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SOME FAMILIES OF COMPONENTWISE LINEAR MONOMIAL IDEALS

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Abstract. Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Let $J = \{j_1, \ldots, j_t\}$ be a subset of $\{1, \ldots, n\}$, and let $\mathfrak{m}_J \subset R$ denote the ideal $(x_{j_1}, \ldots, x_{j_t})$. Given subsets $J_1, \ldots, J_s$ of $\{1, \ldots, n\}$ and positive integers $a_1, \ldots, a_s$, we study ideals of the form $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$. These ideals arise naturally, for example, in the study of fat points, tetrahedral curves, and Alexander duality of squarefree monomial ideals. Our main focus is determining when ideals of this form are componentwise linear. Using polymatroidality, we prove that I is always componentwise linear when $s \leq 3$ or when $J_i \cup J_j = [n]$ for all $i \neq j$. When $s \geq 4$, we give examples to show that I may or may not be componentwise linear. We apply these results to ideals of small sets of general fat points in multiprojective space, and we extend work of Fatabbi, Lorenzini, Valla, and the first author by computing the graded Betti numbers in the $s = 2$ case. Since componentwise linear ideals satisfy the Multiplicity Conjecture of Herzog, Huneke, and Srinivasan when $\mathrm{char}(k) = 0$, our work also yields new cases in which this conjecture holds.

§1. Introduction

Let $R = k[x_1, \ldots, x_n]$ be the polynomial ring in n indeterminates over a field k, and let $[n] := \{1, \ldots, n\}$. For a nonempty subset $J = \{j_1, \ldots, j_t\} \subseteq [n]$, we define $\mathfrak{m}_J := (x_{j_1}, \ldots, x_{j_t})$. The goal of this paper is to understand when ideals of the form

$$I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}, \text{ with } J_i \subseteq [n] \text{ and } a_i \in \mathbb{Z}^+,$$

are componentwise linear. We introduce the following definitions.

Definition 1.1. An ideal of the form $\mathfrak{m}_{J_i}^{a_i}$ for some $J_i \subset [n]$ is called a *Veronese ideal* [18]. We call an ideal $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$ an *intersection of Veronese ideals.*

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Let $I \subseteq R$ be a homogeneous ideal, and for a positive integer d, let (I_d) be the ideal generated by all forms in I of degree d. We say that I is *componentwise linear* if for each positive integer d, (I_d) has a linear resolution. Componentwise linear ideals were first introduced by Herzog and Hibi [19] to generalize Eagon and Reiner's result that the Stanley-Reisner ideal I_Δ of a simplicial complex Δ has a linear resolution if and only if the Alexander dual $\Delta^\star$ is Cohen-Macaulay [6]. In particular, Herzog and Hibi [19] and Herzog, Reiner, and Welker [21] showed that the Stanley-Reisner ideal I_Δ is componentwise linear if and only if $\Delta^\star$ is sequentially Cohen-Macaulay. On the algebraic side, in characteristic zero, Aramova, Herzog, and Hibi subsequently proved that I is componentwise linear if and only if it has the same graded Betti numbers as its graded reverse-lex generic initial ideal [1]. Römer used this result in [26] to prove that componentwise linear ideals satisfy the Multiplicity Conjecture of Herzog, Huneke, and Srinivasan [22] in characteristic zero.

Componentwise linearity also arises naturally in the study of several types of ideals from algebraic geometry. In [12], the first author showed that if I is the ideal of at most $n+1$ general fat points in $\mathbb{P}^n$, then I is componentwise linear. Additionally, the first author, Migliore, and Nagel proved that the ideal of a tetrahedral curve is componentwise linear if and only if the curve does not reduce to a complete intersection of type $(2,2)$; see [25] or [13] for an explanation of the reduction process. One of our goals in this paper is to identify more results applicable to geometry.

Our motivation to study intersections of Veronese ideals comes from the observation that in many of the cases in which the componentwise linear property of a monomial ideal has been studied, the ideal is a special case of an intersection of Veronese ideals. The defining ideal of $s \leq n$ fat points in $\mathbb{P}^{n-1}$ in generic position, investigated in [12], is an intersection of Veronese ideals with $J_i = \{1, \ldots, \hat{i}, \ldots, n\}$ for $i = 1, \ldots, s$. Moreover, the ideals of tetrahedral curves, studied in [25] and [13], have the form

$$I = (x_1,x_2)^{a_1} \cap (x_1,x_3)^{a_2} \cap (x_1,x_4)^{a_3} \cap (x_2,x_3)^{a_4} \cap (x_2,x_4)^{a_5} \cap (x_3,x_4)^{a_6} \subset k[x_1, \ldots, x_4],$$

where the a_i are nonnegative integers. Additionally, when each $a_i = 1$, the intersection of Veronese ideals is the Alexander dual of a Stanley-Reisner ideal; here, the minimal generators of the Stanley-Reisner ideal are the product of J_1 variables, the product of the J_2 variables, and so on. Faridi showed that if I is the facet ideal of a simplicial tree (so I is a squarefree monomial ideal), then the Alexander dual $I^\star$ is componentwise linear [9]. Faridi's result was partially generalized by the two authors [14]; they showed that if I is the edge ideal of a chordal graph, then the Alexander dual $I^\star$ is componentwise linear.

We now present the main results of this paper. Our primary tool is Theorem 3.1. We show that if I is an intersection of Veronese ideals in $k[x_1,\ldots,x_n]$, and if $J_i \cup J_j = [n]$ for all $i \neq j$, then (I_d) is a polymatroidal ideal for all d. We shall discuss polymatroidal ideals in the next section of preliminaries, but their most important property for us is that they have linear resolutions. Thus I is componentwise linear in this case since each (I_d) has a linear resolution. As a corollary of Theorem 3.1, we show that when $s=2$, $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$ is always componentwise linear. With some careful analysis of the generators of (I_d), we prove the same result in the case $s=3$ in Section 4. (When $s=1$, i.e., $I=\mathfrak{m}_J^a$, then the fact that I is componentwise linear is simply a corollary of the Eagon-Northcott resolution.) This shows that the ideals of tetrahedral curves that are not componentwise linear given in [13] are the simplest possible examples of intersections of Veronese ideals for which componentwise linearity fails. When $s \geq 4$, we give examples to show that $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$ may or may not be componentwise linear.

In Section 5, we expand upon the $s=2$ case by giving explicit formulas for the graded Betti numbers of $\mathfrak{m}_J^a \cap \mathfrak{m}_K^b$. Our formulas generalize results of Fatabbi [10], Valla [28], Fatabbi and Lorenzini [11], and the first author [12], which give the Betti numbers of ideals of two fat points in $\mathbb{P}^n$.

We conclude in Section 6 with some applications. We extend the first author's work in [12] by showing that if I is the ideal of a small number of general fat points in a multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, then I is componentwise linear. This also gives a new proof of the result in [12]; our technique in this paper is more general. Additionally, we use the results of Section 5 to write down the graded Betti numbers of two general fat points in multiprojective space. We also note that in each case that we show that a class of ideals is componentwise linear, the result solves the Multiplicity Conjecture of Herzog, Huneke, and Srinivasan [22] for that class of ideals (in characteristic zero).

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§2. Preliminaries

In this section, we recall some definitions and results used throughout the paper. As in the introduction, let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over the field k, and for any subset $J = \{j_1, \ldots, j_t\} \subseteq [n]$, we set $\mathfrak{m}_J := (x_{j_1}, \ldots, x_{j_t})$. Our primary interest in this paper is to determine when intersections of Veronese ideals in R, or equivalently, ideals of the form $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$, where the a_i are positive integers, are componentwise linear.

Associated to any homogeneous ideal I of R is a *minimal free graded resolution*

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{h,j}(I)} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}(I)} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}(I)} \longrightarrow I \longrightarrow 0$$

where $R(-j)$ denotes the R-module obtained by shifting the degrees of R by j. The number $\beta_{i,j}(I)$ is the ij-th graded Betti number of I and equals the number of generators of degree j in the i-th syzygy module. The following property of resolutions will be of interest.

Definition 2.1. Suppose I is a homogeneous ideal of R whose generators all have degree d. Then I has a *linear resolution* if for all $i \geq 0$, $\beta_{i,j}(I) = 0$ for all $j \neq i + d$.

Componentwise linearity is closely related to this property. For a homogeneous ideal I, we write (I_d) to denote the ideal generated by all degree d elements of I. Note that (I_d) is different from I_d, which we shall use to denote the vector space of all degree d elements of I. Herzog and Hibi introduced the following definition in [19].

Definition 2.2. A homogeneous ideal I is *componentwise linear* if (I_d) has a linear resolution for all d.

A number of familiar classes of ideals are componentwise linear. For example, all ideals with linear resolutions are componentwise linear. However, there are many nontrivial examples as well, including stable ideals, squarefree strongly stable ideals, and the **a**-stable ideals studied in [15]. The following examples illustrates cases in which our results in this paper give new examples of componentwise linear ideals that are not in any of the classes mentioned above.

EXAMPLE 2.3. Let $R = k[x_1, \dots, x_5]$, and let

$$\begin{aligned} I &= (x_1, x_2, x_3) \cap (x_1, x_4, x_5) \cap (x_2, x_3, x_5) \\ &= (x_1x_2, x_1x_3, x_1x_5, x_2x_4, x_2x_5, x_3x_4, x_3x_5) \subset R. \end{aligned}$$

Then I is clearly not stable since no pure power of x_1 is among the minimal generators, and it is neither squarefree stable nor **a**-stable for any **a** because, for example, x_1x_5 is a minimal generator, but $x_1x_4 \notin I$, though other minimal generators do involve x_4. Our results in Section 4 show that I is componentwise linear; in fact, I has a linear resolution because it is componentwise linear and has all its minimal generators in the same degree.

For an example that is not squarefree, let $J = (x_1, x_2)^3 \cap (x_2, x_3, x_4, x_5)^2 \subset R$. Then

$$\begin{aligned} J = (&x_1x_2^2, x_2^3, x_1^2x_2x_3, x_1^2x_2x_4, x_1^2x_2x_5, x_1^3x_3^2, x_1^3x_3x_4, x_1^3x_3x_5, x_1^3x_4^2, \\ &x_1^3x_4x_5, x_1^3x_5^2), \end{aligned}$$

which is clearly neither stable nor **a**-stable. By our Theorem 3.1 and Corollary 3.2, J is componentwise linear.

The graded Betti numbers of componentwise linear ideals have a particularly good algebraic property. In [1], Aramova, Herzog, and Hibi proved:

THEOREM 2.4. *Let $I \subset k[x_1, \dots, x_n]$ be a homogeneous ideal, and suppose that* $\mathrm{char}(k) = 0$. *Let* $\mathrm{gin}(I)$ *be the generic initial ideal of I with respect to the graded reverse-lex order. Then I is componentwise linear if and only if I and* $\mathrm{gin}(I)$ *have the same graded Betti numbers.*

In general, $\beta_{i,j}(I) \leq \beta_{i,j}(\mathrm{gin}(I))$ for all i and j, but all the inequalities are equalities exactly when I is componentwise linear. Conca observed in [3] that Aramova, Herzog, and Hibi actually proved that I is componentwise linear if and only if I and $\mathrm{gin}(I)$ have the same number of minimal generators. This observation makes the condition even easier to test computationally.

One way to show that an ideal is componentwise linear is to prove that it has linear quotients. We recall Herzog and Hibi's definition from [18] (which is slightly more restrictive than Herzog and Takayama's definition in [23]).

DEFINITION 2.5. Let I be a monomial ideal of R. We say that I has *linear quotients* if for some ordering $u_1, \ldots, u_m$ of the minimal generators of I with $\deg u_1 \leq \deg u_2 \leq \cdots \leq \deg u_m$ and all $i > 1$, $(u_1, \ldots, u_{i-1}) : u_i$ is generated by a subset of $\{x_1, \ldots, x_n\}$.

The following proposition is probably known, but we could not find it recorded explicitly, so we include it for convenience. The case in which I is generated in a single degree is Lemma 4.1 of [4], and that is the case we shall use in this paper.

PROPOSITION 2.6. *If I is a homogeneous ideal with linear quotients, then I is componentwise linear.*

Proof. Suppose that $I \subset R$ has linear quotients with respect to the ordering $u_1, u_2, \ldots, u_m$ of its minimal generators, where $\deg u_{i-1} \leq \deg u_i$ for all i. We induct on m, the number of minimal generators of I. When $m = 1$, $I = (u_1)$ is componentwise linear because it is principal.

Fix some $m > 1$. Assume that the ideal $J = (u_1, \ldots, u_{m-1})$ is componentwise linear, and suppose that $\deg u_m = d$. Let $J' = (J, u_m)$. Note that $J_e = J'_e$ for all $e < d$, so (J'_e) has a linear resolution for all $e < d$. We have a short exact sequence

$$0 \longrightarrow R/(J : u_m)(-d) \xrightarrow{\times u_m} R/J \longrightarrow R/J' \longrightarrow 0.$$

Because $J : u_m$ is generated by linear forms, $\operatorname{reg}(R/(J : u_m)) = 0$. Since $\deg u_m = d$, we have $\operatorname{reg}(R/J') \geq d - 1$. Because R/J is componentwise linear, and $\deg u_{m-1} \leq d$, we know that $\operatorname{reg}(R/J) \leq d - 1$. By [7, Corollary 20.19],

$$\begin{aligned} \operatorname{reg}(R/J') &\leq \max\{\operatorname{reg}(R/(J : u_n)(-d)) - 1, \operatorname{reg}(R/J)\} \\ &= \max\{d - 1, \operatorname{reg}(R/J)\}, \end{aligned}$$

so $\operatorname{reg}(R/J') = d - 1$. Thus (J'_d) has a linear resolution. The same is true for all (J'_e) with $e > d$. The last statement follows from the fact that for any ideal M with regularity d and $e > d$, (M_e) has a linear resolution. This fact follows, for example, from [13, Lemma 2.3] since the graded Betti numbers $\beta_{i,j}(M_e)$ with $j > i + e$ must be zero. ∎

One special type of ideal that has linear quotients is a polymatroidal ideal. For a discussion of this terminology, see [20] and [18].

Definition 2.7. Let I be a monomial ideal generated in a single degree. We say that I is a *polymatroidal ideal* if the minimal generators of I satisfy the following exchange property: If $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ are minimal generators of I, for each i with $a_i > b_i$, there exists j with $a_j < b_j$ such that $x_j u / x_i \in I$.

Herzog and Takayama proved the following result about polymatroidal ideals in Lemma 1.3 of [23].

Theorem 2.8. *Polymatroidal ideals have linear quotients with respect to the descending reverse-lex order, and hence they have linear resolutions.*

We shall use the ascending reverse-lex order at times, so we state the corresponding result for that case, which follows from the proof of [23, Lemma 1.3] in Herzog and Takayama's paper as well as a dual version of the exchange property for monomial ideals in [18, Lemma 2.1].

Proposition 2.9. *Polymatroidal ideals have linear quotients with respect to the ascending reverse-lex order.*

Suppose we have a componentwise linear monomial ideal $I = (m_1, \ldots, m_r)$ in a polynomial ring $R = k[x_1, \ldots, x_n]$. In the following sections, we shall sometimes want to consider the ideal $I = (m_1, \ldots, m_r)$ as an ideal in a larger polynomial ring R'. The following lemma shows that I is still componentwise linear in the larger ring.

Lemma 2.10. *Let $I = (m_1, \ldots, m_r)$ be a componentwise linear monomial ideal in $R = k[x_1, \ldots, x_n]$, and let $I' = (m_1, \ldots, m_r)R'$ be the ideal generated by the same monomials in the larger polynomial ring $R' = k[x_1, \ldots, x_n, x_{n+1}, \ldots, x_p]$. Then I' is a componentwise linear ideal of R'.*

Proof. Suppose d is the lowest degree in which I has generators. Then $(I_d) = (I'_d)$, so (I'_d) has a linear resolution because (I_d) does.

Let $t \geq 0$, and let $\mathfrak{m} = (x_{n+1}, \ldots, x_p)$. The ideal (I'_{d+t}) has a decomposition as

$$(I'_{d+t}) = (I_{d+t}) + \mathfrak{m}(I_{d+t-1}) + \mathfrak{m}^2(I_{d+t-2}) + \cdots + \mathfrak{m}^t(I_d);$$

by (I_{d+u}), we mean the ideal generated by the degree $(d+u)$ elements of I inside R, so the minimal generators involve only the variables $x_1, \ldots, x_n$. We then consider $\mathfrak{m}^v(I_{d+t-v})$ as an ideal of R'.

By hypothesis, (I_{d+t}) has a linear resolution in R, and hence, viewed as an ideal of R', we will have $\operatorname{reg}(R'/(I_{d+t})) = d+t-1$. We order the rest of the minimal generators of (I'_{d+t}) in the following way. First, take all the minimal generators of $\mathfrak{m}(I_{d+t-1})$ in descending graded reverse-lex order (so those monomials divisible by x_p would be last). Next, take all the minimal generators of $\mathfrak{m}^2(I_{d+t-2})$ in descending graded reverse-lex order, and continue in this way. We shall add each of these generators successively to (I_{d+t}) and show that each resulting ideal has regularity $d+t$. This will imply that $\operatorname{reg}(R'/(I'_{d+t})) = d+t-1$ and thus (I'_{d+t}) has a linear resolution.

As the first step, we compute $(I_{d+t}) : x_{n+1}m$, where $m \in I_{d+t-1}$. Multiplying m by any of $x_1, \ldots, x_n$ gives an element divisible by an element of I_{d+t}, and no multiplication by a monomial involving only $x_{n+1}, \ldots, x_p$ can give us an element of (I_{d+t}), so $(I_{d+t}) : x_{n+1}m = (x_1, \ldots, x_n)$. We have a short exact sequence

$$\begin{aligned} 0 \longrightarrow R'/((I_{d+t}) : x_{n+1}m)(-d-t) \xrightarrow{\times x_{n+1}m} R'/(I_{d+t}) \\ \longrightarrow R'/((I_{d+t}), x_{n+1}m) \longrightarrow 0. \end{aligned}$$

By [7, Corollary 20.19] and the fact that $\operatorname{reg}(R'/(x_1, \ldots, x_n)) = 0$, we have

$$\begin{aligned} &\operatorname{reg}(R'/((I_{d+t}), x_{n+1}m)) \\ &\quad \leq \max\{\operatorname{reg}(R'/((I_{d+t}) : x_{n+1}m)(-d-t)) - 1, \operatorname{reg}(R'/(I_{d+t}))\} \\ &\quad = \max\{d+t-1, d+t-1\} = d+t-1. \end{aligned}$$

Since $((I_{d+t}), x_{n+1}m)$ is generated in degree $d+t$, $\operatorname{reg}(R'/((I_{d+t}), x_{n+1}m)) = d+t-1$.

We proceed by induction. Let

$$\begin{aligned} J = (I_{d+t}) + \mathfrak{m}(I_{d+t-1}) + \cdots + \mathfrak{m}^{r-1}(I_{d+t-r+1}) \\ + \text{initial segment of } \mathfrak{m}^r(I_{d+t-r}). \end{aligned}$$

Suppose $m' = x_{n+1}^{b_{n+1}} \cdots x_p^{b_p} m$ is the next monomial in $\mathfrak{m}^r(I_{d+t-r})$ in descending graded reverse-lex order, where $m \in I_{d+t-r}$. First, we will show that $J : m'$ is an ideal generated by a subset of the variables of R'. Multiplying m by any of $x_1, \ldots, x_n$ gives an element of $I_{d+t-r+1}$, and thus $(x_1, \ldots, x_n) \subseteq J : m'$ since $x_{n+1}^{b_{n+1}} \cdots x_p^{b_p} \in \mathfrak{m}^r \subset \mathfrak{m}^{r-1}$. Let l be the maximum index for which $b_l \neq 0$. Then any of $x_{n+1}m', \ldots, x_{l-1}m'$ is in J because $x_{n+1}m'/x_l, \ldots, x_{l-1}m'/x_l$ are all greater than m' in graded reverse-lex order.

Now suppose that $\bar{m}$ is a monomial in only $x_l, \ldots, x_p$. We will show that $\bar{m}m' = \bar{m}x_{n+1}^{b_{n+1}} \cdots x_p^{b_p} m \notin J$. Note that $\bar{m}m' \in (I_{d+t-r})$, the ideal of R' generated by the elements of I_{d+t-r}, but it is not in any (I_{d+t-u}) for any $u < r$. Hence if $\bar{m}m' \in J$, we have $\bar{m}m' \in \mathfrak{m}^r(I_{d+t-r})$. That implies that $\bar{m}m'$ is divisible by some monomial in $\mathfrak{m}^r(I_{d+t-r})$ greater than m' in the reverse-lex order. Because of the way we have ordered the monomials, and since l is the maximum index for which $b_l \neq 0$, this is impossible. Hence

$$J : m' = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{l-1}),$$

an ideal generated by a subset of the variables of R'. We now have an exact sequence

$$0 \longrightarrow R'/(J : m')(-d-t) \xrightarrow{\times m'} R'/J \longrightarrow R'/(J, m') \longrightarrow 0.$$

By [7, Corollary 20.19], induction, and the fact that $\operatorname{reg}(R'/(J : m')) = 0$, we have

$$\begin{aligned} \operatorname{reg}(R'/(J, m')) &\leq \max\{\operatorname{reg}(R'/(J : m')(-d-t)) - 1, \operatorname{reg}(R'/J)\} \\ &= d + t - 1. \end{aligned}$$

Since (J, m') is generated by monomials of degree $d + t$, we have $\operatorname{reg}(R'/(J, m')) = d + t - 1$, or equivalently, $\operatorname{reg}(J, m') = d + t$ as required. □

Remark 2.11. One can shorten the preceding proof considerably by showing that $\operatorname{gin}(I)$ has the same minimal generators as $\operatorname{gin}(I')$, where gin denotes the graded reverse-lex generic initial ideal. However, this approach would require the hypothesis that $\operatorname{char}(k) = 0$ to use the generic initial ideal characterization of componentwise linearity. Instead, we prefer to have a characteristic-free proof.

We begin our investigation of when intersections of Veronese ideals are componentwise linear with a couple of special cases. Let $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$.

We consider the cases in which $s = 1$ and in which the J_i are pairwise disjoint.

When $s = 1$, $I = \mathfrak{m}_J^a$ is a power of a complete intersection. In this case, the Eagon-Northcott complex of I is a minimal free resolution [5]. The graded Betti numbers of I are given below (and could also be computed from the formulas of [17]).

LEMMA 2.12. *Let $J \subseteq [n]$, and let a be any positive integer. Then*

$$\beta_{i,i+a}(\mathfrak{m}_J^a) = \binom{a+|J|-1}{a+i}\binom{a+i-1}{i} \quad \text{for } i = 0,\dots,|J|-1,$$

and $\beta_{i,j}(\mathfrak{m}_J^a) = 0$ for all other $i,j \geq 0$. In particular, $\mathfrak{m}_J^a$ has a linear resolution, and thus is componentwise linear.

We use the above lemma to prove the following result.

THEOREM 2.13. *Let $J_1,\dots,J_s \subseteq [n]$ be s pairwise disjoint nonempty subsets, and let $a_1,\dots,a_s$ be positive integers. Set $I = \mathfrak{m}_{J_1}^{a_1}\mathfrak{m}_{J_2}^{a_2}\cdots\mathfrak{m}_{J_s}^{a_s}$ and $a = a_1+\cdots+a_s$. Then*

$$\beta_{i,i+a}(I) = \sum_{i_1+\cdots+i_s=i}\prod_{j=1}^{s}\binom{a_j+|J_j|-1}{a_j+i_j}\binom{a_j+i_j-1}{i_j} \quad \text{for all } i \geq 0,$$

and $\beta_{i,j}(I) = 0$ otherwise.

Proof. Let $\mathbf{F}_\ell$ denote the graded minimal free resolution of $\mathfrak{m}_{J_\ell}^{a_\ell}$ for $\ell = 1,\dots,s$. Since $I \cong \mathfrak{m}_{J_1}^{a_1}\otimes\cdots\otimes\mathfrak{m}_{J_s}^{a_s}$, the graded minimal free resolution of I is given by $\mathbf{G} = \mathbf{F}_1\otimes\cdots\otimes\mathbf{F}_s$. So G_i, the i-th graded free module in a minimal graded free resolution of I, is $G_i = \bigoplus_{i_1+\cdots+i_s=i} F_{i_1}\otimes\cdots\otimes F_{i_s}$. Thus

$$\begin{aligned}\beta_{i,j}(I) &= \sum_{i_1+\cdots+i_s=i}\dim_k(F_{i_1}\otimes\cdots\otimes F_{i_s})_j\\ &= \sum_{i_1+\cdots+i_s=i}\ \sum_{j_1+\cdots+j_s=j}\beta_{i_1,j_1}(\mathfrak{m}_{J_1}^{a_1})\cdots\beta_{i_s,j_s}(\mathfrak{m}_{J_s}^{a_s}).\end{aligned}$$

But by Lemma 2.12, $\beta_{i_\ell,j_\ell}(\mathfrak{m}_{J_\ell}^{a_\ell}) \neq 0$ only if $j_\ell = i_\ell + a_\ell$. So $\beta_{i,j}(I) = 0$ if $j \neq i+a$, and

$$\beta_{i,i+a}(I) = \sum_{i_1+\cdots+i_s=i}\beta_{i_1,i_1+a_1}(\mathfrak{m}_{J_1}^{a_1})\cdots\beta_{i_s,i_s+a_s}(\mathfrak{m}_{J_s}^{a_s}).$$

By applying the formula of Lemma 2.12 we get the desired conclusion. ∎

When the J_i's are pairwise disjoint nonempty sets as in the above theorem, then $\mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s} = \mathfrak{m}_{J_1}^{a_1}\mathfrak{m}_{J_2}^{a_2}\cdots\mathfrak{m}_{J_s}^{a_s}$. Since this ideal has a linear resolution, we have:

COROLLARY 2.14. *If* $I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$, *with the* $J_1, \ldots, J_s \subseteq [n]$ *pairwise disjoint nonempty subsets, then* I *is componentwise linear.*

Remark 2.15. It is easy to see that $(x_1, \ldots, x_r)^a$ is polymatroidal for any positive integers r and a. Results in [20] and [4] prove that the product of polymatroidal ideals is polymatroidal. Hence the ideals of Theorem 2.13 are polymatroidal and thus have a linear resolution, as is clear from the graded Betti numbers.

EXAMPLE 2.16. We show that if $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$ with $s \geq 4$, then I may or may not be componentwise linear. First, we construct examples of ideals that are not componentwise linear. We begin with the case that $s = 4$. It was observed in [13] that the ideal

$$I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_1) = (x_1x_3, x_2x_4)$$

is not componentwise linear. To see this fact, note that the ideal I is a complete intersection ideal of type $(2, 2)$. Since $I = (I_2)$, (I_2) does not have a linear resolution.

We can extend this example to any $s > 4$. In the polynomial ring $R = k[x_1, \ldots, x_s]$, let

$$I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_1) \cap (x_5)^{a_5} \cap (x_6)^{a_6} \cap \cdots \cap (x_s)^{a_s}.$$

for any positive integers $a_5, \ldots, a_s$. Then $I = x_5^{a_5}x_6^{a_6}\cdots x_s^{a_s}I'$ where $I' = (x_1x_3, x_2x_4)$. Because $\beta_{i,j}(I) = \beta_{i,j-a_5-a_6-\cdots-a_s}(I')$, the ideal I cannot be componentwise linear since $I = (I_{2+a_5+\cdots+a_s})$ does not have a linear resolution.

On the other hand, we can create very simple intersections of Veronese ideals that are componentwise linear for any s. For example, if $J_i = \{i\}$ for $i = 1, \ldots, s$, then I is principal and hence has a linear resolution. Alternatively, start with a componentwise linear intersection of Veronese ideals I in the variables $x_1, \ldots, x_r$, and intersect I with $(x_{r+1})^{a_{r+1}} \cap \cdots \cap (x_s)^{a_s}$.

In the following sections, we consider the cases in which $s = 2$ or $s = 3$ as well as some special cases for general s.

§3. A family of polymatroidal ideals

In this section, we consider a particular family of intersections of Veronese ideals. We show that ideals in this family are polymatroidal. Our main result is the following theorem.

THEOREM 3.1. *Let $J_1, \ldots, J_s$ be subsets of $[n]$ such that $J_i \cup J_j = [n]$ for all $i \neq j$. Let*

$$I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s} \subset R = k[x_1, \ldots, x_n].$$

Then (I_d) is polymatroidal for all d, and hence I is componentwise linear.

Proof. The condition on $J_i \cup J_j = [n]$ means that any $r \in [n]$ is missing from at most one of the J_i; if $r \notin J_i$ and $r \notin J_j$, then $J_i \cup J_j \neq [n]$. Therefore we may partition the variables $x_1, \ldots, x_n$ in the following way: Rename the variables x_i with the symbols $x_{1,1}, \ldots, x_{1,b_1}, \ldots, x_{s,1}, \ldots, x_{s,b_s}$, $x_{\cap,1}, \ldots, x_{\cap,b_\cap}$. The variables $x_{i,j}$ correspond to the integers in $[n]$ *missing* from J_i, and the variables $x_{\cap,j}$ correspond to the integers in $[n]$ present in all J_i.

For example, if

$$I = (x_1, x_2, x_4, x_6)^3 \cap (x_1, x_3, x_5, x_6)^4 \cap (x_2, x_3, x_4, x_5, x_6)^2 \subset k[x_1, \ldots, x_6],$$

then $J_1 = \{1, 2, 4, 6\}$, $J_2 = \{1, 3, 5, 6\}$, and $J_3 = \{2, 3, 4, 5, 6\}$. We would rename the variables x_3 and x_5 as $x_{1,1}$ and $x_{1,2}$ since 3 and 5 are missing from J_1. Similarly, x_2 and x_4 become $x_{2,1}$ and $x_{2,2}$, x_1 is $x_{3,1}$, and x_6 is $x_{\cap,1}$. Note that there may be some i with $1 \leq i \leq s$ for which there are no $x_{i,j}$ variables; that is true if and only if $J_i = [n]$. That causes no problem in the proof below; alternatively, one can avoid this case since a component of $(x_1, \ldots, x_n)^a$ simply makes the ideal formed by the intersection of the other components zero in degrees below a and the same in degrees a and above.

Fix a degree d. Suppose that $m_e \neq m_f$ are two monomials in (I_d) with

$$m_e = x_{1,1}^{e_{1,1}} \cdots x_{1,b_1}^{e_{1,b_1}} \cdots x_{s,1}^{e_{s,1}} \cdots x_{s,b_s}^{e_{s,b_s}} x_{\cap,1}^{e_{\cap,1}} \cdots x_{\cap,b_\cap}^{e_{\cap,b_\cap}},$$

with m_f having a similar expression in terms of $x_{i,j}^{f_{i,j}}$. We need to show that the polymatroidal exchange condition holds for these two monomials. Namely, if some $e_{i,j} > f_{i,j}$ or some $e_{\cap,j} > f_{\cap,j}$, we must show that there exists $e_{u,v} < f_{u,v}$ (with $1 \leq u \leq s$ or $u = \cap$) such that $x_{u,v} m_e / x_{i,j} \in (I_d)$.

Note that the fact that $m_e \in (I_d)$ means exactly that $\deg m_e = d$ and all of the following inequalities hold:

$$
\begin{aligned}
\sum_j e_{2,j} + \cdots + \sum_j e_{s,j} + \sum_j e_{\cap,j} &\geq a_1, \\
\sum_j e_{1,j} + \sum_j e_{3,j} + \cdots + \sum_j e_{s,j} + \sum_j e_{\cap,j} &\geq a_2, \\
&\vdots \\
\sum_j e_{1,j} + \cdots + \sum_j e_{s-1,j} + \sum_j e_{\cap,j} &\geq a_s.
\end{aligned}
$$

There are two main cases to consider. First, suppose that some $e_{\cap,p} > f_{\cap,p}$. If there exists $e_{\cap,j} < f_{\cap,j}$, then $x_{\cap,j} m_e / x_{\cap,p} \in (I_d)$ since none of the left-hand sides of the inequalities above change, and we are done. Otherwise, we have $e_{\cap,j} \geq f_{\cap,j}$ for all j, and $\sum e_{\cap,j} > \sum f_{\cap,j}$ because $e_{\cap,p} > f_{\cap,p}$. Since m_e and m_f have the same degree, there exists some $e_{i,j} < f_{i,j}$ with $1 \leq i \leq s$. Without loss of generality, assume that $e_{1,1} < f_{1,1}$. If

$$
\sum_j e_{2,j} + \cdots + \sum_j e_{s,j} + \sum_j e_{\cap,j} > a_1,
$$

then $x_{1,1} m_e / x_{\cap,p} \in (I_d)$, for the all the left-hand sides of the inequalities but the first stay the same, and the first inequality for the new monomial is

$$
\sum_j e_{2,j} + \cdots + \sum_j e_{s,j} + \sum_j e_{\cap,j} \geq a_1.
$$

(Note that this property is independent of whether we use $x_{1,1}$ or some other $x_{1,v}$ with $e_{1,v} < f_{1,v}$.)

If $\sum e_{2,j} + \cdots + \sum e_{s,j} + \sum e_{\cap,j} \not> a_1$, then

$$
\sum_j e_{2,j} + \cdots + \sum_j e_{s,j} + \sum_j e_{\cap,j} = a_1 \leq \sum_j f_{2,j} + \cdots + \sum_j f_{s,j} + \sum_j f_{\cap,j} \tag{3.1}
$$

since $m_f \in I$. If $e_{2,j} \geq f_{2,j}, \ldots, e_{s,j} \geq f_{s,j}$ for all j, since $\sum e_{\cap,j} > \sum f_{\cap,j}$, we have contradicted (3.1). Therefore, without loss of generality, we may assume that some $e_{2,j} < f_{2,j}$.

We proceed by induction. Suppose that we have either found an $e_{i,j} < f_{i,j}$ such that $x_{i,j} m_e / x_{\cap,p} \in (I_d)$ for some $i \leq t-1$, or for $r = 1, \ldots, t-1$, we have

$$\sum_{\substack{i=1\\i\neq r}}^{s}\sum_{j=1}^{b_i} e_{i,j} + \sum_{j=1}^{b_\cap} e_{\cap,j} = a_r \leq \sum_{\substack{i=1\\i\neq r}}^{s}\sum_{j=1}^{b_i} f_{i,j} + \sum_{j=1}^{b_\cap} f_{\cap,j}. \tag{3.2}$$

(That is, the double sum is the sum of all $e_{i,j}$ with $i \neq r$.)

As part of the induction hypothesis, we may assume that there exists $e_{t,j} < f_{t,j}$. If $x_{t,j}m_e/x_{\cap,p} \in (I_d)$, we are done; otherwise,

$$\sum_{\substack{i=1\\i\neq t}}^{s}\sum_{j=1}^{b_i} e_{i,j} + \sum_{j=1}^{b_\cap} e_{\cap,j} = a_t \leq \sum_{\substack{i=1\\i\neq t}}^{s}\sum_{j=1}^{b_i} f_{i,j} + \sum_{j=1}^{b_\cap} f_{\cap,j}. \tag{3.3}$$

Summing (3.3) and the inequalities (3.2) for all $r = 1, \ldots, t-1$, we obtain

$$\begin{aligned}(t-1)\sum_{i=1}^{t}\sum_{j} e_{i,j} + t\sum_{i=t+1}^{s}\sum_{j} e_{i,j} + t\sum_{j} e_{\cap,j} \\ \leq (t-1)\sum_{i=1}^{t}\sum_{j} f_{i,j} + t\sum_{i=t+1}^{s}\sum_{j} f_{i,j} + t\sum_{j} f_{\cap,j}.\end{aligned} \tag{3.4}$$

Subtracting $(t-1)\deg m_e = (t-1)\deg m_f$, we are left with

$$\sum_{i=t+1}^{s}\sum_{j} e_{i,j} + \sum_{j} e_{\cap,j} \leq \sum_{i=t+1}^{s}\sum_{j} f_{i,j} + \sum_{j} f_{\cap,j}.$$

If $e_{t+1,j} \geq f_{t+1,j}, \ldots, e_{s,j} \geq f_{s,j}$ for all j, then we have a contradiction since $\sum e_{\cap,j} > \sum f_{\cap,j}$. Hence we may assume without loss of generality that some $e_{t+1,j} < f_{t+1,j}$.

Therefore either we find some $e_{i,j} < f_{i,j}$, with $1 \leq i \leq s$, such that $x_{i,j}m_e/x_{\cap,p} \in (I_d)$, or else the exchange property is not true, and (3.2) holds for all $r = 1, \ldots, s$. In the latter case, summing all s inequalities of the form in (3.2), we have

$$(s-1)\sum_{i=1}^{s}\sum_{j} e_{i,j} + s\sum_{j} e_{\cap,j} \leq (s-1)\sum_{i=1}^{s}\sum_{j} f_{i,j} + s\sum_{j} f_{\cap,j}.$$

If we subtract $(s-1)\deg m_e = (s-1)\deg m_f$ from both sides, we have

$$\sum_{j} e_{\cap,j} \leq \sum_{j} f_{\cap,j}.$$

But this contradicts our assumption that

$$\sum_j e_{\cap,j} > \sum_j f_{\cap,j}.$$

Hence there exists some $e_{i,j} < f_{i,j}$ such that $x_{i,j}m_e/x_{\cap,p} \in (I_d)$, and the exchange condition holds.

The second case to consider is when some $e_{i,j} > f_{i,j}$ for some $1 \leq i \leq s$. Without loss of generality, assume that $e_{1,1} > f_{1,1}$. If there exists $e_{1,j} < f_{1,j}$, then $x_{1,j}m_e/x_{1,1} \in (I_d)$. Otherwise, we have $e_{1,j} \geq f_{1,j}$ for all j, and $\sum e_{1,j} > \sum f_{1,j}$. Additionally, note that if any $e_{\cap,j} < f_{\cap,j}$, then $m' = x_{\cap,j}m_e/x_{1,1} \in (I_d)$ since the degrees of m_e and m' in the $J_2, \dots, J_s$ variables are the same, and the degree of m' in the J_1 variables is one higher than that of m_e. Therefore we may also assume that $e_{\cap,j} \geq f_{\cap,j}$ for all j.

Since $\deg m_e = \deg m_f$, there exists some $e_{i,j} < f_{i,j}$, and we may assume that $e_{2,1} < f_{2,1}$. If $x_{2,1}m_e/x_{1,1} \in (I_d)$, we are done; otherwise,

$$\begin{aligned}\sum_j e_{1,j} + \sum_j e_{3,j} + \cdots + \sum_j e_{s,j} + \sum_j e_{\cap,j}\\ = a_2 \leq \sum_j f_{1,j} + \sum_j f_{3,j} + \cdots + \sum_j f_{s,j} + \sum_j f_{\cap,j}.\end{aligned}$$

If $e_{3,j} \geq f_{3,j}, \dots, e_{s,j} \geq f_{s,j}$ for all j, then we have a contradiction since $\sum e_{1,j} > \sum f_{1,j}$ and $\sum e_{\cap,j} \geq \sum f_{\cap,j}$. Therefore we may assume without loss of generality that $e_{3,1} < f_{3,1}$.

Continuing in this way, we apply an almost identical induction argument as in the previous case except that now $\sum e_{1,j} > \sum f_{1,j}$ and $\sum e_{\cap,j} \geq \sum f_{\cap,j}$. Either we find some $e_{i,j} < f_{i,j}$ with $2 \leq i \leq s-1$ such that $x_{i,j}m_e/x_{1,1} \in (I_d)$, or (3.2) holds for all $r = 2, \dots, s-1$, and there exists some $e_{s,j} < f_{s,j}$. If $x_{s,j}m_e/x_{1,1} \in (I_d)$, we are done. Otherwise, (3.2) also holds for $r = s$, and the exchange property fails. Summing the inequalities of (3.2) for all $r = 2, \dots, s$, we obtain

$$\begin{aligned}(s-1)\sum_j e_{1,j} + (s-2)\sum_{i=2}^{s}\sum_j e_{i,j} + (s-1)\sum_j e_{\cap,j}\\ \leq (s-1)\sum_j f_{1,j} + (s-2)\sum_{i=2}^{s}\sum_j f_{i,j} + (s-1)\sum_j f_{\cap,j}.\end{aligned}$$

Now, subtracting $(s-2)\deg m_e = (s-2)\deg m_f$, we have

$$\sum_j e_{1,j} + \sum_j e_{\cap,j} \leq \sum_j f_{1,j} + \sum_j f_{\cap,j}.$$

But we are assuming that $\sum e_{1,j} > \sum f_{1,j}$, and $\sum e_{\cap,j} \geq \sum f_{\cap,j}$, so we have a contradiction, and thus the exchange property must hold. ∎

As a consequence of Theorem 3.1, the intersection of any two Veronese ideals must be componentwise linear.

Corollary 3.2. *Let $J, K \subseteq [n]$ and let a and b be positive integers. Then $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b \subset R = k[x_1, \ldots, x_n]$ is componentwise linear.*

Proof. If $J \cup K = [n]$, then we are done by Theorem 3.1. If not, we may reindex the variables so that $J \cup K = [m]$, and $[m] \cup \{m+1, \ldots, n\} = [n]$. Then, by Theorem 3.1, I is componentwise linear in $k[x_1, \ldots, x_m]$ and so Lemma 2.10 gives the result. ∎

Remark 3.3. It is not true that all ideals $I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2}$ have (I_d) polymatroidal for all d. For example, let

$$I = \mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} = (x_1, x_2, x_3)^2 \cap (x_2, x_3, x_5)^2 \subset R = k[x_1, \ldots, x_5].$$

Note that x_4 does not appear in either of the two components. Both $m_e = x_3^2 x_4$ and $m_f = x_1 x_3 x_5$ are in (I_3). The power on x_3 is greater in m_e than it is in m_f, and the powers on x_1 and x_5 are larger in m_f than in m_e. But $x_1(x_3^2 x_4/x_3) = x_1 x_3 x_4 \notin (I_3)$, and $x_5(x_3^2 x_4/x_3) = x_3 x_4 x_5 \notin (I_3)$. Therefore (I_3) is not polymatroidal. The proof of Theorem 3.1 breaks down because 4 is missing from both J_1 and J_2, and hence we cannot partition the variables the way we did in that argument; the x_4 exponents would be double-counted, causing problems when we subtract a multiple of $\deg m_e = \deg m_f$.

§4. The intersection of three Veronese ideals

We will show that intersection of any three Veronese ideals is always componentwise linear. Throughout this section, we write $\mathcal{G}(I)$ to denote the set of minimal generators of a monomial ideal I.

We begin with an observation. Suppose $J, K \subseteq [n]$, but $J \cup K \subsetneq [n]$. Let $H = [n] \backslash (J \cup K)$, and after relabeling, we can assume $H = \{r+1, \ldots, n\}$

and $J \cup K = [r]$. Let a, b be any positive integers, and let α be the smallest degree of a nonzero element in $(\mathfrak{m}_J^a \cap \mathfrak{m}_K^b)$. If

$$B' = (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b) \cap k[x_i \mid i \in J \cup K] = (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b) \cap k[x_1, \dots, x_r],$$

then the ideal $B = ((\mathfrak{m}_J^a \cap \mathfrak{m}_K^b)_{\alpha+d})$, as an ideal of $R = k[x_1, \dots, x_n]$, has the decomposition

$$B = (B'_{\alpha+d}) + (B'_{\alpha+d-1})\mathfrak{m}_H + (B'_{\alpha+d-2})\mathfrak{m}_H^2 + \cdots + (B'_{\alpha})\mathfrak{m}_H^d,$$

where (B'_i) denotes the ideal generated by elements of degree i of B' but viewed as an ideal of R.

Order the elements of $\mathcal{G}(B)$ as follows: Order the generators of $(B'_{\alpha+d})$ with respect to the ascending reverse-lex ordering. Then add the generators of $(B'_{\alpha+d-1})\mathfrak{m}_H$ in ascending reverse-lex order. We thus add all the monomials divisible by x_n first. Continue by adding the generators $(B'_{\alpha+d+2})\mathfrak{m}_H^2$ in ascending reverse-lex order, and so on.

Lemma 4.1. *Using the above notation, the ideal*

$$B = (B'_{\alpha+d}) + (B'_{\alpha+d-1})\mathfrak{m}_H + (B'_{\alpha+d-2})\mathfrak{m}_H^2 + \cdots + (B'_{\alpha})\mathfrak{m}_H^d$$

has linear quotients with respect to the order prescribed above.

Proof. Let M_i be the i-th element of $\mathcal{G}(B)$, $i \geq 2$, with respect to our ordering. First, suppose that $M_i \in (B'_{\alpha+d})$. We wish to calculate $(M_1, \dots, M_{i-1}) : M_i$, where $(M_1, \dots, M_{i-1})$ is the ideal generated by all monomials in $\mathcal{G}(B)$ smaller than M_i with respect to our ordering. Note that in this case, each M_j is in $(B'_{\alpha+d})$, so each M_j is in $S = k[x_1, \dots, x_r]$. As an ideal of S, the ideal $(B'_{\alpha+d})$ is polymatroidal by Theorem 3.1. So, $(B'_{\alpha+d})$ has linear quotients with respect to the ascending reverse-lex order by Proposition 2.9. Thus, as an ideal of S, $(M_1, \dots, M_{i-1}) : M_i = (x_{i_1}, \dots, x_{i_j})$ for some subset $\{i_1, \dots, i_j\} \subseteq J \cup K$. Because $S \to R$ is a flat ring homomorphism, by [24, Theorem 7.4(iii)], $(M_1, \dots, M_{i-1})R : M_i R = (x_{i_1}, \dots, x_{i_j})R$. (We note that if I is an ideal of S, then we will sometimes abuse notation and write I to denote an ideal of R, where we really mean IR using the flat homomorphism $S \to R$.)

So, suppose now that $M_i \in (B'_{\alpha+d-s})\mathfrak{m}_H^s$ for some $s = 1, \dots, d$. Let

$$I = (B'_{\alpha+d}) + \cdots + (B'_{\alpha+d-s+1})\mathfrak{m}_H^{s-1} + (M \in \mathcal{G}((B'_{\alpha+d-s})\mathfrak{m}_H^s) \mid M < M_i),$$

where $M < M_i$ with respect to the reverse-lex ordering.

Since $M_i \in (B'_{\alpha+d-s})\mathfrak{m}_H^s$, $M_i = M_{i1}M_{i2}$ with $M_{i1} \in (B'_{\alpha+d-s})$ and $M_{i2} \in \mathfrak{m}_H^s$. If we multiply M_i by any x_l with $l \in J \cup K$, then $x_l M_{i1} \in (B'_{\alpha+d-s+1})$. Since $M_{i2} \in \mathfrak{m}_H^s \subseteq \mathfrak{m}_H^{s-1}$, we have $x_l M_i \in (B'_{\alpha+d-s+1})\mathfrak{m}_H^{s-1} \subseteq I$. Thus $\mathfrak{m}_{J\cup K} \subseteq I : M_i$.

Because $M_{i2} \in \mathfrak{m}_H^s$, we have

$$M_{i2} = x_{r+1}^{c_{r+1}} \cdots x_n^{c_n} \text{ with } c_{r+1} + \cdots + c_n = s.$$

Let l be the smallest integer in $\{r+1, \ldots, n\}$ such that $c_l > 0$. Then $x_e M_i \in I$ for $e = l+1, \ldots, n$. To see this note that

$$\begin{aligned} x_e M_i = M_{i1} x_e M_{i2} &= M_{i1} x_l^{c_l} \cdots x_e^{c_e+1} \cdots x_n^{c_n} \\ &= x_l M_{i1} x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n}. \end{aligned}$$

But then $M_{i1} x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n} < M_i$ since $c_e + 1 > c_e$. So

$$\begin{aligned} x_e M_i &= x_l M_{i1} x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n} \\ &\in (M \in \mathcal{G}((B'_{\alpha+d-s})\mathfrak{m}_H^s) \mid M < M_i) \subseteq I, \end{aligned}$$

from which we deduce that $(x_{l+1}, \ldots, x_n) \subseteq I : M_i$.

Let m now be any monomial not in $\mathfrak{m}_{J\cup K} + (x_{l+1}, \ldots, x_n)$ and suppose $mM_i \in I$. The monomial m can only be divisible by the variables $x_{r+1}, \ldots, x_l$; suppose $\deg m = z$. Since $mM_i \in I$, there exists a monomial $M_j \in I$ such that $mM_i = m'M_j$ for some monomial m'. Since $M_i \in \mathfrak{m}_H^s$, $mM_i \in \mathfrak{m}_H^{s+z}$. If $M_j \in (B'_{\alpha+d-i})\mathfrak{m}_H^i$ for some $i < s$, then $m'M_j$ cannot be in $\mathfrak{m}_H^{s+z}$ since in $m'M_j$, the exponents of $x_{r+1}, \ldots, x_n$ can add up to at most $i+z$. Thus, we must have $M_j \in (B_{\alpha+d-s})\mathfrak{m}_H^s$, and so $M_j < M_i$ with respect to the reverse-lex ordering.

If we write $M_j = M_{j1}M_{j2}$ with $M_{j1} \in (B_{\alpha+d-s})$ and $M_{j2} \in \mathfrak{m}_H^s$, then since m is a monomial in the variables $x_{r+1}, \ldots, x_l$ only, we must have $M_{j1} = M_{i1}$ and $M_{j2} < M_{i2}$. If $M_{j2} = x_{r+1}^{f_{r+1}} \cdots x_n^{f_n}$, there must be some e such that $f_e > c_e$ but $f_{e+1} = c_{e+1}, \ldots, f_n = c_n$. Furthermore, since $M_{i2} = x_l^{c_l} \cdots x_n^{c_n}$, we must have $l < e \leq n$. Indeed, if $e \leq l$, then

$$s = f_{r+1} + \cdots + f_n \geq f_e + \cdots f_n > c_e + \cdots + c_n = c_l + \cdots + c_n = s.$$

Thus, for $mM_i = m'M_j$ to be true, both sides must be divisible by $x_e^{f_e}$. But since M_i is not divisible by x_e^v with $v > c_e$, we must have m divisible by x_e.

But this contradicts the fact that m is not in the ideal $\mathfrak{m}_{J\cup K}+(x_{l+1},\dots,x_n)$. We then arrive at the conclusion

$$I : M_i = \mathfrak{m}_{J\cup K} + (x_{l+1},\dots,x_n).$$

So, B has linear quotients. □

Remark 4.2. Lemma 4.1 gives a second proof that $\mathfrak{m}_J^a \cap \mathfrak{m}_K^b$ is componentwise linear for all J, K, a, and b.

We thank the referee for suggestions that significantly simplified the proof of the following theorem.

THEOREM 4.3. *Let $J_1, J_2, J_3 \subseteq [n]$ be three sets, and let a_1, a_2, a_3 be three positive integers. Then $\mathfrak{m}_{J_1}^{a_1} \cap \mathfrak{m}_{J_2}^{a_2} \cap \mathfrak{m}_{J_3}^{a_3}$ is componentwise linear.*

Proof. If $J_i = J_j$ for some i and j, then $\mathfrak{m}_{J_i}^{a_i} \cap \mathfrak{m}_{J_j}^{a_j} = \mathfrak{m}_{J_i}^{\max\{a_i,a_j\}}$, and thus we are in the case of Corollary 3.2. So, we may assume that all the J_i's are distinct. Next we may assume by Lemma 2.10 that $J_1 \cup J_2 \cup J_3 = [n]$. If the hypotheses of Theorem 3.1 are satisfied, we done. So we may further assume that there exists a pair of sets J_i and J_j such that $J_i \cup J_j \subsetneq [n]$.

For ease of exposition, we shall use J, K, L for J_1, J_2, J_3, we shall use a, b, c for a_1, a_2, a_3, and we shall assume that $J \cup K \subsetneq [n]$ and that J, K, and L are distinct. After relabeling, we can also assume that $L = \{t, t+1, \dots, n\}$. We also set $H_1 = L \cap (J \cup K)$ and $H_2 = L \backslash (J \cup K)$. After relabeling again, we may further assume that $H_1 = \{t, \dots, r\}$ and $H_2 = \{r+1, \dots, n\}$.

Let α be the smallest degree of a nonzero element in $\mathfrak{m}_J^a \cap \mathfrak{m}_K^b$. Because $(\mathfrak{m}_J^a \cap \mathfrak{m}_K^b \cap \mathfrak{m}_L^c)_e \subseteq (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b)_e$ for all e, $(\mathfrak{m}_J^a \cap \mathfrak{m}_K^b \cap \mathfrak{m}_L^c)_e = (0)$ if $e < \alpha$, and thus has a linear resolution.

Now fix a $d \geq 0$, and set $A = ((\mathfrak{m}_J^a \cap \mathfrak{m}_K^b \cap \mathfrak{m}_L^c)_{\alpha+d})$ and $B = ((\mathfrak{m}_J^a \cap \mathfrak{m}_K^b)_{\alpha+d})$. We shall show that A has linear quotients, and hence A has a linear resolution. It will then follow that $\mathfrak{m}_J^a \cap \mathfrak{m}_K^b \cap \mathfrak{m}_L^c$ is componentwise linear.

Set

$$B' = (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b) \cap k[x_i \mid i \in J \cup K] = (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b) \cap k[x_1,\dots,x_r].$$

Note that the ideal B' has the same generators as $\mathfrak{m}_J^a \cap \mathfrak{m}_K^b$, but we are now considering B' as an ideal in a smaller ring. The ideal B then has the following decomposition:

$$B = (B'_{\alpha+d}) + (B'_{\alpha+d-1})\mathfrak{m}_{H_2} + (B'_{\alpha+d-2})\mathfrak{m}_{H_2}^2 + \cdots + (B'_{\alpha})\mathfrak{m}_{H_2}^d$$

where by (B'_i) we mean the ideal generated by the degree i part of B' in $k[x_1, \ldots, x_r]$, but considering the ideal as an ideal of R.

Since $A \subseteq B$, each generator of A belongs to some $(B'_{\alpha+d-i})\mathfrak{m}^i_{H_2}$ for some $i = 0, \ldots, d$. Set

$$A_i = \{M \mid M \in \mathcal{G}(A) \text{ and } M \in (B'_{\alpha+d-i})\mathfrak{m}^i_{H_2}\} \quad \text{for each } i = 0, \ldots, d.$$

So $\mathcal{G}(A) = A_0 \cup A_1 \cup \cdots \cup A_d$.

Order the elements of $\mathcal{G}(A)$ as follows: Begin by adding the elements of A_0 in ascending reverse-lex order. Then, add the elements of A_1, after all the elements in A_0, in ascending reverse-lex order. We then add the elements of A_2 in ascending reverse-lex order, and so on. The ordering could also be described as follows: Write out the generators of B in the same order as in Lemma 4.1. Then simply remove any element of $\mathcal{G}(B)$ that is not in $\mathcal{G}(A)$. We will show that A has linear quotients with respect to this ordering.

Let M_i be the i-th monomial of $\mathcal{G}(A)$ with respect to our ordering with $i \geq 2$. Set $I = (M_1, \ldots, M_{i-1})$, the ideal generated by all the monomials in $\mathcal{G}(A)$ smaller than M_i with respect our ordering. Furthermore, let $D = (M \in \mathcal{G}(B) \mid M < M_i)$ with respect to our ordering. Since $I \subseteq D$, by Lemma 4.1 we have

$$I : M_i \subseteq D : M_i = (x_{i_1}, \ldots, x_{i_j})$$

because B has linear quotients with respect to this order.

If $M_i \in A_0$, then we will show that $I : M_i = D : M_i$. Take any $x_e \in \{x_{i_1}, \ldots, x_{i_j}\}$. Then $x_e M_i = x_f M_j$ for some $M_j \in D$. We wish to show that M_j is in I. Suppose $M_j \notin I$, i.e., $M_j \notin A$. Then

$$M_j = x_1^{d_1} \cdots x_t^{d_t} \cdots x_n^{d_n} \text{ with } d_t + \cdots + d_n < c.$$

On the other hand, $M_i \in A$, so

$$M_i = x_1^{c_1} \cdots x_t^{c_t} \cdots x_n^{c_n} \text{ with } c_t + \cdots + c_n \geq c.$$

Since $M_i \in A_0 \subseteq (B'_{\alpha+d})$, we have $M_j \in (B'_{\alpha+d})$ as well, and $M_j < M_i$ with respect to reverse-lex order. Hence there must exist some l such that $d_l > c_l$, but $d_{l+1} = c_{l+1}, \ldots, d_n = c_n$. Moreover, $l \in \{t, \ldots, n\}$, for if $l < t$, then

$$c > d_t + \cdots + d_n = c_t + \cdots + c_n \geq c.$$

Thus, for $x_eM_i = x_fM_j$ to be true, $x_e = x_l$, since the exponent of x_l is higher in M_j than in M_i. So the exponents of $x_t, \dots, x_n$ in x_eM_i must add up at least $c+1$. However, the exponents of $x_t, \dots, x_n$ in x_fM_j can add up to at most c, a contradiction. So M_j must be in A and thus is in I.

Suppose now that $M_i \in A_s$ with $s \in \{1, \dots, d\}$. So $M_i \in (B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$. Write M_i as $M_i = M_{i1}M_{i2}$ with $M_{i1} \in (B'_{\alpha+d-s})$ and $M_{i2} \in \mathfrak{m}_{H_2}^s$. Then

$$M_{i2} = x_{r+1}^{c_{r+1}} \cdots x_n^{c_n} \text{ with } c_{r+1} + \cdots + c_n = s.$$

Let l be the smallest integer in the set $\{r+1, \dots, n\}$ such that $c_l > 0$. As shown in the proof of Lemma 4.1,

$$D : M_i = \mathfrak{m}_{J \cup K} + (x_{l+1}, \dots, x_n).$$

Since $I : M_i \subseteq D : M_i$, the above fact implies that no monomial of the form $x_{r+1}^{c_{r+1}} \cdots x_l^{c_l}$ can belong to $I : M_i$.

Next, we show that $(x_{l+1}, \dots, x_n) \subseteq I : M_i$. Let $x_e \in \{x_{l+1}, \dots, x_n\}$. Then

$$x_eM_i = M_{i1}x_l^{c_l} \cdots x_e^{c_e+1} \cdots x_n^{c_n} = x_lM_{i1}x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n},$$

and $M_{i1}x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n} \in (B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$. Also, it is clear that

$$M_{i1}x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n} \in A_s$$

since the exponents of $x_t, \dots, x_n$ still add up to at least c. Now

$$M_j = M_{i1}x_l^{c_l-1} \cdots x_e^{c_e+1} \cdots x_n^{c_n} < M_i$$

with respect to the reverse-lex order, so $M_j \in I$, and hence $x_e \in I : M_i$.

Now suppose that $x_e \in \{x_1, \dots, x_r\} = \{x_i \mid i \in J \cup K\}$. Since $x_e \in D : M_i$, we have that x_eM_i is divisible by some monomial $M \in D$ with M less than M_i with respect to our ordering. The monomial M may or may not be in I. We thus partition $J \cup K$ into the following two sets:

$$P_1 = \{e \in J \cup K \mid x_eM_i \text{ is divisible by some } M \in D \text{ with } M \in I\}$$
$$P_2 = \{e \in J \cup K \mid \text{for every } M \in D \text{ such that } M|x_eM_i,\ M \notin I\}.$$

It follows immediately that if $e \in P_1$, then $x_e \in I : M_i$, so $\mathfrak{m}_{P_1} \subseteq I : M_i$.

We will now show (through many steps) that if m is any monomial in the variables $\{x_e \mid e \in P_2\}$, then $m \notin I : M_i$. It then follows that

$$I : M_i = \mathfrak{m}_{P_1} + (x_{l+1}, \dots, x_n)$$

so that I has linear quotients.

Suppose that $e \in P_2$. Then $x_e M_i = x_f M_j$ for some $M_j \in D$, and also $M_j \notin I$. Furthermore, let

$$M_i = x_1^{c_1} \cdots x_n^{c_n} \text{ and } M_j = x_1^{d_1} \cdots x_n^{d_n}.$$

We begin with some facts that must be true about x_e, x_f, M_i and M_j in this case. First, since $M_j \notin I$, $d_t + \cdots + d_n < c$, but $M_i \in I$ means that $c_t + \cdots + c_n \geq c$. Since $x_f M_j = x_e M_i$, this implies that $x_f \in \{x_t, \ldots, x_n\}$ and $x_e \in \{x_1, \ldots, x_{t-1}\}$. We also observe that this must imply that

$$d_t + \cdots + d_n = c - 1 \text{ and } c_t + \cdots + c_n = c.$$

Second, the monomial $M_j \notin (B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$. Observe that if $M_j \in (B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$, then $M_j < M_i$ with respect to the reverse-lex order. So there exists some index p such that $d_p > c_p$ but $d_{p+1} = c_{p+1}, \ldots, d_n = c_n$. Now because $x_e M_i = x_f M_j$, we must have $x_e = x_p$. But since $x_p \in \{x_1, \ldots, x_{t-1}\}$ we would have

$$c > d_t + \cdots + d_n = c_t + \cdots + c_n \geq c,$$

which is a contradiction. Note that this argument applies to any monomial $M \in D$ with the property that $M | x_e M_i$ but $M \notin I$.

Now, let m be any monomial in the variables $\{x_e \mid e \in P_2\}$, and suppose that $mM_i \in I$; that is, $m \in I : M_i$. Then there exists a monomial $m' \in R$ and $M \in \mathcal{G}(I)$ such that $mM_i = m'M$. If $M = x_1^{b_1} \cdots x_n^{b_n}$, then $b_t + \cdots + b_n \geq c$ since $M \in I$. Since m is not divisible by any element of $\{x_t, \ldots, x_n\}$ (this follows since any variable in $\{x_e \mid e \in P_2\}$ must be in $\{x_1, \ldots, x_{t-1}\}$), the exponents of $x_t, \ldots, x_n$ in mM_i are the same as those of M_i, and thus, the exponents of $x_t, \ldots, x_n$ in mM_i add up to c since $c_t + \cdots + c_n = c$. Thus, any variable that divides m' must also be in $\{x_1, \ldots, x_{t-1}\}$; otherwise, the exponents of $x_t, \ldots, x_n$ in $m'M$ add up to a number greater than c.

Since m and m' are only divisible by the variables $x_1, \ldots, x_{t-1}$, we must therefore have $b_t = c_t, \ldots, b_n = c_n$. In particular, $b_{r+1} = c_{r+1}, \ldots, b_n = c_n$. Thus, if we let $M^\star = x_{r+1}^{b_{r+1}} \cdots x_n^{b_n}$, then $M_i = M_i' M^\star$ and $M = M' M^\star$. Because $M^\star \in \mathfrak{m}_{H_2}^s$, we have $M, M_i \in (B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$. It follows that $M_i', M' \in (B'_{\alpha+d-s})$. Since M is in I, we have $M < M_i$ with respect to the reverse-lex ordering. This implies that $M' < M_i'$ with respect to the reverse-lex ordering.

Let D' be the ideal of $S = k[x_1, \ldots, x_r]$ generated by all generators of $(B'_{\alpha+d-s})$ less than M_i' with respect to the reverse-lex order. Since $(B'_{\alpha+d-s})$

is polymatroidal in this ring by Theorem 3.1, it has linear quotients with respect to the ascending reverse-lex ordering (by Proposition 2.9). So

$$D' : M_i' = (x_{i_1}, \dots, x_{i_g}).$$

Now $mM_i = m'M$ implies that $mM_i' = mM'$. Since m, m' can be viewed as elements of S, we have $m \in D' : M_i'$ since $M' \in D'$. So there exists some $x_e \in \{x_{i_1}, \dots, x_{i_g}\}$ such that $x_e | m$. Note that $e \in P_2$. We thus must have some $M'' \in D' \subseteq (B'_{\alpha+d-s})$ with $M'' < M_i'$ such that $x_e M_i' = x_f M''$ for some x_f. But then

$$x_e M_i = x_e M_i' M^\star = x_f M'' M^\star.$$

Now $M''M^\star \in D$ since $M''M^\star < M_i'M^\star = M_i$. We must have that $M''M^\star \notin I$, because if $M''M^\star \in I$, this would imply that $x_e \in P_1$ since $M''M^\star | x_e M_i$. But we also have that $M''M^\star \in (B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$, contradicting the fact that every $M \in D$ with the property $M | x_e M_i$ but $M \notin I$ cannot be in $(B'_{\alpha+d-s})\mathfrak{m}_{H_2}^s$.

Thus $m \notin I : M_i$, and the conclusion follows. ∎

Remark 4.4. Combining Corollary 3.2 and Theorem 4.3, we conclude that

$$I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_1, x_4)$$

is the simplest intersection of Veronese ideals that is not componentwise linear. It is the ideal of a tetrahedral curve; see [25] and [13] for studies of these ideals and their resolutions, including a characterization of which curves are componentwise linear. Note that to form an intersection of Veronese ideals that is not componentwise linear, by our earlier results, we must have $s \geq 4$. By analyzing the possible cases for three variables, it is not hard to see that we must also work in a ring with at least four variables: The presence of any ideal $(x_1, x_2, x_3)^a \subset k[x_1, x_2, x_3]$ in the intersection is irrelevant to componentwise linearity, and hence one needs only show that

$$(x_1)^{a_1} \cap (x_2)^{a_2} \cap (x_3)^{a_3} \cap (x_1, x_2)^{a_4} \cap (x_1, x_3)^{a_5} \cap (x_2, x_3)^{a_6},$$

where the $a_i \geq 0$, can be expressed as

$$(x_1^{a_1} x_2^{a_2} x_3^{a_3}) \Big((x_1, x_2)^{\max\{a_4 - (a_1+a_2), 0\}} \cap (x_1, x_3)^{\max\{a_5 - (a_1+a_3), 0\}} \cap (x_2, x_3)^{\max\{a_6 - (a_2+a_3), 0\}} \Big).$$

Theorem 4.3 tells us that this ideal is componentwise linear.

The proof of Theorem 4.3 gives some insight into why there are ideals with $s = 4$ that fail to be componentwise linear. The ideals

$$B' = (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b) \cap k[x_i \mid i \in J \cup K] = (\mathfrak{m}_J^a \cap \mathfrak{m}_K^b) \cap k[x_1, \ldots, x_r]$$

play a prominent role in the proof. We use the fact that the ideals $(B'_{\alpha+d-s})$ are polymatroidal in $k[x_1, \ldots, x_r]$ by Theorem 3.1, which shows that they have linear quotients with respect to ascending reverse-lex order. If, in trying to prove the $s = 4$ case, we defined the B' as the intersection of three ideals $\mathfrak{m}_J^a$, $\mathfrak{m}_K^b$, and $\mathfrak{m}_L^c$, intersected with the appropriate ring, this step would fail without extra hypotheses on J, K, and L.

§5. Resolutions of $\mathfrak{m}_J^a \cap \mathfrak{m}_K^b$

In this section we provide a thorough analysis of the graded Betti numbers of ideals of the form $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$ with $a \geq b \geq 1$. We derive formulas for the Betti numbers of these intersections of Veronese ideals that enable us in the next section to recapture the formulas of Valla [28], Fatabbi and Lorenzini [11], and the first author [12] for the graded Betti numbers of two fat points in $\mathbb{P}^n$. In fact, we can extend their results to compute the $\mathbb{N}$-graded Betti numbers of two fat points in the multiprojective space $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$.

To compute the graded Betti numbers of $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$, we generalize the approach given by the first author in [12]. Our proof hinges on the fact that I is an example of a splittable monomial ideal. As in the previous section, for a monomial ideal I we let $\mathcal{G}(I)$ denote the unique set of minimal generators of I.

Definition 5.1. (see [8]) A monomial ideal I is *splittable* if I is the sum of two nonzero monomial ideals J and K, that is, $I = J + K$, such that

(1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$.
(2) there is a *splitting function*

$$\begin{aligned} \mathcal{G}(J \cap K) &\longrightarrow \mathcal{G}(J) \times \mathcal{G}(K) \\ w &\longmapsto (\phi(w), \psi(w)) \end{aligned}$$

satisfying

(a) for all $w \in \mathcal{G}(J \cap K)$, $w = \operatorname{lcm}(\phi(w), \psi(w))$.
(b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\operatorname{lcm}(\phi(S))$ and $\operatorname{lcm}(\psi(S))$ strictly divide $\operatorname{lcm}(S)$.

If J and K satisfy the above properties, then we shall say $I = J + K$ is a *splitting* of I.

When $I = J + K$ is a splitting of a monomial ideal I, then there is a relation between $\beta_{i,j}(I)$ and the graded Betti numbers of the smaller ideals.

THEOREM 5.2. (Eliahou-Kervaire [8], Fatabbi [10]) *Suppose I is a splittable monomial ideal with splitting $I = J + K$. Then for all $i, j \geq 0$,*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

The following lemma (for a proof see [23, Lemma 1.5] or [12, Lemma 2.3]) will allows us to determine when a resolution built via a mapping cone construction is in fact minimal.

LEMMA 5.3. *Let $I \subseteq R = k[x_1, \ldots, x_n]$ be a homogeneous ideal with the regularity of R/I at most $d-1$. Let m be a monomial of degree d not in I such that $I : m = \mathfrak{m}_J$ for some $J \subseteq [n]$. Then the mapping cone resolution of $R/(I, m)$ is minimal.*

With these tools we can now turn to the graded Betti numbers of $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$. The resolution depends upon how the two subsets $J, K \subseteq [n]$ intersect. There are four possible cases, as listed below, and we shall deal with each case separately.

CASE 1: $J \cap K = \emptyset$.

If $J \cap K = \emptyset$, then $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b = \mathfrak{m}_J^a \mathfrak{m}_K^b$. The resolution of I is then a corollary of Theorem 2.13. For completeness we record the formula here:

$$\beta_{i,i+a+b}(I) = \sum_{i_1+i_2=i} \binom{|J|+a-1}{a+i_1}\binom{a+i_1-1}{i_1}\binom{|K|+b-1}{b+i_2}\binom{b+i_2-1}{i_2}$$

and $\beta_{i,j}(I) = 0$ for all other $i, j \geq 0$.

CASE 2: $J \backslash K = \emptyset$ (i.e., $J \subseteq K$).

In this case $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b = \mathfrak{m}_J^a$. By Lemma 2.12, the resolution of I is then given by

$$\beta_{i,i+a}(I) = \binom{|J|+a-1}{a+i}\binom{a+i-1}{i} \text{ and } \beta_{i,j}(I) = 0 \text{ otherwise.}$$

Case 3: $K \backslash J = \emptyset$ (i.e., $K \subseteq J$).

Set $A = J \backslash K$, and let $\mathfrak{m}_A$ denote the corresponding ideal. In this situation $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b = \mathfrak{m}_K^a + \mathfrak{m}_A \mathfrak{m}_K^{a-1} + \cdots + \mathfrak{m}_A^{a-b} \mathfrak{m}_K^b$. We will postpone describing $\beta_{i,j}(I)$ in this case since these numbers will be a byproduct of our work in the final case.

Case 4: $J \cap K, J \backslash K, K \backslash J \neq \emptyset$.

Set $A = J \backslash K$, $B = K \backslash J$ and $C = J \cap K$. Let $\mathfrak{m}_A$, $\mathfrak{m}_B$ and $\mathfrak{m}_C$ be the corresponding monomial ideals.

Notation 5.4. Since the generators of $\mathfrak{m}_A$, $\mathfrak{m}_B$ and $\mathfrak{m}_C$ are disjoint subsets of indeterminates of R, for ease of exposition we write $\mathfrak{m}_A = \langle x_1, \ldots, x_{t_1} \rangle$, $\mathfrak{m}_B = \langle y_1, \ldots, y_{t_2} \rangle$, and $\mathfrak{m}_C = \langle z_1, \ldots, z_{t_3} \rangle$.

With this notation, we set

$$\begin{aligned} U &= \mathfrak{m}_C^a + \mathfrak{m}_A \mathfrak{m}_C^{a-1} + \cdots + \mathfrak{m}_A^{a-b} \mathfrak{m}_C^b \\ V &= \mathfrak{m}_B \mathfrak{m}_A^{a-b+1} \mathfrak{m}_C^{b-1} + \mathfrak{m}_B^2 \mathfrak{m}_A^{a-b+2} \mathfrak{m}_C^{b-2} + \cdots + \mathfrak{m}_B^b \mathfrak{m}_A^a. \end{aligned}$$

To find the graded Betti numbers of I, we will exploit the fact that U and V form a splitting of I (as we prove below).

Theorem 5.5. *Suppose that J, K are subsets of $[n]$ that belong to Case 4, and let U and V be as above. Then $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$ is a splittable ideal with splitting $I = U + V$.*

Proof. It is easy to check that $I = U + V$. The containment $U + V \subseteq I$ follows directly from the definitions of U and V, and the other containment is a consequence of the fact that $\mathfrak{m}_A$, $\mathfrak{m}_B$, and $\mathfrak{m}_C$ are generated by disjoint monomials.

The definition of U and V gives $\mathcal{G}(U) \cap \mathcal{G}(V) = \emptyset$. To show that U and V is a splitting, we first make the observation that

$$U \cap V = \mathfrak{m}_B \mathfrak{m}_A^{a-b+1} \mathfrak{m}_C^b,$$

and hence

$$\mathcal{G}(U \cap V) = \{ y_i m_1 m_2 \mid y_i \in \mathcal{G}(\mathfrak{m}_B),\ m_1 \in \mathcal{G}(\mathfrak{m}_A^{a-b+1}),\ m_2 \in \mathcal{G}(\mathfrak{m}_C^b) \}.$$

We define our splitting function as follows:

$$\begin{aligned} \mathcal{G}(U \cap V) &\longrightarrow \mathcal{G}(U) \times \mathcal{G}(V) \\ m = y_i m_1 m_2 &\longmapsto (\phi(m), \psi(m)) = ((m_1/x_{\max(m_1)}) m_2, y_i m_1 (m_2/z_{\max(m_2)})) \end{aligned}$$

where $\max(m_1) = \max\{i \mid x_i|m_1\}$ and $\max(m_2) = \max\{i \mid z_i|m_2\}$. It is immediate that $\operatorname{lcm}(\phi(m), \psi(m)) = m$, so our splitting function satisfies the first condition.

To verify the second condition, let $S \subseteq \mathcal{G}(U \cap V)$. It is straightforward to check that both $\operatorname{lcm}(\phi(S))$ and $\operatorname{lcm}(\psi(S))$ divide $\operatorname{lcm}(S)$. Moreover, $\operatorname{lcm}(\phi(S))$ strictly divides $\operatorname{lcm}(S)$ since $\operatorname{lcm}(S)$ is divisible by some y_ℓ, but $\operatorname{lcm}(\phi(S))$ is not. To see that $\operatorname{lcm}(\psi(S))$ strictly divides $\operatorname{lcm}(S)$, let $m = y_i m_1 m_2 \in S$ be the monomial with largest $\max(m_2)$, and among all monomials $m' \in S$ divisible by $z_{\max(m_2)}$, the power of $z_{\max(m_2)}$ in m, say d, is the largest. Hence $z^d_{\max(m_2)} | \operatorname{lcm}(S)$, but $z^d_{\max(m_2)}$ does not divide $\operatorname{lcm}(\psi(S))$. This implies that $\operatorname{lcm}(\psi(S))$ strictly divides $\operatorname{lcm}(S)$.

So, $I = U + V$ is a splitting of I. ∎

Since U and V is a splitting of I, by Theorem 5.2 we only need to compute the graded Betti numbers of U, V, and $U \cap V$. As noted within the previous proof

$$U \cap V = \mathfrak{m}_B \mathfrak{m}_A^{a-b+1} \mathfrak{m}_C^b.$$

The graded Betti numbers of $U \cap V$ can be computed using Theorem 2.13.

We now generalize the proof in [12] to compute the graded Betti numbers of U and V.

THEOREM 5.6. *With the notation as above, for all $i \geq 0$,*

$$\beta_{i,i+a}(U) = \binom{|C|+a-1}{a+i}\binom{a+i-1}{i} + \sum_{j=1}^{a-b}\sum_{k=0}^{|A|-1}\binom{k+j-1}{j-1}\binom{|C|+a-j-1}{a-j}\binom{|C|+k}{i}$$

and $\beta_{i,j}(U) = 0$ for all other $i, j \geq 0$.

Proof. To compute the graded Betti numbers of U, first note that we know the graded Betti numbers of $\mathfrak{m}_C^a$ by Lemma 2.12. We shall add the remaining generators of U to $\mathfrak{m}_C^a$, one at a time, and at each intermediate step, compute the graded Betti numbers of the resulting ideal using Lemma 5.3. After adding the last generator, we will arrive at the desired formula.

We add the remaining generators of U to $\mathfrak{m}_C^a$ in the following order: First, we add the generators of $\mathfrak{m}_A \mathfrak{m}_C^{a-1}$, then those of $\mathfrak{m}_A^2 \mathfrak{m}_C^{a-2}$, and so on. When adding the generators of $\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$, we shall add the generators in descending lexicographic order with respect to the ordering $x_1 > \cdots > x_{t_1} > z_1 > \cdots > z_{t_3}$. Let m_ℓ denote the ℓ-th monomial added to $\mathfrak{m}_C^a$, and set $U_\ell = \mathfrak{m}_C^a + (m_1, \ldots, m_\ell)$.

For each $m = x_1^{a_1} \cdots x_{t_1}^{a_{t_1}} z_1^{c_1} \cdots z_{t_3}^{c_{t_3}} \in \mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$, we associate to m the following number:

$$k_x(m) := \max\{k \mid x_{k+1} \text{ divides } x_1^{a_1} \cdots x_{t_1}^{a_{t_1}}\}.$$

For example $k_x(x_1^4 z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}) = 0$ since x_1 divides x_1^4, while $k_x(x_1^2 x_2 x_3 z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}) = 2$ because x_3 divides $x_1^2 x_2 x_3$. This notation is needed to prove:

Claim. *If m_ℓ, the ℓ-th monomial to be added $\mathfrak{m}_C^a$, belongs to $m_\ell \in \mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ and $k = k_x(m_\ell)$ then*

$$U_{\ell-1} : m_\ell = \mathfrak{m}_C + (x_1, \ldots, x_k).$$

Proof. By construction,

$$U_{\ell-1} = \mathfrak{m}_C^a + \mathfrak{m}_A \mathfrak{m}_C^{a-1} + \cdots + \mathfrak{m}_A^{t-1} \mathfrak{m}_C^{a-t+1} + (m \in \mathcal{G}(\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}) \mid m > m_\ell).$$

Since multiplying m_ℓ by any z_i gives $z_i m_\ell \in \mathfrak{m}_A^t \mathfrak{m}_C^{a-t+1} \subseteq \mathfrak{m}_A^{t-1} \mathfrak{m}_C^{a-t+1}$, it immediately follows that $\mathfrak{m}_C \subseteq U_{\ell-1} : m_\ell$.

If $k = 0$, then $m_\ell = x_1^t z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}$. Multiplying m_ℓ by any monomial $m \in R$ not divisible by z_i does not land you in $\mathfrak{m}_C^{a-i}$ for $i = 0, \ldots, t-1$. So, if $m m_\ell \in U_{\ell-1}$, then $m m_\ell \in \mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$. That is $m m_\ell$ must be divisible by a monomial in $\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ greater than m_ℓ. But the only elements greater than m_ℓ must have the form $x_1^t z_1^{d_1} \cdots z_{t_3}^{d_{t_3}}$ with $z_1^{d_1} \cdots z_{t_3}^{d_{t_3}} > z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}$. No element of this form can divide $m m_\ell$. So, if $k = 0$, $U_{\ell-1} : m_\ell = \mathfrak{m}_C$.

If $k > 0$, to show that $x_1, \ldots, x_k \in U_{\ell-1} : m_\ell$, we note that $m_\ell = x_1^{a_1} \cdots x_{k+1}^{a_{k+1}} z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}$. Then for each $i = 1, \ldots, k$,

$$x_i m_\ell = (x_i x_1^{a_1} \cdots x_{k+1}^{a_{k+1}-1} z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}) x_{k+1} = m' x_{k+1}.$$

Now $m' > m_\ell$ with respect to our ordering, so $x_i m_\ell \in U_{\ell-1}$. So $\mathfrak{m}_C + (x_1, \ldots, x_k) \subseteq U_{\ell-1} : m_\ell$.

To prove the reverse inclusion, let m be any monomial of R not divisible by either the z_is or $x_1, \ldots, x_k$. If $m m_\ell \in U_{\ell-1}$, then $m m_\ell$ must be in $\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ since $m m_\ell \notin \mathfrak{m}_A^i \mathfrak{m}_C^{a-i}$ for $i = 0, \ldots, t-1$. For $m m_\ell$ to be both in $U_{\ell-1}$ and $\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$, it must be divisible by some monomial $m' \in \mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ with $m' > m_\ell$. For m' to be larger than $m_\ell = x_1^{a_1} \cdots x_{k+1}^{a_{k+1}} z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}$, the exponent of one of $x_1, \ldots, x_{k+1}, z_1, \ldots, z_{t_3}$ must be larger in m'. Let g be the first index where the exponent of some x or z variable is bigger in m' than in m_ℓ.

We claim that $g \neq k+1$. Since $m' \in \mathcal{G}(\mathfrak{m}_A^t \mathfrak{m}_C^{a-t})$, we can write m' as

$$m' = x_1^{b_1} \cdots x_{t_1}^{b_{t_1}} z_1^{d_1} \cdots z_{t_3}^{d_{t_3}},$$

where $b_1 + \cdots + b_{t_1} = a$. If $g = k+1$, then $b_{k+1} > a_{k+1}$. By the definition of g, $a_i = b_i$ for $i = 1, \ldots, k$. But then we have

$$a = a_1 + \cdots + a_{k+1} < b_1 + \cdots + b_{k+1} \leq b_1 + \cdots + b_{t_1} = a,$$

a contradiction.

Hence m' is divisible either by some z_i or one of $x_1, \ldots, x_k$, and thus, m would also have this property, providing us with a contradiction. So the only monomials in $U_{\ell-1} : m_\ell$ are those in $\mathfrak{m}_C + (x_1, \ldots, x_k)$. $\square$

We now compute the graded Betti numbers of U_ℓ for each ℓ. When $\ell = 0$, $U_0 = \mathfrak{m}_C^a$, and the graded Betti numbers are given by Lemma 2.12:

$$\beta_{i,i+a}(U_0) = \binom{|C| + a - 1}{a+i} \binom{a+i-1}{i}$$

and $\beta_{i,j}(U_0) = 0$ for all other $i, j \geq 0$. Observe that this formula implies that the regularity of R/U_0 is $a - 1$.

Suppose now that $\ell > 0$, and that m_ℓ is the ℓ-th monomial. Furthermore, suppose that $m_\ell \in \mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ with $k = k_x(m_\ell)$. Applying the claim, we have a short exact sequence

$$\begin{aligned} 0 \longrightarrow R/(U_{\ell-1} : m_\ell)(-a) = R/(\mathfrak{m}_C + (x_1, \ldots, x_k))(-a) &\xrightarrow{\times m_\ell} R/U_{\ell-1} \\ &\longrightarrow R/U_\ell \longrightarrow 0. \end{aligned}$$

By Lemma 5.3, the mapping construction gives a minimal graded resolution of R/U_ℓ. Thus

$$\beta_{i,i+a}(U_\ell) = \beta_{i,i+a}(U_{\ell-1}) + \binom{|C| + k}{i}.$$

and $\beta_{i,j}(U_\ell) = 0$ for all other $i, j \geq 0$. So, each new generator m that we add to U_0 contributes $\binom{|C| + k_x(m)}{i}$ to $\beta_{i,i+a}(U)$.

For each $t = 1, \ldots, a-b$, there are $\binom{|C|+a-t-1}{a-t}$ generators of $\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ with $k_x(m) = 0$. These are the elements of $x_1^t \mathfrak{m}_C^{a-t}$. Also, for each $t = 1, \ldots, a-b$, there are

$$\binom{k+t-1}{t-1}\binom{|C|+a-t-1}{a-t}$$

generators of $\mathfrak{m}_A^t \mathfrak{m}_C^{a-t}$ with $k_x(m) = k$ as $1 \leq k \leq |A|-1$. To see this, we first need to count the number of elements of $\mathfrak{m}_A^t$ of the form $x_1^{a_1} \cdots x_{k+1}^{a_{k+1}}$ with $a_{k+1} \geq 1$. This is equivalent to counting the number of nonnegative integer solutions to

$$a_1 + \cdots + a_{k+1} = t \text{ with } a_{k+1} > 0.$$

Standard techniques in combinatorics imply that this equals $\binom{k+t-1}{t-1}$. For each monomial $m \in \mathfrak{m}_A^t$ of this form, every monomial $m'' \in m\mathfrak{m}_C^{a-t}$ has $k_x(m'') = k$. So we get $\binom{k+t-1}{t-1}\binom{|C|+a-t-1}{a-t}$ generators with $k_x(m) = k$. By the discussion in the previous paragraph, each generator contributes $\binom{|C|+k}{i}$ to $\beta_{i,i+a}(U)$. The formula in the statement of the theorem then comes by summing over all t and k. ∎

Note that when $K \subseteq J$ as in Case 3, $C = K \cap J = K$ and $A = J \backslash K$. So $I = U$ when $K \subseteq J$. The above theorem provides the following formula for I in Case 3.

Corollary 5.7. *Suppose $J, K \subseteq [n]$ with $K \subseteq J$. If $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$, then*

$$\beta_{i,i+a}(I) = \binom{|K|+a-1}{a+i}\binom{a+i-1}{i} + \sum_{j=1}^{a-b}\sum_{k=0}^{|J\backslash K|-1}\binom{k+j-1}{j-1}\binom{|K|+a-j-1}{a-j}\binom{|K|+k}{i}$$

and $\beta_{i,j}(I) = 0$ for all other $i, j \geq 0$.

The formula for the graded Betti numbers of V is proved similarly.

THEOREM 5.8. *With the notation as above, for $i \geq 0$ and $2 \leq j \leq b$,*

$$\beta_{i,i+a+1}(V) = \sum_{i_1+i_2+i_3=i} \binom{|B|}{1+i_1}\binom{|A|+a-b}{a-b+1+i_2}\binom{a-b+i_2}{i_2} \times \binom{|C|+b-2}{b-1+i_3}\binom{b+i_3-2}{i_3}$$

$$\beta_{i,i+a+j}(V) = \sum_{k_1=0}^{|A|-1}\sum_{k_2=0}^{|B|-1} \binom{|C|+b-j-1}{b-j}\binom{k_1+a-b+j-1}{a-b+j-1} \times \binom{k_2+j-1}{j-1}\binom{|C|+k_1+k_2}{i}.$$

Proof. Set $V_0 = \mathfrak{m}_B\mathfrak{m}_A^{a-b+1}\mathfrak{m}_C^{b-1}$. We add the remaining generators of V to V_0, one at a time, and after adding a new generator, we compute the graded Betti numbers of the resulting ideal.

We shall add the remaining generators of V in the following order: First, we add the generators of $\mathfrak{m}_B^2\mathfrak{m}_A^{a-b+2}\mathfrak{m}_C^{b-2}$, then those of $\mathfrak{m}_B^3\mathfrak{m}_A^{a-b+3}\mathfrak{m}_C^{b-3}$, and so on. When adding the generators of $\mathfrak{m}_B^t\mathfrak{m}_A^{a-b+t}\mathfrak{m}_C^{b-t}$, we will add them in descending lexicographic order with respect to $y_1 > \cdots > y_{t_2} > x_1 > \cdots > x_{t_1} > z_1 > \cdots > z_{t_3}$. We let m_ℓ denote the ℓ-th monomial added to V_0, and define $V_\ell := V_0 + (m_1, \dots, m_\ell)$.

To each monomial $m = y_1^{b_1}\cdots y_{t_2}^{b_{t_2}} x_1^{a_1}\cdots x_{t_1}^{a_{t_1}} z_1^{c_1}\cdots z_{t_3}^{c_{t_3}} \in \mathfrak{m}_B^t\mathfrak{m}_A^{a-b+t}\mathfrak{m}_C^{b-t}$ we associate the following two numbers:

$$k_x(m) := \max\{k \mid x_{k+1} \text{ divides } x_1^{a_1}\cdots x_{t_1}^{a_{t_1}}\}.$$
$$k_y(m) := \max\{k \mid y_{k+1} \text{ divides } y_1^{b_1}\cdots y_{t_2}^{b_{t_2}}\}.$$

Using this notation, we shall prove:

CLAIM. *Suppose that m_ℓ is the ℓ-th monomial added to V_0, and that $m_\ell \in \mathfrak{m}_B^t\mathfrak{m}_A^{a-b+t}\mathfrak{m}_C^{b-t}$ with $k_y = k_y(m_\ell)$ and $k_x = k_x(m_\ell)$. Then*

$$V_{\ell-1} : m_\ell = \mathfrak{m}_C + (x_1, \dots, x_{k_x}, y_1, \dots, y_{k_y}).$$

Proof. By definition

$$V_{\ell-1} = \mathfrak{m}_B\mathfrak{m}_A^{a-b+1}\mathfrak{m}_C^{b-1} + \cdots + \mathfrak{m}_B^{t-1}\mathfrak{m}_A^{a-b+t-1}\mathfrak{m}_C^{b-t+1} + (m \in \mathcal{G}(\mathfrak{m}_B^t\mathfrak{m}_A^{a-b+t}\mathfrak{m}_C^{b-t}) \mid m > m_\ell).$$

It is straightforward to check that $\mathfrak{m}_C \subseteq V_{\ell-1} : m_\ell$.

If $k_x = k_y = 0$, then $m_\ell = y_1^t x_1^{a-b+t} m'$ where $m' \in \mathfrak{m}_C^{b-t}$. Multiplying m_ℓ by any monomial $m \in R$ not divisible by z_i does not land you in $\mathfrak{m}_C^{b-i}$ for $i = 1, \ldots, t-1$. So, if $mm_\ell \in U_{\ell-1}$, then $mm_\ell \in \mathfrak{m}_B^t \mathfrak{m}_A^{a-b+t} \mathfrak{m}_C^{b-t}$. That is, mm_ℓ must be divisible by a monomial in $\mathfrak{m}_B^t \mathfrak{m}_A^{a-b+t} \mathfrak{m}_C^{b-t}$ greater than m_ℓ. But the only elements greater than m_ℓ must have the form $y_1^t x_1^{a-b+t} m''$ with $m'' > m'$. No element of this form can divide mm_ℓ. So, if $k_x = k_y = 0$, $V_{\ell-1} : m_\ell = \mathfrak{m}_C$.

If $k_y > 0$, then $m_\ell = y_1^{b_1} \cdots y_{k_y+1}^{b_{k_y+1}} x_1^{a_1} \cdots x_{t_1}^{a_{t_1}} z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}$. Then for each $i = 1, \ldots, k_y$,

$$y_i m_\ell = \big(y_i y_1^{b_1} \cdots y_{k_y+1}^{b_{k_y+1}-1} x_1^{a_1} \cdots x_{t_1}^{a_{t_1}} z_1^{c_1} \cdots z_{t_3}^{c_{t_3}}\big) y_{k_y+1} = m' y_{k_y+1}$$

But $m' > m_\ell$, so $m' \in V_{\ell-1}$, thus $y_i \in V_{\ell-1} : m_\ell$. If $k_x > 0$, a similar argument implies that $x_1, \ldots, x_{k_x} \in V_{\ell-1} : m_\ell$. Hence $\mathfrak{m}_C + (x_1, \ldots, x_{k_x}, y_1, \ldots, y_{k_y}) \subseteq V_{\ell-1} : m_\ell$.

The opposite containment follows from an argument similar to the one in Theorem 5.6. ∎

We now compute the graded Betti numbers of V_ℓ for each ℓ. When $\ell = 0$, $V_0 = \mathfrak{m}_B \mathfrak{m}_A^{a-b+1} \mathfrak{m}_C^{b-1}$. The graded Betti numbers follow from Theorem 2.13:

$$\beta_{i,i+a+1}(V_0) = \sum_{i_1+i_2+i_3=i} \binom{|B|}{1+i_1} \binom{|A|+a-b}{a-b+1+i_2} \binom{a-b+i_2}{i_2} \times \binom{|C|+b-2}{b-1+i_3} \binom{b+i_3-2}{i_3}$$

and $\beta_{i,j}(V_0) = 0$ otherwise.

Suppose that $\ell > 0$ and let m_ℓ be the ℓ-th monomial with $m_\ell \in \mathfrak{m}_B^t \mathfrak{m}_A^{a-b+t} \mathfrak{m}_C^{b-t}$. We have the short exact sequence

$$0 \longrightarrow R/(V_{\ell-1} : m_\ell)(-a-t) \xrightarrow{\times m_\ell} R/V_{\ell-1} \longrightarrow R/V_\ell \longrightarrow 0.$$

Note that $\operatorname{reg}(R/V_0) = a$, and inductively, for $\ell - 1 \geq 0$, $\operatorname{reg}(R/V_{\ell-1}) = a + t - 1$ since $V_{\ell-1} : m_\ell$ is generated by a subset of the variables. Therefore by Lemma 5.3, the mapping cone construction gives a minimal graded free resolution of R/V_ℓ. If $k_x = k_x(m_\ell)$ and $k_y = k_y(m_\ell)$, then the claim implies $R/(V_{\ell-1} : m_\ell) = R/(\mathfrak{m}_C + (x_1, \ldots, x_{k_x}, y_1, \ldots, y_{k_y}))$. So, each generator $m \in \mathfrak{m}_B^t \mathfrak{m}_A^{a-b+t} \mathfrak{m}_C^{b-t}$ contributes $\binom{|C|+k_x(m)+k_y(m)}{i}$ to $\beta_{i,i+a+t}(V)$.

Counting as in Theorem 5.6 and summing over all possible t, k_x, and k_y, we obtain the final formulas; we leave the details to the reader. ∎

THEOREM 5.9. *Suppose $J, K \subseteq [n]$ are such that $J \cap K, J \backslash K, K \backslash J \neq \emptyset$. If $I = \mathfrak{m}_J^a \cap \mathfrak{m}_K^b$, then*

$$\begin{aligned}
\beta_{i,i+a}(I) &= \beta_{i,i+a}(U) \\
\beta_{i,i+a+1}(I) &= \beta_{i,i+a+1}(V) + \beta_{i-1,i+a+1}(\mathfrak{m}_B \mathfrak{m}_A^{a-b+1} \mathfrak{m}_C^b) \\
\beta_{i,i+a+j}(I) &= \beta_{i,i+a+j}(V) \ \text{ for } j = 2, \ldots, b.
\end{aligned}$$

where U and V are as defined above.

Proof. Since $I = U + V$ is a splitting, the formulas are a consequence of Theorem 5.2 and the fact that $\beta_{i-1,i+a+1}(U \cap V) = \beta_{i-1,i+a+1}(\mathfrak{m}_B \mathfrak{m}_A^{a-b+1} \mathfrak{m}_C^b)$. ∎

§6. Applications: multiplicity, combinatorics, and fat points in multiprojective space

In our final section, we present some applications of our results in the earlier sections. First, we discuss some cases of the Multiplicity Conjecture of Herzog, Huneke, and Srinivasan. In addition, we use our componentwise linearity results and Alexander duality to prove a corollary about the sequential Cohen-Macaulayness of some simplicial complexes. Finally, we apply our earlier results to investigate the resolutions of some sets of fat points in multiprojective space. The main result of [12] is that ideals of small sets of general fat points in $\mathbb{P}^n$ are componentwise linear; we generalize this theorem to multiprojective space. Furthermore, we extend work from [10], [28], [11], [12] to describe the graded Betti numbers of ideals of small sets of fat points in linear general position in multiprojective space.

6.1. Multiplicity Conjecture

The Multiplicity Conjecture of Herzog, Huneke, and Srinivasan (see, e.g., [22]) proposes bounds for the multiplicity of an ideal in terms of the shifts in its graded free resolution. The explicit statement is given below.

CONJECTURE 6.1. *Let R/I be a homogeneous k-algebra with resolution of the form*

$$0 \longrightarrow \bigoplus_{j=1}^{b_r} R(-d_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_{j=1}^{b_1} R(-d_{1j}) \longrightarrow R \longrightarrow R/I \longrightarrow 0.$$

Set $m_i = \min\{d_{ij} \mid j = 1, \ldots, b_i\}$ *and* $M_i = \max\{d_{ij} \mid j = 1, \ldots, b_i\}$. *If* $\operatorname{codim}(I) = c$ *and* $e(R/I)$ *denotes the multiplicity of* R/I*, then*

$$e(R/I) \leq \frac{\prod_{i=1}^{c} M_i}{c!}.$$

Furthermore, if R/I *is Cohen-Macaulay, then*

$$\frac{\prod_{i=1}^{c} m_i}{c!} \leq e(R/I).$$

In [26], Römer proved that when the characteristic of k is zero, componentwise linear ideals satisfy the above Multiplicity Conjecture. As a consequence of Theorem 3.1, Corollary 3.2, Theorem 4.3, and Römer's result, we have:

COROLLARY 6.2. *Suppose* $\operatorname{char}(k) = 0$. *Let* $I = \mathfrak{m}_{J_1}^{a_1} \cap \cdots \cap \mathfrak{m}_{J_s}^{a_s}$, *and suppose either that* $s \leq 3$, *or* $J_i \cup J_j = [n]$ *for all* $i \neq j$. *Then* I *satisfies the upper bound of the Multiplicity Conjecture.*

Note that we only know that the upper bound is true since in general, R/I may not be Cohen-Macaulay. If it is, then the lower bound holds as well. (Römer states his result only for the upper bound, but his proof is based on the fact that if I is componentwise linear, then I and the reverse-lex generic initial ideal $\operatorname{gin}(I)$ have the same graded Betti numbers in characteristic zero. Both bounds of the conjecture hold for all Cohen-Macaulay generic initial ideals in characteristic zero since the bounds are true for all Cohen-Macaulay strongly stable ideals. Since the reverse-lex gin preserves depth and dimension, if R/I is Cohen-Macaulay, $R/\operatorname{gin}(I)$ is as well, so the lower bound holds in that case.)

6.2. The sequentially Cohen-Macaulay property

The notion of componentwise linearity is intimately related to the concept of sequential Cohen-Macaulayness.

DEFINITION 6.3. Let $R = k[x_1, \ldots, x_n]$. A graded R-module M is called *sequentially Cohen-Macaulay* if there exists a finite filtration of M by graded R-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

We say that a simplicial complex Δ is sequentially Cohen-Macaulay if R/I_Δ is sequentially Cohen-Macaulay, where I_Δ is the Stanley-Reisner ideal of Δ.

Stanley introduced sequential Cohen-Macaulayness in connection with developments in the theory of shellability; see, e.g., [27] for a definition of shellable. A shellable pure simplicial complex (that is, a shellable simplicial complex whose maximal faces all have the same dimension) is Cohen-Macaulay, but if one extends the definition of shellability to allow nonpure simplicial complexes, one obtains simplicial complexes that are not Cohen-Macaulay. However, they are sequentially Cohen-Macaulay.

The theorem connecting sequentially Cohen-Macaulayness to componentwise linearity is based on the idea of Alexander duality. We define Alexander duality for squarefree monomial ideals and then state the fundamental result of Herzog and Hibi [19] and Herzog, Reiner, and Welker [21].

Definition 6.4. If $I = (x_{1,1}x_{1,2}\cdots x_{1,t_1}, \ldots, x_{s,1}x_{s,2}\cdots x_{s,t_s})$ is a squarefree monomial ideal, then the *Alexander dual* of I, denoted $I^\star$, is the monomial ideal

$$I^\star = (x_{1,1}, \ldots, x_{1,t_1}) \cap \cdots \cap (x_{s,1}, \ldots, x_{s,t_s}).$$

If Δ is a simplicial complex and $I = I_\Delta$ its Stanley-Reisner ideal, then the simplicial complex $\Delta^\star$ with $I_{\Delta^\star} = I_\Delta^\star$ is the *Alexander dual* of Δ.

Theorem 6.5. *Let Δ be a simplicial complex with Stanley-Reisner ideal I_Δ. Let $\Delta^\star$ be the Alexander dual of Δ. Then R/I_Δ is sequentially Cohen-Macaulay if and only if $I_\Delta^\star = I_{\Delta^\star}$ is componentwise linear.*

Our results in this paper yield the following corollary.

Corollary 6.6. *Let Δ be a simplicial complex on n vertices, and let I_Δ be its Stanley-Reisner ideal, minimally generated by squarefree monomials $m_1, \ldots, m_s$. If $s \leq 3$, so that Δ has at most three minimal nonfaces, or if $\operatorname{Supp}(m_i) \cup \operatorname{Supp}(m_j) = \{x_1, \ldots, x_n\}$ for all $i \neq j$, then Δ is sequentially Cohen-Macaulay.*

Proof. I_Δ is a squarefree monomial ideal; suppose it is minimally generated by monomials $\{x_{1,1}\cdots x_{1,t_1},\ldots,x_{s,1}\cdots x_{s,t_s}\}$. Then

$$I_\Delta^\star = I_{\Delta^\star} = (x_{1,1},\ldots,x_{1,t_1})\cap\cdots\cap(x_{s,1},\ldots,x_{s,t_s}).$$

By Theorem 3.1, Corollary 3.2, or Theorem 4.3, $I_\Delta^\star$ is componentwise linear, and so Theorem 6.5 gives the result. ∎

Example 6.7. Let Δ be a simplicial complex on six vertices. Suppose the minimal nonfaces of Δ are $\{145, 126, 135\}$. Then

$$I_{\Delta^\star} = (x_1,x_4,x_5)\cap(x_1,x_2,x_6)\cap(x_1,x_3,x_5)\subset R = k[x_1,\ldots,x_6]$$

is componentwise linear by Theorem 4.3, and thus Δ is sequentially Cohen-Macaulay. Note that Δ is not Cohen-Macaulay since $\operatorname{codim} I_\Delta = 1$, while the projective dimension of R/I_Δ is two.

6.3. Fat points in multiprojective space

We begin by recalling some of the relevant definitions for points in multiprojective space (for more on this topic see [29], [30], [31]). The coordinate ring of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ is the $\mathbb{N}^r$-graded polynomial ring $R = k[x_{1,0},\ldots,x_{1,n_1},\ldots,x_{r,0},\ldots,x_{r,n_r}]$ with $\deg x_{i,j} = e_i$, the i-th basis vector of $\mathbb{N}^r$. The defining ideal of a point $P = P_1\times\cdots\times P_r \in \mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_r}$ is the prime ideal $I_P = (L_{1,1},\ldots,L_{1,n_1},\ldots,L_{r,1},\ldots,L_{r,n_r})$ with $\deg L_{i,j} = e_i$. The forms $L_{i,1},\ldots,L_{i,n_i}$ are the generators of the defining ideal of $P_i\in\mathbb{P}^{n_i}$.

Definition 6.8. A set of points $X\subseteq\mathbb{P}^n$ is said to be in *linear general position* if no more than two points lie on a line, no more than three points line in a plane, …, no more than n points lie in an $(n-1)$-plane.

Observe that the above definition is equivalent to the fact that if $\mathcal{L}_d$ is any linear subspace of $\mathbb{P}^n$ of dimension d with $d = 0,\ldots,n-1$, then the intersection of $\mathcal{L}_d$ and X contains at most $d+1$ points of X. When $d = 0$, $\mathcal{L}_d$ is a point, so this simply says that the intersection of a point and X is at most one point. To extend this to a multigraded context, we say that $\mathcal{L}$ is $(d_1,\ldots,d_r)$-linear subspace of $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_k}$ if $\mathcal{L} = \mathcal{L}_{d_1}\times\cdots\times\mathcal{L}_{d_r}$, where each $\mathcal{L}_{d_i}$ is a linear subspace of $\mathbb{P}^{n_i}$ of dimension d_i with $d_i = 0,\ldots,n_i$ (so $\mathcal{L}_{n_i} = \mathbb{P}^{n_i}$ is allowed) and there exists at least one $j\in[r]$ such that $d_j < n_j$.

Definition 6.9. A set of points $X \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is in *linear general position* if for every $(d_1, \ldots, d_r)$-linear subspace $\mathcal{L}$, the intersection of $\mathcal{L}$ and X contains at most $d+1$ points of X where $d = \min\{d_1, \ldots, d_r\}$.

We point out that if $\mathcal{L}$ is $(d_1, \ldots, d_r)$-linear subspace with $d = d_i = 0$, then $\mathcal{L}_{d_i}$ is a point. So if X is in linear general position, this means that at most one point of X can intersect $\mathcal{L}$, which, in turn, implies that at most one point of X can have i-th coordinate equal to $\mathcal{L}_{d_i}$. It follows from this observation that for any two points $P, Q \in X$ with X in linear general position in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, we must have $P_i \neq Q_i$ for $i = 1, \ldots, r$. In other words if $\pi_i : \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \to \mathbb{P}^{n_i}$ denotes the projection morphism for $i = 1, \ldots, r$, and if $\{Q_1, \ldots, Q_t\} \in \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ is in linear general position, then the sets of the projections $\{\pi_i(Q_1), \ldots, \pi_i(Q_t)\}$ are in linear general position in $\mathbb{P}^{n_i}$ for each i. In particular, we require that $\pi_i(Q_j) \neq \pi_i(Q_l)$ for all i and all $j \neq l$; see Remark 6.12 for what can go wrong without this condition.

Definition 6.10. Let $\{P_1, \ldots, P_s\} \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ be a set of points with the defining ideal of P_i denoted I_{P_i} and let $a_1, \ldots, a_s$ be positive integers. The scheme $Z \subseteq \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ defined by

$$I_Z = I_{P_1}^{a_1} \cap \cdots \cap I_{P_s}^{a_s}$$

is scheme of *fat points*, and is sometimes denoted $Z = \{(P_1, a_1), \ldots, (P_s, a_s)\}$. We call a_i the *multiplicity* of the point P_i. The points $\{P_1, \ldots, P_s\}$ are referred to as the *support* of Z.

By a small set of linear general fat points in $\mathbb{P}^n$, we mean that the support has at most $n+1$ points in linear general position. This restriction allows us to make a change of coordinates to move all the points to the coordinate vertices, and we can take the ideal corresponding to the set of fat points to be an intersection of monomial ideals

$$I = (x_1, \ldots, x_n)^{a_0} \cap (x_0, x_2, \ldots, x_n)^{a_1} \cap \cdots \cap (x_0, \ldots, x_{s-1}, x_{s+1}, x_n)^{a_s}.$$

If we are working in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, we would like to change coordinates to work with a set of fat points at the coordinate vertices so that the corresponding ideals are monomial ideals. Therefore, a small set of fat points can consist of no more than $1 + \min\{n_1, \ldots, n_r\}$ points. The set of fat points is general if the points in the support are in linear general position.

Suppose that I is the ideal of a small set of general fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$. As a consequence of Theorem 3.1, we can generalize the componentwise linearity result for the $r = 1$ case from [12] (and obtain a different proof for that case).

THEOREM 6.11. *Let I be the ideal of $s+1$ general fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$, where $s \leq \min\{n_1, \ldots, n_r\}$. Then for all d, (I_d) is polymatroidal, and I is componentwise linear.*

Proof. Because I is the ideal of a small set of general fat points in multiprojective space, we may assume that I has the form

$$I = (x_{1,1}, \ldots, x_{1,n_1}, x_{2,1}, \ldots, x_{2,n_2}, \ldots, x_{r,1}, \ldots, x_{r,n_r})^{a_0} \cap \cdots \cap \\ (x_{1,0}, \ldots, \hat{x_{1,s}}, \ldots, x_{1,n_1}, x_{2,0}, \ldots, \hat{x_{2,s}}, \ldots, x_{2,n_2}, \ldots, \\ x_{r,0}, \ldots, \hat{x_{r,s}}, \ldots, x_{r,n_r})^{a_s} \subset R,$$

where $\hat{x_{i,s}}$ denotes that $x_{i,s}$ is left out. Note that the union of the variables appearing in any two of the components is all the variables of R. Hence the result follows immediately from Theorem 3.1. ∎

As in Corollary 6.2, when the char$(k) = 0$, Theorem 6.11 implies that ideals of small sets of general fat points in multiprojective space satisfy the Multiplicity Conjecture of Herzog, Huneke, and Srinivasan. Note that if $r > 1$, the ideal will not be Cohen-Macaulay (for example, see [29], [30]), so we may only conclude that the conjectured upper bound is true.

We conclude this discussion with a remark about how we defined the notion of a "general" set of fat points.

Remark 6.12. In our definition of what it means for a set of fat points $Q_1, \ldots, Q_s$ in multiprojective space to be general, we required that for all i and all $j \neq l$, the projections $\pi_i(Q_j) \neq \pi_i(Q_l)$. If that condition is not satisfied, the corresponding ideal may not be componentwise linear.

Consider the points $[1:0] \times [1:0]$, $[1:0] \times [0:1]$, $[0:1] \times [1:0]$, and $[0:1] \times [0:1]$ in $\mathbb{P}^1 \times \mathbb{P}^1$, and suppose each point has multiplicity one. The ideal corresponding to the set of four points in $R = k[x_0, x_1, y_0, y_1]$ is

$$I = (x_1, y_1) \cap (x_1, y_0) \cap (x_0, y_1) \cap (x_0, y_0) = (x_0x_1, y_0y_1).$$

This ideal is a complete intersection of degree two polynomials, and hence it is not componentwise linear; in particular, $I = (I_2)$ does not have a linear resolution. The problem is that the union of the variables appearing in, for example, the first two components, is not all of $\{x_0, x_1, y_0, y_1\}$.

We turn now to the graded Betti numbers of two general fat points in multiprojective space. As an application of Theorem 5.9 we can compute the $\mathbb{N}$-graded Betti numbers of the defining ideal of two fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ in linear general position.

COROLLARY 6.13. *Let $Z = \{(P, a), (Q, b)\}$ be two fat points in $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r}$ with $a \geq b$. Set $N = n_1 + \cdots + n_r$, and let I_Z denote the defining ideal of Z. If P and Q are in linear general position, then*

$$\beta_{i,i+a}(I_Z) = \binom{N-r+a-1}{a+i}\binom{a+i-1}{i} + \sum_{j=1}^{a-b}\sum_{k=0}^{r-1}\binom{k+j-1}{j-1}\binom{N-r+a-j-1}{a-j}\binom{N-r+k}{i}$$

$$\begin{aligned}\beta_{i,i+a+1}(I_Z) = &\sum_{i_1+i_2+i_3=i}\binom{r}{1+i_1}\binom{r+a-b}{a-b+1+i_2}\binom{a-b+i_2}{i_2}\\ &\qquad\times\binom{N-r+b-2}{b-1+i_3}\binom{b+i_3-2}{i_3}\\ &+\sum_{i_1+i_2+i_3=i-1}\binom{r}{1+i_1}\binom{r+a-b}{a-b+1+i_2}\binom{a-b+i_2}{i_2}\\ &\qquad\times\binom{N-r+b-1}{b+i_3}\binom{b+i_3-1}{i_3}\end{aligned}$$

$$\begin{aligned}\beta_{i,i+a+j}(I_Z) = &\sum_{k_1=0}^{r-1}\sum_{k_2=0}^{r-1}\binom{N-r+b-j-1}{b-j}\binom{k_1+a-b+j-1}{a-b+j-1}\\ &\qquad\times\binom{k_2+j-1}{j-1}\binom{N-r+k_1+k_2}{i}\end{aligned}$$

for $j = 2, \ldots, b$.

and $\beta_{i,j}(I_Z) = 0$ for all other $i, j \geq 0$.

Proof. Since P and Q are in linear general position, we may assume (after a change of coordinates) that $P = [1 : 0 : \cdots : 0] \times \cdots \times [1 : 0 : \cdots : 0]$ and $Q = [0 : 1 : 0 : \cdots : 0] \times \cdots \times [0 : 1 : 0 \cdots : 0]$. So, the defining ideal of

I_Z has the form

$$\begin{aligned} I_Z = (x_{1,1},\ldots,x_{1,n_1},\ldots,x_{r,1},\ldots,x_{r,n_r})^a \\ \cap\,(x_{1,0},x_{1,2},\ldots,x_{1,n_1},\ldots,x_{r,0},x_{r,2},\ldots,x_{r,n_r})^b \end{aligned}$$

The graded Betti numbers of I_Z are then a consequence of Theorem 5.9 with $|C| = N - r$ and $|A| = |B| = r$. □

Remark 6.14. When $r = 1$ in the previous corollary, we recover the formulas of Valla [28] and first author [12] for two fat points in $\mathbb{P}^n$. When $r > 1$, then I_Z also has a multigraded resolution of the form

$$0 \longrightarrow \bigoplus_{\underline{j}\in\mathbb{N}^r} R(-\underline{j})^{\beta_{h,\underline{j}}(I_Z)} \longrightarrow \cdots \longrightarrow \bigoplus_{\underline{j}\in\mathbb{N}^r} R(-\underline{j})^{\beta_{0,\underline{j}}(I_Z)} \longrightarrow I_Z \longrightarrow 0.$$

Corollary 6.13 gives us some information on the multigraded Betti numbers $\beta_{i,\underline{j}}(I_Z)$ because of the identity $\beta_{i,j}(I_Z) = \sum_{|\underline{j}|=j} \beta_{i,\underline{j}}(I_Z)$.

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