -\omega'_M. \tag{3.5}$$

From here by (3.2) we obtain

$$H^{0,r}_{(2)}((M \setminus C_M)', E) = 0 \quad \text{for } r < n. \tag{3.6}$$

3.4. In the proof we also use the following result.

Lemma 3.2. *Let h be a nonnegative piecewise continuous function on M equals 0 in some neighbourhood of C_M and bounded on every compact subset of $M \setminus C_M$. Then there exists a smooth plurisubharmonic function $\hat{g}$ on M such that*

$$\hat{g}(z) \geq h(z) \quad \textit{for all } z \in M.$$

Proof. Without loss of generality we identify $M \setminus C_M$ with $X_M \setminus \bigcup_{1\le j\le l}\{x_j\}$. Also, we identify X_M with a closed subvariety of $\mathbb{C}^{2n+1}$ as in Section 3.2. Let U be a neighbourhood of $\bigcup_{1\le j\le l}\{x_j\}$ such that $h|_U \equiv 0$. By $\Delta_r \subset \mathbb{C}^{2n+1}$ we denote the open polydisk of radius r centered at $0 \in \mathbb{C}^{2n+1}$. Assume without loss of generality that $0 \in X_M \setminus U$. Consider the monotonically increasing function

$$v(r) := \sup_{\Delta_r \cap X_M} h, \quad r \ge 0. \tag{3.7}$$

By v_1 we denote a smooth monotonically increasing function satisfying $v_1 \ge v$ (such v_1 can be easily constructed by v). Let us determine

$$v_2(r) := \int_0^{r+1} 2v_1(2t)\, dt, \quad r \ge 0.$$

By the definition v_2 is smooth, convex and monotonically increasing. Moreover,

$$v_2(r) \ge \int_{\frac{r+1}{2}}^{r+1} 2v_1(2t)\, dt \ge (r+1)v(r+1).$$

Next we define a smooth plurisubharmonic function v_3 on $\mathbb{C}^{2n+1}$ by the formula

$$v_3(z_1, \dots, z_{2n+1}) := \sum_{j=1}^{2n+1} v_2(|z_j|).$$

Then the pullback of v_3 to M is a smooth plurisubharmonic function on M. This is the required function $\hat{g}$. Indeed, under the identification described at the beginning of the proof for $|z|_\infty := \max_{1\le i\le 2n+1} |z_i|$ we have

$$\begin{aligned}\hat{g}(z) = v_3(z) &\ge (|z|_\infty + 1)v(|z|_\infty + 1)\\ &\ge \sup_{\Delta_{|z|_\infty+1}\cap X_M} h \ge h(z) \quad \text{for all } z \in M. \qquad \square\end{aligned}$$

3.5. In the proof of Theorem 1.3 we will assume without loss of generality that C_M is a *divisor with normal crossings*. Indeed, according to the Hironaka theorem, there is a *modification* $m : N_H \to N$ of N from Section 1.3 such that $m^{-1}(C_M)$ is a divisor with normal crossings and $m : N_H \setminus m^{-1}(C_M) \to N \setminus C_M$ is biholomorphic. By the definition $M_H := m^{-1}(M) \subset N_H$ is strongly pseudoconvex. Further, since M is a complex manifold, m induces an isomorphism of fundamental groups

$m_* : \pi_1(M_H) \to \pi_1(M)$. Thus for an unbranched covering $r : M' \to M$ of M there are a covering $r_H : M'_H \to M_H$ and a modification $m' : M'_H \to M'$ such that $r \circ m' = m \circ r_H$ and m' induces an isomorphism of the corresponding fundamental groups.

Assume now that a $(0,1)$-form $\eta \in \mathcal{E}_2(M')$ satisfies the hypotheses of Theorem 1.3. Consider its pullback $\widetilde{\eta} := (m')^*\eta$ on M'_H. Clearly, $\widetilde{\eta}$ also satisfies the hypotheses of Theorem 1.3 with M replaced by M_H. Now, suppose that Theorem 1.3 is valid for M'_H, i.e., there is a continuous function $\widetilde{f} \in \mathcal{L}_2(M'_H)$ such that $\overline{\partial}\widetilde{f} = \widetilde{\eta}$ and $\widetilde{f}$ vanishes in a neighbourhood of bM'_H. Since by the definition of η the function $\widetilde{f}$ is holomorphic in a neighbourhood of $(r \circ m')^{-1}(C_M) \subset M'_H$ and $m' : M_{H'} \to M'$ is a modification of M', there is a function $f \in \mathcal{L}_2(M')$ such that $\widetilde{f} = (m')^* f$. Obviously, f satisfies the required statements of the theorem.

3.6. Let $U_q \subset\subset M$ be a simply connected coordinate chart of $q \in C_M$ with complex coordinates $z = (z_1, \dots, z_n)$, $n = \dim_{\mathbb{C}} M$, such that $z_1(q) = \cdots = z_n(q) = 0$ and

$$C_M \cap U_q = \{f_q(z) = 0\}, \quad f_q(z) := z_1 \cdots z_k. \tag{3.8}$$

(Such coordinates exist by the definition of a divisor with normal crossings.)

Let $\hat{f}$ be a function on $M \setminus C_M$ such that $r^*\hat{f} = f$, see Section 3.3. From the definition of f we obtain

Lemma 3.3. *$e^{\hat{f}}$ extended by 0 to C_M is a continuous function on M such that $e^{\hat{f}}/|f_q|$ is unbounded on $U_q \setminus C_M$.* □

Let ω be the associated $(1,1)$-form of a hermitian metric g_N on N. Since by the definition $\omega_M \geq \omega_e$ and the latter form vanishes on C_M, we have locally near $C_M \cap U_q$

$$\omega_M^n \geq c'|f_q|^{2m'}\omega^n \tag{3.9}$$

for some $c' > 0$, $m' \in \mathbb{N}$. This and Lemma 3.3 imply that locally near $C_M \cap U_q$

$$e^{\hat{f}}\omega_M^n \geq c|f_q|^{2m}\omega^n \tag{3.10}$$

for some $c > 0$, $m \in \mathbb{N}$.

By $E_n(M)$ we denote a holomorphic line vector bundle on M determined by the divisor nC_M, $n \in \mathbb{N}$. Let s_1 be a holomorphic section of $E_1(M)$

defined in local coordinates on U_q by functions f_q from (3.8). Then $(r^*s_1)^n$ is a holomorphic section of the bundle $E'_n(M) := r^*E_n(M)$ on M'.

Since the hermitian bundle E from Section 3.3 is holomorphically trivial, we naturally identify sections of E with functions on $(M \setminus C_M)'$. Here and below we set $X' := r^{-1}(X)$ for $X \subset M$. Also, the Banach space $\mathcal{L}_2(X')$ of continuous functions on X' is defined similarly to $\mathcal{L}_2(M')$, see Section 1.3.

Let $(U_i)_{i\in I}$ be a finite open cover of a neighbourhood $\overline{M}$ ($\subset\subset N$) by simply connected coordinate charts $U_i \subset\subset N$.

PROPOSITION 3.4. *Suppose $h \in L_2((M \setminus C_M)', E)$ is such that for any U'_i there is a continuous function $h_i \in \mathcal{L}_2(U'_i)$ so that $c_i := h - h_i \in \mathcal{O}((U_i \setminus C_M)')$. Then there is an integer $n \in \mathbb{N}$ independent of h such that $h\cdot(r^*s_1)^n$ admits an extension $\hat{h} \in C(M', E'_n(M))$. Moreover, $h|_{O'} \in \mathcal{L}_2(O')$ for every $O \subset\subset M \setminus C_M$.*

Proof. Let U_q be a simply connected coordinate chart of $q \in C_M$ with the local coordinates satisfying (3.8). We naturally identify U'_q with $U_q \times S$ where S is the fibre of r. Then the hypotheses of the proposition imply that

$$\int_{z\in U_q\setminus C_M}\left(\sum_{s\in S}|h(z,s)|^2\right)e^{\hat{f}(z)+\hat{g}(z)}\omega_M^n(z) < \infty \tag{3.11}$$

where $\hat{g}$ is a smooth plurisubharmonic function on M such that $r^*\hat{g} = g$. Diminishing if necessary U_q assume that $\hat{f}$, ω_M^n satisfy (3.10) there. Also, on U_q we clearly have $\hat{g} \sim 1$. From here and (3.11) we obtain on U_q

$$\int_{z\in U_q\setminus C_M}\left(\sum_{s\in S}|h(z,s)|^2\right)|f_q(z)|^{2m}\omega^n(z) < \infty \tag{3.12}$$

where f_q is defined by (3.8).

Further, according to the hypothesis of the proposition, there is a continuous function $h_q \in \mathcal{L}_2(U'_q)$ such that $c_q := h - h_q \in \mathcal{O}((U_q \setminus C_M)')$. This and (3.12) imply that every $f_q^m\cdot c_q(\,\cdot\,,s)$, $s \in S$, is L^2 integrable with respect to the volume form $(\sqrt{-1})^n \bigwedge_{i=1}^n dz_i\wedge d\overline{z}_i$. Using these facts and the Cauchy integral formulas for coefficients of the Laurent expansion of $f_q^m c_q(\,\cdot\,,s)$, one obtains easily that every $f_q^m c_q(\,\cdot\,,s)$ can be extended holomorphically to U_q. In turn, this gives a continuous extension $\hat{h}$ of $h\cdot(r^*f_q)^m$ to U'_q.

Let $V_q \subset\subset U_q$ be another connected neighbourhood of q. By the Bergman inequality for holomorphic functions, see, e.g., [GR, Chapter 6, Theorem 1.3], we have

$$|h(y,s)f_q^m(y)|^2 \leq A \int_{z\in U_q} |h(z,s)f_q^m(z)|^2 \omega^n(z) \quad \text{for all } (y,s) \in W_q' \tag{3.13}$$

with some constant A depending on U_q, W_q and ω only. Therefore from (3.12) and (3.13) we obtain

$$\sup_{z\in V_q} \left(\sum_{s\in S} |\hat{h}(z,s)|^2 \right)^{1/2} < \infty.$$

Equivalently, $\hat{h}|_{V_q'} \in \mathcal{L}_2(V_q')$.

Next assume that $U_q \subset (U_i)_{i\in I}$ is a simply connected coordinate neighbourhood of a point $q \in M \setminus C_M$. Without loss of generality we may assume that all such U_q are relatively compact in $M\setminus C_M$. Identifying U_q' with $U_q \times S$ we have anew inequality of type (3.11) for $h|_{U_q'}$. Since $U_q \subset\subset M \setminus C_M$ and $\hat{f}$, $\hat{g}$ and ω_M^n are smooth on $M \setminus C_M$ by their definitions, we obviously have on U_q

$$e^{\hat{f}+\hat{g}} \cdot \omega_M^n \sim \omega^n.$$

Similarly to (3.12)–(3.13) (with $f_q = 1$) this implies that $h|_{V_q'} \in \mathcal{L}_2(V_q')$ for any connected neighbourhood $V_q \subset\subset U_q$ of q. Choose the above neighbourhoods V_q so that they form a finite cover of a set $O \subset\subset M \setminus C_M$. Then from the implications $h|_{V_q'} \in \mathcal{L}_2(V_q')$ we obtain that $h|_{O'} \in \mathcal{L}_2(O')$. Now, choosing the neighbourhoods V_q, $q \in C_M$, so that they form a finite cover of C_M and taking as the n the maximum of the numbers m in the powers of f_q, see (3.10), we obtain that $h \cdot (r^*s_1)^n$ admits an extension $\hat{h} \in C(M', E_n'(M))$. By our construction n is independent of h. □

3.7. Proof of Theorem 1.3

Assume that a $(0,1)$-form η belongs to $\mathcal{E}_2(M')$, is $\overline{\partial}$-closed and $r(\operatorname{supp}\eta) \subset O \subset\subset M \setminus C_M$.

Let us define the function g in the definition of the bundle E from Section 3.3 by Lemma 3.2. Namely, fix a neigbourhood $U \subset\subset M$ of C_M and consider the function h on M defined by the formula

$$h(z) := \frac{\chi_{U^c}(z)}{\operatorname{dist}(z, bM)} \tag{3.14}$$

where χ_{U^c} is the characteristic function of $U^c := M \setminus U$ and the distance to the boundary is defined by the path metric d_N on N induced by the Riemannian metric g_N. Further, according to Lemma 3.2 we can find a C^∞ plurisubharmonic function $\hat{g}$ on M such that $\hat{g}(z) \geq h(z)$ for all $z \in M$. Then in the definition of the metric on E we choose $g := r^*\hat{g}$.

LEMMA 3.5. *The form η belongs to $L_2^{0,1}((M \setminus C_M)', E)$.*

Proof. We retain the notation of Proposition 3.4. Consider the set $U_q' \cong U_q \times S$ on M' for some $q \in M$ such that $U_q \subset\subset M \setminus C_M$. Since $\eta \in \mathcal{E}_2(M')$, $r(\operatorname{supp}\eta) \subset O \subset\subset M \setminus C_M$ and $\hat{g}$, $\hat{f}$ and ω_M^n are bounded on O, for every such U_q we have

$$\int_{z \in U_q \setminus C_M} \left(\sum_{s \in S} |\eta|_{(z,s)}^2 \right) e^{\hat{f}(z)+\hat{g}(z)} \omega_M^n(z) < \infty. \tag{3.15}$$

(Recall that $|\eta|_{(z,s)}^2$ stands for the norm of η at $(z,s) \in M'$ defined by the natural hermitian metric on the fibres of the cotangent bundle T^*M' on M'.) Taking a finite open cover of O by such sets U_q we get the required statement. □

From Lemma 3.5 and the fact that $\overline{\partial}\eta = 0$ we obtain by (3.6) that there exists a function $F' \in L_2((M \setminus C_M)', E)$ such that $\overline{\partial}F' = \eta$. Moreover, by the definition of η, this function is holomorphic on $(M \setminus C_M)' \setminus r^{-1}(\overline{O})$. Also, since $\eta \in \mathcal{E}_2(M')$ the equation $\overline{\partial}G = \eta$ has local (continuous) solutions $f_U \in \mathcal{L}_2(U')$ for every $U \subset\subset M$ biholomorphic to an open Euclidean ball of $\mathbb{C}^n$. (In fact, since $U' \cong U \times S$, we can rewrite the equation $\overline{\partial}G = \eta$ on U' as a $\overline{\partial}$-equation on U with a measurable Hilbert valued $(0,1)$-form on the right-hand side. Then we apply the formula presented in the proof of Lemma 3.4 of [Br3] (see also [H, Section 4.2]) to solve this equation and to get a solution from $\mathcal{L}_2(U')$, for similar arguments see [Br1, Appendix A].)

Let us prove now

LEMMA 3.6. *There is a neighbourhood $U \subset M$ of bM such that $F'|_{r^{-1}(U)} = 0$.*

Proof. Let $q \in bM$ and $U_q \subset\subset N \setminus C_M$ be a simply connected coordinate chart of q. Since $\pi_1(M) = \pi_1(N)$ by our assumption, the covering M' of M is contained in the corresponding covering $r : N' \to N$ of N. Thus

$r^{-1}(U_q) \subset N'$ can be naturally identified with $U_q \times S$ where S is the fibre of r. Further, without loss of generality we may identify U_q with an open Euclidean ball B in $\mathbb{C}^n$. In this identification, on each component $U_q \times \{s\}$, $s \in S$, the path metric d on N' is equivalent to the Euclidean metric on B with the constants of equivalence independent of s.

Next, for some $s \in S$ let us consider the restriction F'_s of F' to $U_q \times \{s\} = B$. We set $M'_s := M' \cap (U_q \times \{s\})$ and $bM'_s := bM' \cap (U_q \times \{s\})$. Diminishing if necessary U_q, without loss of generality we may assume that these sets are connected. Also by dv we denote the Euclidean volume form on $\mathbb{C}^n$. By the constructions of $\omega_{M'}$, see Section 3.2, and f, see Section 3.3, we clearly have

$$f|_{U_q \times \{s\}} \geq c \quad \text{and} \quad \omega^n_{M'}|_{U_q \times \{s\}} \geq c\, dv \tag{3.16}$$

for some $c > 0$ independent of $s \in S$.

Further, by the definition $F'_s \in L_2(M'_s, E)$. So by the choice of g in the definition of the hermitian metric on E using (3.16) we obtain

$$\int_{z \in M'_s} |F'_s(z)|^2 e^{\frac{1}{\operatorname{dist}(z, bM'_s)}}\, dv(z) < \infty. \tag{3.17}$$

Without loss of generality we may assume that $U_q \cap r(\operatorname{supp} \eta) = \emptyset$. Thus F'_s is holomorphic on M'_s for each $s \in S$. Now, from (3.17) using the mean-value property for the plurisubharmonic function $|F'_s|^2$ defined on M'_s we easily obtain that for any $y \in bM'_s$

$$\lim_{z \to y} F'_s(z) = 0. \tag{3.18}$$

Indeed, for a point z sufficiently close to $y \in bM'_s$ consider a Euclidean ball B_z centered at z of radius $r_z := \operatorname{dist}(z, bM'_s)/2$. Choosing z closer to y we may assume that $B_z \subset\subset M'_s$. Then by the triangle inequality for the metric d_N we have

$$\operatorname{dist}(w, bM'_s) \leq 3r_z/2 \quad \text{for all } w \in B_z.$$

Now from (3.17) by the mean-value property we get for some $c_n > 0$ depending on n only:

$$\begin{aligned} c_n r_z^{2n} e^{2/(3r_z)} |F'_s(z)|^2 &\leq e^{2/(3r_z)} \int_{w \in B_z} |F'_s(w)|^2\, dv(w) \\ &\leq \int_{w \in B_z} |F'_s(w)|^2 e^{\frac{1}{\operatorname{dist}(w, bM'_s)}}\, dv(w) \leq A < \infty. \end{aligned}$$

Hence,

$$\lim_{z\to y}|F_s'(z)|^2\le \lim_{z\to y}\frac{Ae^{-2/(3r_z)}}{c_n r_z^{2n}}=0.$$

Thus (3.18) is true for any $y\in bM_s'$.

Next, since M_s' is connected, (3.18) implies that $F_s'\equiv 0$ on M_s' for each $s\in S$. Actually, let $z\in bM_s'$. Consider a complex line l_z passing through z and containing the normal to bM_s' at z (recall that bM_s' is smooth). Then l_z intersects bM_s' transversely in a neighbourhood of z in bM_s'. This implies that there is a simply connected domain $W_z\subset l_s\cap M_s'$ whose boundary bW contains z such that $F_s'|_{\overline{W}_z}\in C(\overline{W}_z)$ and it equals 0 on an open subset of bW_z. Thus by the uniqueness property for univariate holomorphic functions we have $F_s'=0$ on W_z. Observe that if z varies along bM_s' the union of the connected components of $l_z\cap M_s'$ containing W_z contains an open subset of M_s'. This implies that $F_s'\equiv 0$ on M_s'.

Finally, taking a finite open cover of bM by the above sets U_q and using similar arguments we obtain the required neighbourhood U of bM (as the union of such U_q intersected with M). This completes the proof of the lemma. ∎

Let us finish the proof of the theorem. As established above, the function F' satisfies conditions of Proposition 3.4. According to this proposition there is a number $n\in\mathbb{N}$ independent of F' such that $F'\cdot(r^*s_1)^n$ is extended to a continuous section of $E_n'(M)$ equals 0 on $r^{-1}(U)$. Moreover, $F'|_O\in\mathcal{L}_2(O')$ for any $O\subset\subset M\setminus C_M$.

We set

$$\widetilde{F}:=e^{F'}-1\quad\text{and}\quad \widetilde{\eta}:=\overline{\partial}\widetilde{F}=\widetilde{F}\eta.$$

By the definitions of η and F' we have $\operatorname{supp}\widetilde{\eta}=\operatorname{supp}\eta$ and $\widetilde{F}$ is bounded on $\operatorname{supp}\eta$. In particular, $\widetilde{\eta}$ is $\overline{\partial}$-closed and belongs to $L_2^{0,1}((M\setminus C_M)',E)$, as well. Then by (3.6) there is a function $\widetilde{F}'\in L_2((M\setminus C_M)',E)$ such that $\overline{\partial}\widetilde{F}'=\widetilde{\eta}$. Applying to $\widetilde{F}'$ the same arguments as to F' we conclude that $\widetilde{F}'|_{r^{-1}(U)}\equiv 0$ for some neigbourhood $U\subset M$ of bM. Since by the definition $\widetilde{F}-\widetilde{F}'$ is holomorphic on $(M\setminus C_M)'$ and equals zero on a neighbourhood of bM', from the connectedness of M' we get $\widetilde{F}=\widetilde{F}'$. Also, as in the case of F', $\widetilde{F}'\cdot(r^*s_1)^n$ is extended to a continuous section of $E_n(M')$.

Let q be a regular point of $C_M':=r^{-1}(C_M)$. The above properties of F' and $\widetilde{F}$ imply that for suitable complex coordinates $z=(z_1,\dots,z_n)$ in a neighbourhood U_q of q we have $C'_M \cap U_q = \{z_1 = 0\}$ and

$$e^{F'(z)} = z_1^{-n} A(z), \quad F'(z) = z_1^{-n} B(z), \quad z \in U_q \setminus C_M,$$

where $A, B \in \mathcal{O}(U_q)$. Suppose that $A(z) = z_1^l A'(z)$ for some $0 \le l < n$ with $A' \in \mathcal{O}(U_q)$ not identically 0 on $C'_M \cap U_q$. Then there is a point $p \in C'_M \cap U_q$ and its neighbourhood $W \subset U_q$ so that $A'(z) \neq 0$ for all $z \in \overline{W}$. Thus we can introduce complex coordinates $y = (y_1, \dots, y_n)$ on W by $y_1 := z_1(A'(z))^{1/(n-l)}$, $y_2 = z_2, \dots, y_n = z_n$. In these coordinates we have $e^{F'(y)} = y_1^{l-n}$, $y \in W \setminus C_M$. Since $F' \in \mathcal{O}(W \setminus C_M)$, the latter is impossible. This contradiction shows that $l \ge n$ and so $e^{F'}$ admits a holomorphic extension to U_q. From here we obtain easily that F' admits a holomorphic extension to U_q, as well.

Taking an open cover of regular points of C'_M by such neighbourhoods U_q, from the above arguments we obtain that F' is extended holomorphically to C'_M (it is extended to nonregular points of C'_M by the Hartogs theorem because the complex codimension of the set of such points in M' is ≥ 2).

Finally, the extended function F (i.e., the extension of F') belongs to $\mathcal{L}_2(M')$. Indeed, by the definition $F|_{O'} \in \mathcal{L}_2(O')$ for every $O \subset\subset N \setminus C_M$. Assume now that $q \in C_M$. Let U be a simply connected coordinate chart of q with complex coordinates $z = (z_1, \dots, z_n)$ such that $z_1(q) = \cdots = z_n(q) = 0$, $C_M \cap M = \{z_1 \cdots z_k = 0\}$ and $\overline{U} = \{z \in M : \max_{1 \le k \le n} |z_k| \le 1\}$. We identify $\overline{U}$ with the unit polydisk in $\mathbb{C}^n$ and by $\mathbb{T}^n$ we denote its boundary torus. Also, we naturally identify $(\overline{U})' \subset M'$ with $\overline{U} \times S$ where S is the fibre of $r : M' \to M$. Diminishing, if necessary, U we will assume that F is holomorphic in a neighbourhood $O' := r^{-1}(O)$ of $(\overline{U})'$ where O is a neighbourhood of $\overline{U}$.

Let $\{S_l\}_{l \in \mathbb{N}} \subset S$ be an increasing sequence of finite subsets of S such that $\bigcup_l S_l = S$. Then from the Cauchy integral formula we obtain

$$\lim_{l \to \infty} \left(\sum_{s \in S_l} |F(y,s)|^2 \right) \le \left(\frac{1}{2\pi} \right)^n \int_{x \in \mathbb{T}^n} \sum_{s \in S} \frac{|F(x,s)|^2}{(1 - |z_1(y)|) \cdots (1 - |z_n(y)|)} \, dx, \quad y \in U,$$

where dx is the volume form on $\mathbb{T}^n$. Since $\mathbb{T}^n \subset\subset M \setminus C_M$, $F|_{\mathbb{T}^n} \in \mathcal{L}_2(\mathbb{T}^n \times S)$. This implies that $F|_{r^{-1}(y)} \in l_2(S)$ for all $y \in V_q := \{z \in U : \max_{1 \le k \le n} |z_k| \le 1/2\}$ and the l_2 norms $|\cdot|_y$ of these functions are uniformly bounded. Choosing a finite cover of C_M by such V_q and taking into account that $F|_{O'} \in \mathcal{L}_2(O')$ for every $O \subset\subset N \setminus C_M$, from the above we obtain that $F \in \mathcal{L}_2(M')$. Also, $\overline{\partial} F = \eta$.

This completes the proof of the theorem. □

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