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CARLESON MEASURES FOR WEIGHTED HARDY-SOBOLEV SPACES

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Abstract. We obtain characterizations of positive Borel measures μ on $\mathbf{B}^n$ so that some weighted Hardy-Sobolev are imbedded in $L^p(d\mu)$, where w is an A_p weight in the unit sphere of $\mathbf{C}^n$.

§1. Introduction

The purpose of this paper is the study of the positive Borel measures μ on $\mathbf{S}^n$, the unit sphere in $\mathbf{C}^n$, for which the weighted Hardy-Sobolev space $H_s^p(w)$ is imbedded in $L^p(d\mu)$, that is, the Carleson measures for $H_s^p(w)$.

The weighted Hardy-Sobolev space $H_s^p(w)$, $0 < s, p < +\infty$, consists of those functions f holomorphic in $\mathbf{B}^n$ such that if $f(z) = \sum_k f_k(z)$ is its homogeneous polynomial expansion, and $(I+R)^s f(z) = \sum_k (1+k)^s f_k(z)$, we have that

$$\|f\|_{H_s^p(w)} = \sup_{0<r<1} \|(I+R)^s f_r\|_{L^p(w)} < +\infty,$$

where $f_r(\zeta) = f(r\zeta)$.

We will consider weights w in A_p classes in $\mathbf{S}^n$, $1 < p < +\infty$, that is, weights in $\mathbf{S}^n$ satisfying that there exists $C > 0$ such that for any non-isotropic ball $B \subset \mathbf{S}^n$, $B = B(\zeta, r) = \{\eta \in \mathbf{S}^n \,;\, |1 - \zeta\overline{\eta}| < r\}$,

$$\left(\frac{1}{|B|}\int_B w\, d\sigma\right)\left(\frac{1}{|B|}\int_B w^{\frac{-1}{p-1}}\, d\sigma\right)^{p-1} \le C,$$

where σ is the Lebesgue measure on $\mathbf{S}^n$ and $|B|$ the Lebesgue measure of B. We will use the notation $\zeta\overline{\eta}$ to indicate the complex inner product in $\mathbf{C}^n$ given by $\zeta\overline{\eta} = \sum_{i=1}^n \zeta_i\overline{\eta_i}$, if $\zeta = (\zeta_1, \ldots, \zeta_n)$, $\eta = (\eta_1, \ldots, \eta_n)$.

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If $0 < s < n$, any function f in $H^p_s(w)$ can be expressed as

$$f(z) = C_s(g)(z) := \int_{\mathbf{S}^n} \frac{g(\zeta)}{(1 - z\overline{\zeta})^{n-s}}\, d\sigma(\zeta),$$

where $d\sigma$ is the normalized Lebesgue measure on the unit sphere $\mathbf{S}^n$ and $g \in L^p(w)$, and consequently, μ is Carleson for $H^p_s(w)$ if there exists $C > 0$ such that

$$\|C_s f\|_{L^p(d\mu)} \leq C\|f\|_{L^p(w)}.$$

We denote by K_s the nonisotropic potential operator defined by

$$K_s[f](z) = \int_{\mathbf{S}^n} \frac{f(\eta)}{|1 - z\overline{\eta}|^{n-s}}\, d\sigma(\eta), \quad z \in \overline{\mathbf{B}}^n.$$

The problem of characterizing the positive Borel measures μ on $\mathbf{B}^n$ for which there exists $C > 0$ such that

$$\|K_s[f]\|_{L^p(d\mu)} \leq C\|f\|_{L^p(d\sigma)}, \tag{1.1}$$

that is, the characterization of the Carleson measures for the space $K_s[L^p(d\sigma)]$ has been very well studied and there exist different characterizations (see for instance [Ma], [AdHe], [KeSa]).

The representation of the functions in H^p_s in terms of the operator C_s gives that in dimension 1 the Carleson measures for $K_s[L^p(d\sigma)]$ coincide with the Carleson measures for the Hardy-Sobolev space H^p_s simply because the real part of $1/(1 - z\overline{\zeta})^{1-s}$ is equivalent to $1/|1 - z\overline{\zeta}|^{1-s}$. This representation also shows that in any dimension every Carleson measure for $K_s[L^p(d\sigma)]$ is also a Carleson measure for H^p_s. The coincidence fails to be true for $n > 1$ in general, as it is shown in [AhCo] (see also [CaOr2]).

Of course, when $n - sp < 0$, the space H^p_s consists of continuous functions on $\overline{\mathbf{B}}^n$, and in particular, the Carleson measures in this case are just the finite measures. But for $n - sp \geq 0$, and $n > 1$, the characterization of the Carleson measures for H^p_s still remains open. In the case where we are "near" the regular case, that is when $n - sp < 1$ it is shown in [AhCo], [CohVe1] and [CohVe2], that the Carleson measures for H^p_s and $K_s[L^p(d\sigma)]$ are the same, and any of the different characterizations of the Carleson measures for the last ones also hold for H^p_s.

One of the main purposes of this paper is to extend this situation to $H^p_s(w)$ for w a weight in A_p. If $E \subset \mathbf{S}^n$ is measurable, we define

$$W(E) = \int_E w\, d\sigma.$$

A weight w satisfies a doubling condition of order τ, if there exists $\tau > 0$ such that for any nonisotropic ball B in $\mathbf{S}^n$, $W(2^k B) \le C 2^{k\tau} W(B)$.

It is well known that any weight in A_p satisfies a doubling condition of some order τ strictly less than np. We begin observing that if $\tau - sp < 0$, the space $H^p_s(w)$ consists of continuous functions on $\overline{\mathbf{B}}^n$, and consequently, the Carleson measures are just the finite ones. If $\tau - sp < 1$, we show that the Carleson measures for $H^p_s(w)$ and $K_s[L^p(w)]$ coincide, whereas if $\tau - sp \ge 1$, this coincidence may fail.

As it happens in the unweighted case (see [CohVe1]), the proof of the characterization of the Carleson measures for $H^p_s(w)$ will be based in the construction of weighted holomorphic potentials, with control of their $H^p_s(w)$-norm. In fact, technical reasons give that it is convenient to deal with weighted Triebel-Lizorkin spaces which, on the other hand, have interest on their own. In the second section we study these spaces. If $s \ge 0$, we will write $[s]^+$ the integer part of s plus 1. Let $1 < p < +\infty$, $1 \le q \le +\infty$, and $s \ge 0$. The weighted holomorphic Triebel-Lizorkin space $HF^{pq}_s(w)$ when $q < +\infty$ is the space of holomorphic functions f in $\mathbf{B}^n$ for which

$$\begin{aligned}\|f\|_{HF^{pq}_s(w)} &= \left(\int_{\mathbf{S}^n}\left(\int_0^1 |((I+R)^{[s]^+} f)(r\zeta)|^q (1-r^2)^{([s]^+ - s)q - 1}\, dr\right)^{p/q} w(\zeta)\, d\sigma(\zeta)\right)^{1/p} \\ &< +\infty,\end{aligned}$$

whereas when $q = +\infty$,

$$\begin{aligned}\|f\|_{HF^{p\infty}_s(w)} &= \left(\int_{\mathbf{S}^n}\left(\sup_{0<r<1} |((I+R)^{[s]^+} f)(r\zeta)|(1-r^2)^{[s]^+ - s}\right)^{p} w(\zeta)\, d\sigma(\zeta)\right)^{1/p} < +\infty,\end{aligned}$$

where I denotes the identity operator.

The Section 2 is devoted to the general theory of weighted holomorphic Triebel-Lizorkin spaces. We give different equivalent definitions of the spaces $HF^{pq}_s(w)$ in terms of admissible area functions, we give duality theorems on these spaces, we study some relations of inclusion among them and we also obtain that when $q = 2$, the weighted Triebel-Lizorkin space $HF^{p2}_s(w)$ coincides with the weighted Hardy-Sobolev space $H^p_s(w)$.

The main result in Section 3 is the characterization of the Carleson measures for $H^p_s(w)$, when $0 < \tau - sp < 1$, in terms of a positive kernel.

THEOREM C. *Let $1 < p < +\infty$, w an A_p-weight, and μ a finite positive Borel measure on $\mathbf{B}^n$. Assume that w is doubling of order τ, for some $\tau < 1 + sp$. We then have that the following statements are equivalent*:

(*i*) $\|K_s(f)\|_{L^p(d\mu)} \leq C\|f\|_{L^p(w)}$.

(*ii*) $\|f\|_{L^p(d\mu)} \leq C\|f\|_{H^p_s(w)}$.

The proof relies on the construction of weighted holomorphic potentials, with control of their weighted Hardy-Sobolev norm.

We also give examples of the sharpness of the above theorem. We show that if $p = 2$ and $\tau > 1 + sp$, $n < \tau < n + 1$, then there exists w in $A_2 \cap D_\tau$ and a measure μ on $\mathbf{S}^n$ which is Carleson for $H^2_s(w)$, but it is not Carleson for $K_s[L^2(w)]$.

Finally, the usual remark on notation: we will adopt the convention of using the same letter for various absolute constants whose values may change in each occurrence, and we will write $A \preceq B$ if there exists an absolute constant M such that $A \leq MB$. We will say that two quantities A and B are equivalent if both $A \preceq B$ and $B \preceq A$, and, in that case, we will write $A \simeq B$.

§2. Weighted holomorphic Triebel-Lizorkin spaces

In this section we will introduce weighted holomorphic Triebel-Lizorkin spaces, and we will obtain characterizations in terms of Littlewood-Paley functions and admissible area functions. These characterizations, known in the unweighted case, will be used in the following sections.

We begin recalling some simple facts about A_p weights that we will need later. It is well known that $A_\infty = \bigcup_{1<p<+\infty} A_p$ and that any A_p weight satisfies a doubling condition. We recall that a weight w satisfies a doubling condition of order τ, $\tau > 0$, if there exists $C > 0$, such that for any nonisotropic ball $B \subset \mathbf{S}^n$, and any $k \geq 0$, $W(2^k B) \leq C 2^{\tau k} W(B)$. We will say that this weight w is in D_τ. In fact, if $w \in A_p$, there exists $p_1 < p$ such that w is also in A_{p_1}, and consequently we have that $w \in D_\tau$ for $\tau = np_1 < np$, (see [StrTo]).

Examples of A_p weights can be obtained as follows: if $\zeta = (\zeta', \zeta_n)$, and $w(\zeta) = (1 - |\zeta'|^2)^\varepsilon$, we then have that $w \in A_p$ if $-1 < \varepsilon < p - 1$. We also have that for this weight, $w \in D_\tau$, $\tau = n + \varepsilon$.

The following lemma gives the natural relationships between the spaces $L^p(w)$, $w \in A_p$, and the Lebesgue spaces $L^q(d\sigma)$.

LEMMA 2.1. *Let $1<p<+\infty$, and w be an A_p-weight. We then have:*
(i) There exists $1<p_1<p$ such that $L^p(w)\subset L^{p_1}(d\sigma)$.
(ii) There exists $p_2>p$ such that $L^{p_2}(d\sigma)\subset L^p(w)$.

We now proceed to study the weighted holomorphic Triebel-Lizorkin spaces $H^{pq}_s(w)$ already defined in the introduction. We begin with some definitions. If $1<q\le+\infty$, k an integer such that $k>s\ge 0$, and $\zeta\in\mathbf{S}^n$, the Littlewood-Paley type functions are given by

$$A_{1,k,q,s}(f)(\zeta)=\left(\int_0^1 |(I+R)^k f(r\zeta)|^q(1-r^2)^{(k-s)q-1}\,dr\right)^{1/q},$$

when $q<+\infty$, and

$$A_{1,k,\infty,s}(f)(\zeta)=\sup_{0<r<1}|(I+R)^k f(r\zeta)|(1-r^2)^{k-s},$$

when $q=+\infty$.

If $\alpha>1$, $\zeta\in\mathbf{S}^n$, we denote by $D_\alpha(\zeta)$, $\alpha>1$ the admissible region given by $D_\alpha(\zeta)=\{z\in\mathbf{B}^n\ ;\ |1-z\overline{\zeta}|<\alpha(1-|z|)\}$. We introduce the admissible area function

$$A_{\alpha,k,q,s}(f)(\zeta)=\left(\int_{D_\alpha(\zeta)}|(I+R)^k f(z)|^q(1-|z|^2)^{(k-s)q-n-1}\,dv(z)\right)^{1/q},$$

when $q<+\infty$, where dv is the Lebesgue measure on $\mathbf{B}^n$, and in case $q=+\infty$,

$$A_{\alpha,k,\infty,s}(f)(\zeta)=\sup_{z\in D_\alpha(\zeta)}|(I+R)^k f(z)|(1-|z|^2)^{k-s}.$$

Our first goal is to obtain that if $1<p<+\infty$, $1<q<+\infty$ and w is an A_p weight, then an holomorphic function f is in $HF^{p,q}_s(w)$ if and only if $A_{\alpha,k,q,s}(f)\in L^p(w)$, for some (and then for all) $\alpha\ge 1$ and $k>s$. We will follow the ideas in [OF]. For the sake of completeness, we will sketch the modifications needed to obtain the weighted case.

If $1<p<+\infty$, $1<q\le+\infty$ we denote by

$$L^p(w)(L^q_1)=L^p(w)\left(L^q\left(\frac{2nr^{2n-1}}{1-r^2}\,dr\right)\right)$$

the mixed-norm space of measurable functions f in $\mathbf{S}^n \times [0,1]$ such that

$$\|f\|_{p,q,w} = \left(\int_{\mathbf{S}^n}\left(\int_0^1 |f(r\zeta)|^q \frac{2nr^{2n-1}}{1-r^2}\,dr\right)^{p/q} w(\zeta)\,d\sigma(\zeta)\right)^{1/p} < +\infty.$$

Also if $\alpha > 1$, and $E_\alpha(z) = \left(\int_{\mathbf{S}^n} \chi_{D_\alpha(\zeta)}(z)\,d\sigma(\zeta)\right)^{-1} \simeq (1-|z|^2)^{-n}$, we denote by $L^p(w)(L^q_\alpha)$ the mixed-norm space of measurable functions f defined in $\mathbf{S}^n \times \mathbf{B}^n$ such that

$$\|f\|_{\alpha,p,q,w} = \left(\int_{\mathbf{S}^n}\left(\int_{\mathbf{B}^n} |f(\zeta,z)|^q \frac{E_\alpha(z)}{(1-|z|^2)}\,dv(z)\right)^{p/q} w(\zeta)\,d\sigma(\zeta)\right)^{1/p} < +\infty.$$

We denote by $F^{\alpha,p,q}(w)$ the space of measurable functions on $\mathbf{B}^n$ such that

$$J_\alpha f(\zeta,z) = \chi_{D_\alpha(\zeta)}(z) f(z)$$

is in $L^p(w)(L^q_\alpha)$, normed with the norm induced by $\|\cdot\|_{\alpha,p,q,w}$. We also introduce the space $F^{1,p,q}(w)$ of measurable functions on $\mathbf{B}^n$ such that $J_1 f(\zeta,r) = f(r\zeta)$ is in $L^p(w)(L^q_1)$.

The representation of the dual of a mixed-norm space, see [BeLo], gives that if $1 < p,q < +\infty$, the dual space of $L^p(w)(L^q_1)$ is $L^{p'}(w)(L^{q'}_1)$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, and that if $f \in F^{1,p,q}(w)$, $g \in F^{1,p',q'}(w)$ the pairing is given by

$$(f,g) = \int_{\mathbf{S}^n}\left(\int_0^1 f(r\zeta)\overline{g(r\zeta)}\frac{2nr^{2n-1}}{1-r^2}\,dr\right) w(\zeta)\,d\sigma(\zeta).$$

Analogously, the dual space of $L^p(w)(L^q_\alpha)$ is $L^{p'}(w)(L^{q'}_\alpha)$, and if $f \in F^{\alpha,p,q}(w)$, $g \in F^{\alpha,p',q'}(w)$ the pairing is given by

$$\begin{aligned}(f,g)_\alpha &= \int_{\mathbf{B}^n}\int_{\mathbf{S}^n} f(z)\overline{g(z)}\chi_{D_\alpha(\zeta)}(z) w(\zeta)\,d\sigma(\zeta)\frac{dv(z)}{(1-|z|^2)^{n+1}}\\ &= \int_{\mathbf{B}^n} f(z)\overline{g(z)}\frac{E^w_\alpha(z)}{(1-|z|^2)^{n+1}}\,dv(z),\end{aligned}$$

where $E^w_\alpha(z) = \int_{\mathbf{S}^n} \chi_{D_\alpha(\zeta)}(z) w(\zeta)\,d\sigma(\zeta)$.

Observe that if we write $z_0 = z/|z|$, the doubling property of w gives that $E^w_\alpha(z) \simeq W(B(z_0,(1-|z|)))$. From now on we will write $B_z = B(z_0,(1-|z|))$.

We begin with two lemmas that are weighted versions of Lemmas 2.2. and 2.3 in [OF], and whose proofs we omit. We recall that if ψ is a measurable function on $\mathbf{S}^n$, the weighted Hardy-Littlewood maximal function is given by

$$M^w_{HL}(\psi)(\zeta) = \sup_{B\ni\zeta} \frac{1}{W(B)} \int_B |\psi(\eta)| w(\eta)\, d\sigma(\eta).$$

LEMMA 2.2. *There exist $C > 0$, $N_0 > 0$ such that for any $z \in D_\alpha(\zeta)$, $N \geq N_0$,*

$$\frac{(1-|z|^2)^{n+N}}{W(B_z)} \int_{\mathbf{S}^n} \frac{|\psi(\eta)|}{|1-z\overline{\eta}|^{n+N}} w(\eta)\, d\sigma(\eta) \leq C M^w_{HL}(\psi)(\zeta).$$

LEMMA 2.3. *Let $\alpha > 1$. There exists $C > 0$, such that for any $z \in D_\alpha(\zeta)$,*

$$\frac{1}{W(B_z)} \int_{\mathbf{S}^n} \chi_{D_\alpha(\eta)}(z) |\psi(\eta)| w(\eta)\, d\sigma(\eta) \leq C M^w_{HL}(\psi)(\zeta).$$

THEOREM 2.4. *Let $1 < p < +\infty$, $1 \leq q \leq +\infty$, and $\alpha \geq 1$. Then the space $F^{\alpha,p,q}(w)$ is a retract of $L^p(w)(L^q_\alpha)$.*

Proof of Theorem 2.4. The fact that J_1 is an isometry between $F^{1,p,q}(w)$ and $L^p(w)(L^q_1)$ gives the theorem for the case $\alpha = 1$.

If $\alpha > 1$, we introduce the averaging operator

$$A_\alpha(\varphi)(z) = \frac{1}{E^w_\alpha(z)} \int_{\mathbf{S}^n} \chi_{D_\alpha(\eta)}(z) \varphi(\eta, z) w(\eta)\, d\sigma(\eta).$$

The definition of $E^w_\alpha(z)$ gives that $A_\alpha \circ J_\alpha$ is the identity operator on $F^{\alpha,p,q}(w)$. So, in order to finish the theorem, we need to show that A_α maps $L^p(w)(L^q_\alpha)$ to $F^{\alpha,p,q}(w)$. We consider first the case $1 \leq q \leq p < +\infty$. Let $m = p/q \geq 1$ and let m' be the conjugate exponent of m. We then have by duality that

$$\begin{aligned}
&\|A_\alpha(\varphi)\|^q_{\alpha,p,q,w} \\
&\quad = \sup_{\|\psi\|_{L^{m'}(w)}\leq 1} \left| \int_{\mathbf{S}^n} \int_{D_\alpha(\zeta)} |A_\alpha(\varphi)(z)|^q \frac{dv(z)}{(1-|z|^2)^{n+1}} \psi(\zeta) w(\zeta)\, d\sigma(\zeta) \right|.
\end{aligned}$$

Now Hölder's inequality gives that

$$|A_\alpha(\varphi)(z)|^q \leq \frac{1}{E^w_\alpha(z)} \int_{\mathbf{S}^n} |\varphi(\eta, z)|^q \chi_{D_\alpha(\eta)}(z) w(\eta)\, d\sigma(\eta).$$

Hence, by Lemma 2.3

$$\begin{aligned}
&\|A_\alpha(\varphi)\|^q_{\alpha,p,q,w} \\
&\quad\preceq \sup_{\|\psi\|_{L^{m'}(w)}\leq 1} \int_{\mathbf{S}^n} \int_{\mathbf{B}^n} \frac{1}{E^w_\alpha(z)} \chi_{D_\alpha(\zeta)}(z) \int_{\mathbf{S}^n} \chi_{D_\alpha(\eta)}(z) |\varphi(\eta, z)|^q w(\eta)\, d\sigma(\eta) \\
&\qquad\qquad \times \frac{dv(z)}{(1-|z|^2)^{n+1}} |\psi(\zeta)| w(\zeta)\, d\sigma(\zeta) \\
&\quad\preceq \sup_{\|\psi\|_{L^{m'}(w)}\leq 1} \int_{\mathbf{S}^n} \int_{\mathbf{B}^n} |\varphi(\eta, z)|^q \frac{dv(z)}{(1-|z|^2)^{n+1}} w(\eta) M^w_{HL}(\psi)(\eta)\, d\sigma(\eta).
\end{aligned}$$

Next, Hölder's inequality with exponent $m = p/q$ gives that the above is bounded by

$$\begin{aligned}
&\sup_{\|\psi\|_{L^{m'}(w)}\leq 1} \|M^w_{HL}\psi\|_{L^{m'}(w)} \\
&\qquad \times \left(\int_{\mathbf{S}^n} \left(\int_{\mathbf{B}^n} |\varphi(\eta, z)|^q \frac{dv(z)}{(1-|z|^2)^{n+1}} \right)^{p/q} w(\eta)\, d\sigma(\eta) \right)^{q/p} \\
&\leq \sup_{\|\psi\|_{L^{m'}(w)}\leq 1} \|\psi\|_{L^{m'}(w)} \|\varphi\|^q_{\alpha,\, p,q,w},
\end{aligned}$$

where we have used that since w is a doubling measure, the weighted Hardy-Littlewood maximal function is bounded from $L^{m'}(w)$ to $L^{m'}(w)$. That finishes the proof of the theorem when $q \leq p$.

So we are lead to deal with the case $1 < p < q \leq +\infty$, which can be easily obtained from the previous case using the duality in the mixed-norm spaces $L^p(w)(L^q_\alpha)$. ∎

This result can be used as in the unweighted case to obtain a characterization of the dual spaces of the weighted spaces $F^{\alpha,\,p,q}(w)$.

Corollary 2.5. *Let $1 < p < +\infty$, $1 < q < +\infty$, $\alpha > 1$, and w an A_p-weight. Then the dual of $F^{\alpha,\,p,q}(w)$ is $F^{\alpha,\,p',q'}(w)$ with the pairing given by $(f, g)_\alpha$.*

The following proposition will be needed in the proof of the main theorem in this section. If $N > 0$, $M > 0$, we consider the operators defined by

$$P^{N,M} f(y) = \int_{\mathbf{B}^n} f(z) \frac{(1-|z|^2)^N (1-|y|^2)^M}{|1-z\overline{y}|^{n+1+N+M}}\, dv(z), \quad y \in \mathbf{B}^n.$$

THEOREM 2.6. *Let $1 < p < +\infty$, $1 \leq q < +\infty$, $\alpha, \beta \geq 1$, and w an A_p weight. Then there exists $N_0 > 0$ such that for any $N \geq N_0$ and any $M > 0$, the operator $P^{N,M}$ is continuous from $F^{\alpha,p,q}(w)$ to $F^{\beta,p,q}(w)$.*

Proof of Theorem 2.6. We begin with the case $\alpha, \beta > 1$. The case where $1 \leq q \leq p < +\infty$ can be deduced following the scheme of [OF], using Lemma 2.2.

In the case $1 < p < q < +\infty$ we apply duality in the mixed norm space and obtain

$$\begin{aligned} (2.1) \qquad & \|P^{N,M}(f)\|^q_{\beta,p,q,w} \\ &= \sup_{\|g\|_{\beta,p'q',w} \leq 1} \left| \int_{\mathbf{B}^n} P^{N,M}(f)(y)\overline{g(y)} \frac{E^w_\beta(y)}{(1-|y|^2)^{n+1}}\, dv(y) \right| \\ &\leq \sup_{\|g\|_{\beta,p'q',w} \leq 1} (f, \widetilde{P}^{M-1,N+1}(g))_\alpha, \end{aligned}$$

where

$$\begin{aligned} (2.2) \qquad & \widetilde{P}^{R,S}(g)(z) \\ &= \int_{\mathbf{B}^n} \frac{(1-|y|^2)^R (1-|z|^2)^S g(y)}{|1-y\overline{z}|^{n+1+R+S}} \frac{E^w_\beta(y)}{(1-|y|^2)^n} \frac{(1-|z|^2)^n}{E^w_\alpha(z)}\, dv(y). \end{aligned}$$

Observe that when $w \equiv 1$, then $\widetilde{P}^{M,N}(f) \simeq P^{M,N}(f)$. Here we are led to obtain that the operator $\widetilde{P}^{M-1,N+1}$ maps boundedly $F^{\beta,p',q'}$ to $F^{\alpha,p',q'}$, provided $p < q$. If we claim this proposition, we finish the proof of the theorem. Using (2.1), and applying Hölder's inequality,

$$\begin{aligned} \|P^{N,M}(f)\|^q_{\beta,p,q,w} &= \sup_{\|g\|_{\alpha,p'q',w} \leq 1} (f, \widetilde{P}^{M-1,N-1}(g))_\alpha \\ &\leq \sup_{\|g\|_{\alpha,p'q',w} \leq 1} \|f\|_{\alpha,p,q,w} \|\widetilde{P}^{M-1,N-1}(g)\|_{\alpha,p',q',w} \\ &\leq C \sup \|f\|_{\alpha,p,q,w}. \end{aligned}$$

The cases $\alpha = 1$ and $\beta = 1$ are proved in a simmilar way.

To finish the theorem we will prove the claim. Changing the notation, it is enough to prove:

Proposition 2.7. *Let $1 < q < p < +\infty$, $\alpha, \beta \geq 1$, and w an A_p weight. We then have that there exists $N_0 > 0$ such that for any $N \geq N_0$ and any $M \geq 0$,*

(i) $\widetilde{P}^{M,N}(1) < +\infty$.

(ii) The operator $\widetilde{P}^{M,N}$ is continuous from $F^{\alpha,p,q}(w)$ to $F^{\beta,p,q}(w)$.

Proof of Proposition 2.7. Let us begin with (i). From the definition of $E_\alpha^w(z)$ and Fubini's theorem,

$$\begin{aligned}
&\int_{\mathbf{B}^n} \frac{(1-|z|^2)^M}{|1-z\overline{y}|^{n+1+M+N}} \frac{E_\alpha^w(z)}{(1-|z|^2)^n}\, dv(z) \\
&= \int_{\mathbf{S}^n} \int_{D_\alpha(z)} \frac{(1-|z|^2)^M}{|1-z\overline{y}|^{n+1+M+N}} \frac{dv(z)}{(1-|z|^2)^n} w(\zeta)\, d\sigma(\zeta) \\
&\preceq \int_{\mathbf{S}^n} \frac{1}{|1-y\overline{\zeta}|^{n+N}} w(\zeta)\, d\sigma(\zeta),
\end{aligned}$$

where in last inequality we have used Lemma 2.7 in [OF] since $M > -1$.

Next, let $B_k = B(y_0, 2^k(1-|y|^2))$, $k \geq 0$, where $y_0 = y/|y|$. Since w is doubling and $E_\alpha^w(y) \simeq W(B_0)$ we have that $W(B_k) \leq C^k E_\alpha^w(y)$. Consequently

$$\begin{aligned}
\int_{\mathbf{S}^n} \frac{1}{|1-y\overline{\zeta}|^{n+N}} w(\zeta)\, d\sigma(\zeta) &\preceq \sum_k \int_{B_k} \frac{w(\zeta)\, d\sigma(\zeta)}{(2^k(1-|y|^2))^{n+N}} \\
&\preceq \frac{E_\alpha^w(y)}{(1-|y|^2)^{n+N}} \sum_k \frac{C^k}{2^{k(n+N)}} \preceq \frac{E_\alpha^w(y)}{(1-|y|^2)^{n+N}},
\end{aligned}$$

if N is chosen sufficiently large. That finishes the proof of (i).

Since $m = p/q > 1$, duality gives that

$$\begin{aligned}
(2.3)\quad &\|\widetilde{P}^{M,N}(f)\|_{\beta,p,q,w}^q \\
&= \sup_{\|\psi\|_{L^{m'}(w)} \leq 1} \left| \int_{\mathbf{S}^n} \int_{D_\beta(\zeta)} |\widetilde{P}^{M,N} f(y)|^q \frac{dv(y)}{(1-|y|^2)^{n+1}} \overline{\psi(\zeta)} w(\zeta)\, d\sigma(\zeta) \right|.
\end{aligned}$$

Next, Hölder's inequality shows that if $0<\varepsilon<N$ then

$$\begin{aligned}
&|\widetilde{P}^{M,N}(f)(y)|^q\\
&\quad\le \int_{\mathbf{B}^n}|f(z)|^q\frac{(1-|z|^2)^M(1-|y|^2)^{N-\varepsilon}}{|1-z\overline{y}|^{n+1+M+N-\varepsilon}}\frac{E^w_\alpha(z)}{(1-|z|^2)^n}\frac{(1-|y|^2)^n}{E^w_\alpha(y)}\,dv(z)\\
&\qquad\times\left(\int_{\mathbf{B}^n}\frac{(1-|z|^2)^M(1-|y|^2)^{N+\varepsilon\frac{q'}{q}}}{|1-z\overline{y}|^{n+1+M+N+\varepsilon\frac{q'}{q}}}\frac{E^w_\alpha(z)}{(1-|z|^2)^n}\frac{(1-|y|^2)^n}{E^w_\alpha(y)}\,dv(z)\right)^{q/q'}\\
&\quad\preceq \int_{\mathbf{B}^n}|f(z)|^q\frac{(1-|z|^2)^M(1-|y|^2)^{N-\varepsilon}}{|1-z\overline{y}|^{n+1+N+M-\varepsilon}}\frac{E^w_\alpha(z)}{(1-|z|^2)^n}\frac{(1-|y|^2)^n}{E^w_\alpha(y)}\,dv(z),
\end{aligned}$$

where in last inequality we have used (i).

Consequently,

(2.4)

$$\begin{aligned}
&\|\widetilde{P}^{M,N}(f)\|^q_{\beta,p,q,w}\\
&\quad\le C\sup_{\|\psi\|_{L^{m'}(w)}\le 1}\left|\int_{\mathbf{S}^n}\int_{y\in D_\beta(\zeta)}\int_{\mathbf{B}^n}\frac{|f(z)|^q(1-|z|^2)^M(1-|y|^2)^{N-\varepsilon}}{|1-z\overline{y}|^{n+1+N+M-\varepsilon}}\right.\\
&\qquad\qquad\left.\times\frac{E^w_\alpha(z)}{(1-|z|^2)^n}\frac{(1-|y|^2)^n}{E^w_\alpha(y)}\,dv(z)\frac{dv(y)}{(1-|y|^2)^{n+1}}\psi(\zeta)w(\zeta)\,d\sigma(\zeta)\right|\\
&\quad= C\sup_{\|\psi\|_{L^{m'}(w)}\le 1}\left|\int_{\mathbf{S}^n}\int_{\mathbf{B}^n}\int_{D_\beta(\zeta)}\frac{(1-|y|^2)^{N+n-\varepsilon}}{|1-z\overline{y}|^{n+1+N+M-\varepsilon}}\frac{dv(y)}{E^w_\alpha(y)(1-|y|^2)^{n+1}}\right.\\
&\qquad\qquad\left.\times|f(z)|^q(1-|z|^2)^{M-n}E^w_\alpha(z)\,dv(z)\psi(\zeta)w(\zeta)\,d\sigma(\zeta)\right|.
\end{aligned}$$

Next, if $y\in D_\beta(\zeta)$, $E^w_\alpha(y)\simeq W(B_y)\simeq W(B(\zeta,(1-|y|^2))$, and $|1-z\overline{y}|\simeq(1-|y|^2)+|1-z\overline{\zeta}|$.

Assume first that $|1-z\overline{\zeta}|\le 1$. Hence,

$$\begin{aligned}
(2.5)\qquad &\int_{D_\beta(\zeta)}\frac{(1-|y|^2)^{N+n-\varepsilon}}{|1-z\overline{y}|^{n+1+N+M-\varepsilon}}\frac{dv(y)}{E^w_\alpha(y)(1-|y|^2)^{n+1}}\\
&\quad\simeq\int_{\mathbf{B}^n}\frac{(1-|y|^2)^{N-\varepsilon}}{((1-|y|^2)+|1-z\overline{\zeta}|)^{n+1+N+M-\varepsilon}}\chi_{D_\beta(\zeta)}(y)\\
&\qquad\qquad\times\frac{(1-|y|^2)^n}{W(B(\zeta,1-|y|^2))}\frac{dv(y)}{(1-|y|^2)^{n+1}},
\end{aligned}$$

which by integration in polar coordinates is bounded by

$$\begin{aligned}&\int_0^1 \frac{r^{2n-1}(1-r^2)^{N+n-\varepsilon}}{((1-r^2)+|1-z\overline{\zeta}|)^{n+1+N+M-\varepsilon}}\frac{dr}{(1-r^2)W(B(\zeta,C(1-r^2)))}\\&\quad\simeq\int_0^{|1-z\overline{\zeta}|}\frac{t^{N+n-\varepsilon-1}}{(t+|1-z\overline{\zeta}|)^{n+1+N+M-\varepsilon}}\frac{dt}{W(B(\zeta,t))}\\&\qquad\qquad+\int_{|1-z\overline{\zeta}|}^1\frac{t^{N+n-\varepsilon-1}}{(t+|1-z\overline{\zeta}|)^{n+1+N+M-\varepsilon}}\frac{dt}{W(B(\zeta,t))}=I+II.\end{aligned}$$

In I we have that $(t+|1-z\overline{\zeta}|)\simeq|1-z\overline{\zeta}|$, and, since $w\in A_p$,

$$\frac{t^n}{W(B(\zeta,t))}\simeq\left(\frac{1}{t^n}\int_{B(\zeta,t)}w^{-(p'-1)}\right)^{p-1}.$$

Thus we obtain that

$$\begin{aligned}I&\simeq\int_0^{|1-z\overline{\zeta}|}\frac{t^{N-\varepsilon-1}}{|1-z\overline{\zeta}|^{n+1+N+M-\varepsilon}}\left(\frac{1}{t^n}\int_{B(\zeta,t)}w^{-(p'-1)}\right)^{p-1}dt\\&\preceq\left(\int_{B(\zeta,|1-z\overline{\zeta}|)}w^{-(p'-1)}\right)^{p-1}\frac{1}{|1-z\overline{\zeta}|^{n+1+N+M-\varepsilon}}\int_0^{|1-z\overline{\zeta}|}t^{N-\varepsilon-n(p'-1)-1}\,dt\\&\preceq\frac{1}{|1-z\overline{\zeta}|^{M+1}}\frac{1}{W(B(z_0,|1-z\overline{\zeta}|))},\end{aligned}$$

where we have used that $N>0$ is chosen big enough, and that w satisfies the A_p condition.

In II, $(t+|1-z\overline{\zeta}|)\simeq t$, and since $M+1>0$, we have

$$\begin{aligned}II&\simeq\int_{|1-z\overline{\zeta}|}^1\frac{1}{t^{M+2}}\frac{dt}{W(B(\zeta,t))}\leq\int_{|1-z\overline{\zeta}|}^1\frac{1}{t^{M+2}}\frac{dt}{W(B(\zeta,|1-z\overline{\zeta}|))}\\&\preceq\frac{1}{|1-z\overline{\zeta}|^{M+1}}\frac{1}{W(B(z_0,|1-z\overline{\zeta}|))}.\end{aligned}$$

If $|1-z\overline{\zeta}|>1$, then we have that $(1-r^2)+|1-z\overline{\zeta}|\simeq 1$. We return to (2.5) and obtain

$$\begin{aligned}&\int_0^1\frac{(1-r^2)^{N+n-\varepsilon-1}\,dr}{((1-r)^2+|1-z\overline{\zeta}|)^{n+1+N+M-\varepsilon}W(B(\zeta,1-r^2))}\\&\quad\preceq\left(\int_{B(\zeta,1)}w^{-p'/p}\right)^{p/p'}\int_0^1 t^{N-\varepsilon-n\frac{p}{p'}-1}\,dt\\&\quad\preceq\frac{1}{|1-z\overline{\zeta}|^{M+1}}\frac{1}{W(B(z_0,|1-z\overline{\zeta}|))}.\end{aligned}$$

Then we have in any case that (2.5) is bounded by

$$\frac{1}{|1-z\overline{\zeta}|^{M+1}}\frac{1}{W(B(z_0,|1-z\overline{\zeta}|))}.$$

In consequence, we return to (2.4) and we obtain

$$\begin{aligned}(2.6)\quad &\|\widetilde{P}^{M,N}(f)\|^q_{\beta,p,q,w}\\ &\preceq \sup_{\|\psi\|_{L^{m'}(w)}\le 1}\left|\int_{\mathbf{S}^n}\int_{\mathbf{B}^n}|f(z)|^q\frac{(1-|z|^2)^{M-n}E^w_\alpha(z)}{|1-z\overline{\zeta}|^{M+1}W(B(z_0,|1-z\overline{\zeta}|))}\right.\\ &\qquad\left.\times\,\psi(\zeta)\,dv(z)w(\zeta)\,d\sigma(\zeta)\right|\\ &\preceq \sup_{\|\psi\|_{L^{m'}(w)}\le 1}\left|\int_{\mathbf{S}^n}\int_{\mathbf{B}^n}|f(z)|^q(1-|z|^2)^{M-n}\chi_{D_\alpha(\eta)}(z)\right.\\ &\qquad\left.\times\int_{\mathbf{S}^n}\frac{\psi(\zeta)w(\zeta)\,d\sigma(\zeta)}{|1-z\overline{\zeta}|^{M+1}W(B(z_0,|1-z\overline{\zeta}|))}\,dv(z)w(\eta)\,d\sigma(\eta)\right|.\end{aligned}$$

Next, if $z\in D_\alpha(\eta)$, $B(\eta,|1-z\overline{\zeta}|)\subset B(z_0,C|1-z\overline{\zeta}|)$, and if $B_k=B(\eta,2^k(1-|z|^2))$, $k\ge 0$ and $\zeta\in B_{k+1}\setminus B_k$, $|1-z\overline{\zeta}|\simeq 2^k(1-|z|^2)$. Thus

$$\begin{aligned}&\int_{\mathbf{S}^n}\frac{|\psi(\zeta)|w(\zeta)\,d\sigma(\zeta)}{|1-z\overline{\zeta}|^{M+1}W(B(z_0,|1-z\overline{\zeta}|))}\\ &\quad\preceq\frac{1}{(1-|z|^2)^{M+1}W(B(\eta,1-|z|^2))}\int_{B_0}|\psi(\zeta)|w(\zeta)\,d\sigma(\zeta)\\ &\qquad+\sum_{k\ge 1}\frac{1}{2^{k(M+1)}(1-|z|^2)^{M+1}W(B(\eta,2^k(1-|z|^2)))}\int_{B_k}|\psi(\zeta)|w(\zeta)\,d\sigma(\zeta)\\ &\quad\preceq\frac{1}{(1-|z|^2)^{M+1}}\sum_{k\ge 0}\frac{1}{2^{k(M+1)}}M^w_{HL}(\psi)(\eta)\preceq\frac{1}{(1-|z|^2)^{M+1}}M^w_{HL}(\psi)(\eta).\end{aligned}$$

Plugging the above estimate in (2.6) and using Hölder's inequality with exponent $m=p/q$, we obtain

$$\begin{aligned}\|\widetilde{P}^{M,N}(f)\|^q_{\beta,p,q,w}&\preceq\sup_{\psi\in L^{m'}(w)}\int_{\mathbf{S}^n}\int_{\mathbf{B}^n}|f(z)|^q\frac{1}{(1-|z|^2)^{n+1}}\chi_{D_\alpha(\eta)}(z)\,dv(z)\\ &\qquad\times M^w_{HL}(\psi)(\eta)w(\eta)\,d\sigma(\eta)\\ &\preceq\sup_{\psi\in L^{m'}(w)}\|f\|^q_{\alpha,p,q,w}\|M^w_{HL}(\psi)\|^q_{L^{m'}(w)}\preceq\|f\|^q_{\alpha,p,q,w}.\end{aligned}$$ ∎

We deduce from the previous theorem the following characterization of the weighted holomorphic Triebel-Lizorkin spaces. If $f \in H(\mathbf{B}^n)$, $s, t > 0$, let

$$L_s^t f(z) = (1 - |z|^2)^{t-s}(I + R)^t f(z).$$

THEOREM 2.8. *Let* $1 < p < +\infty$, $1 < q < +\infty$, $t > s \geq 0$ *and* $\alpha \geq 1$. *Let*

$$HF_s^{\alpha,\, t,\, p,q}(w) = \{f \in H(\mathbf{B}^n)\ ;\ \|L_s^t f\|_{\alpha,p,q} < +\infty\}.$$

Then $HF_s^{\alpha,\, t,\, p,q}(w) = HF_s^{pq}(w)$.

Proof of Theorem 2.8. If $s < t_0 < t_1$, $\alpha, \beta \geq 1$, we just need to check that $HF_s^{\alpha,\, t_0,\, p,q}(w) = HF_s^{\beta,\, t_1,\, p,q}(w)$. Any holomorphic function f on $\mathbf{B}^n$ satisfying that $L_s^t f(z) \in F^{\alpha,\, p,q}(w)$ is in $A^{-\infty}(\mathbf{B}^n)$, the space of holomorphic functions in $\mathbf{B}^n$ for which there exists $k > 0$ such that $\sup_z (1-|z|^2)^k |f(z)| < +\infty$. Consequently, f and its derivatives have a representation formula via the reproducing kernel $c_N \frac{(1-|z|^2)^N}{(1-\bar{z}y)^{n+1+N}}$, for $N > 0$ sufficiently large and an adequate constant c_N. Once we have made this observation, we can reproduce the arguments in [OF] and obtain

$$(I + R)^{t_0} f(y) = C_N \int_{\mathbf{B}^n} (I + R)^{t_1} f(z)(I + R_y)^{t_0 - t_1} \frac{(1 - |z|^2)^N}{(1 - y\overline{z})^{n+1+N}}\, dv(z).$$

Since for $m > 0$ we have that

$$(I + R)^{-m} g(y) = \frac{1}{\Gamma(m)} \int_0^1 \left(\log \frac{1}{r}\right)^{m-1} g(ry)\, dr, \tag{2.7}$$

we obtain

$$\begin{aligned} \|L_s^{t_0} f\|_{\alpha,p,q,w} &\preceq \left\| \int_{\mathbf{B}^n} |(I + R)^{t_1} f(z)| \frac{(1 - |z|^2)^N (1 - |y|^2)^{t_0 - s}}{|1 - \overline{z}y|^{n+1+N+t_0-t_1}}\, dv(z) \right\|_{\alpha,p,q,w} \\ &= \|P^{N-t_1+s,t_0-s}(|L_s^{t_1} f|)\|_{\alpha,p,q,w}, \end{aligned}$$

and we just have to apply Theorem 2.6 to finish the proof. ∎

THEOREM 2.9. *Let* $1 < p < +\infty$, $1 < q < +\infty$, w *an* A_p*-weight, and* f *a holomorphic function. Then the following assertions are equivalent*:

(*i*) f *is in* $HF_s^{pq}(w)$.

(*ii*) $A_{\alpha,k,q,s}(f) \in L^p(w)$, *for some* $\alpha \geq 1$ *and* $k > s$.

(*iii*) $A_{\alpha,k,q,s}(f) \in L^p(w)$, *for all* $\alpha \geq 1$ *and* $k > s$.

Our next result studies some inclusion relationships between different weighted holomorphic Triebel-Lizorkin spaces.

THEOREM 2.10. *Let $1 < p < +\infty$, $1 \le q_0 \le q_1 \le +\infty$, $s \ge 0$ and let w be an A_p-weight. We then have*

$$HF_s^{pq_0}(w) \subset HF_s^{pq_1}(w).$$

Proof of Theorem 2.10. We begin with the case $q_1 = +\infty$. Let $0 < \varepsilon < 1$. If $L_s^k f(z) = (1-|z|^2)^{k-s}(I+R)^k f(z)$, the fact that $(I+R)^k f$ is holomorphic gives that

$$|L_s^k f(r\zeta)| \preceq \left(\frac{1}{(1-r^2)^{n+1}}\int_{K(r\zeta,c(1-r^2))} |(I+R)^k f(z)|^\varepsilon \, dv(z)\right)^{1/\varepsilon} (1-r^2)^{k-s},$$

where for $y \in \mathbf{B}^n$ $K(y,t)$ is the nonisotropic ball in $\mathbf{B}^n$ given by

$$K(y,t) = \{z \in \mathbf{B}^n \,;\, |\overline{z}(z-y)| + |\overline{y}(y-z)| < t\}.$$

In [OF] it is obtained that

$$|L_s^k f(r\zeta)| \preceq \left(M_{HL}\left(\int_0^1 |(I+R)^k f(t\eta)|^q (1-t^2)^{(k-s)q-1}\, dt\right)^{\varepsilon/q}(\zeta)\right)^{1/\varepsilon}.$$

Thus

$$\begin{aligned}\|f\|^p_{HF_s^{p\infty}(w)} &= \int_{\mathbf{S}^n} \sup_{0<r<1} |L_s^k f(r\zeta)|^p w(\zeta)\, d\sigma(\zeta)\\ &\preceq \int_{\mathbf{S}^n}\left(M_{HL}\left(\int_0^1 |(I+R)^k f(t\eta)|^q (1-t^2)^{(k-s)q-1}\, dt\right)^{\varepsilon/q}(\zeta)\right)^{p/\varepsilon}\\ &\qquad \times w(\zeta)\, d\sigma(\zeta).\end{aligned}$$

Since $p/\varepsilon > p$, and w is an A_p-weight, w is in $A_{p/\varepsilon}$, and in consequence the unweighted Hardy-Littlewood maximal function is a bounded map $L^{p/\varepsilon}(w)$ to itself. Hence the above is bounded by

$$\begin{aligned}&C\int_{\mathbf{S}^n}\left(\int_0^1 |(I+R)^k f(t\zeta)|^q (1-t^2)^{(k-s)q-1}\, dt\right)^{p/q} w(\zeta)\, d\sigma(\zeta)\\ &= C\|f\|^p_{HF_s^{pq}(w)}.\end{aligned}$$

In order to finish the theorem, we will prove that if $q_0 < q_1 < +\infty$, then

$$\|f\|_{HF_s^{pq_1}(w)} \leq \|f\|_{HF_s^{pq_0}(w)}^{\frac{q_0}{q_1}} \|f\|_{HF_s^{p\infty}(w)}^{1-\frac{q_0}{q_1}}.$$

Since

$$\begin{aligned}\|f\|_{HF_s^{pq_1}(w)}^p \leq \int_{\mathbf{S}^n} &\left(\sup_{0<r<1} |(I+R)^k f(r\zeta)|(1-r)^{k-s}\right)^{(q_1-q_0)p/q_1} \\ &\times \left(\int_0^1 |(I+R)^k f(r\zeta)|^{q_0}(1-r^2)^{(k-s)q_0-1}\, dr\right)^{p/q_1} w(\zeta)\, d\sigma(\zeta),\end{aligned}$$

Hölder's inequality with exponent $q_1/q_0 > 1$, gives that the above is bounded by

$$C\|f\|_{HF_s^{pq_0}(w)}^{p\frac{q_0}{q_1}} \|f\|_{HF_s^{p\infty}(w)}^{p(1-\frac{q_0}{q_1})}. \qquad \square$$

We now consider the weighted Hardy space $H^p(w)$, for $1 < p < +\infty$, and w an A_p weight. It is shown in [Lu] that $f \in H^p(w)$ if and only if $f = C[f^*]$, where $f^*(\zeta) = \lim_{r\to 1} f(r\zeta) \in L^p(w)$ is the radial limit, C is the Cauchy-Szegö kernel. In addition, $f = P[f^*]$, where P is the Poisson-Szegö kernel. It follows also that $\|f\|_{H^p(w)}^p \simeq \|f^*\|_{L^p(w)}$.

It is immediate to deduce from this that $f \in H^p(w)$ if and only if for any $\alpha \geq 1$, $M_\alpha(f) \in L^p(w)$, where M_α is the α-admissible maximal operator given by

$$M_\alpha(f)(\zeta) = \sup_{z\in D_\alpha(\zeta)} |f(z)|.$$

In addition $\|f\|_{H^p(w)} \simeq \|M_\alpha(f)\|_{L^p(w)}$, with constant that depends on α. Indeed, since $|f(r\zeta)| \leq M_\alpha(f)(\zeta)$, we have that $\|f\|_{H^p(w)} \leq \|M_\alpha(f)\|_{L^p(w)}$. On the other hand, assume that $f \in H^p(w)$. Then $f = P[f^*]$, $f^* \in L^p(w)$ and since $M_\alpha(f) \leq CM_{HL}(f^*)$, (see for instance [Ru]), we deduce that

$$\begin{aligned}\int_{\mathbf{S}^n} (M_\alpha(f)(\zeta))^p w(\zeta)\, d\sigma(\zeta) &\preceq \int_{\mathbf{S}^n} (M_{HL}(f^*)(\zeta))^p w(\zeta)\, d\sigma(\zeta) \\ &\preceq \int_{\mathbf{S}^n} |f^*(\zeta)|^p w(\zeta)\, d\sigma(\zeta) \preceq \|f\|_{H^p(w)}^p,\end{aligned}$$

where we have used that since w in an A_p-weight, the Hardy-Littlewood maximal operator maps $L^p(w)$ continuously to itself.

Our next result gives a proof for the weighted nonisotropic case of the fact that the spaces $H^p(w)$ can also be defined in terms of admissible area functions. Similar results, but using a different approach based on localized good-lambda inequalities, have been obtained in [StrTo] for weighted isotropic Hardy spaces in $\mathbf{R}^n$.

THEOREM 2.11. *Let $1 < p < +\infty$, and w be an A_p-weight. Let f be an holomorphic function on $\mathbf{B}^n$. Then the following assertions are equivalent*:

(i) f is in $H^p(w)$.

(ii) There exists $\alpha \geq 1$, $k > 0$, such that $A_{\alpha,k,2,0}(f) \in L^p(w)$.

(iii) For every $\alpha \geq 1$, and $k > 0$, $A_{\alpha,k,2,0}(f) \in L^p(w)$.

In addition, there exists $C > 0$ such that for any $f \in H^p(w)$,

$$\frac{1}{C}\|f\|_{H^p(w)} \leq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} \leq C\|f\|_{H^p(w)}.$$

Proof of Theorem 2.11. We already know that (ii) and (iii) are equivalent, so we only have to check the equivalence of (i) and (ii) for the case $k = 1$. The proof of (i) implies (ii) is given in [KaKo], using the arguments of [St2]. For the proof of (ii) implies (i), we will follow some ideas of [AhBrCa].

Without loss of generality we may assume that $f(0) = 0$. Let us assume first that $f \in H(\overline{\mathbf{B}^n})$. Then $f = P[f^*]$ where $f^* \in \mathcal{C}(\mathbf{S}^n)$. We want to check that

$$\|f^*\|_{L^p(w)} \leq C\|A_{\alpha,1,2,0}(f)\|_{L^p(w)}.$$

We will use that the dual space of $L^p(w)$ can be identified with $L^{p'}(w^{-(p'-1)})$ if the duality is given by

$$\langle f, g\rangle = \int_{\mathbf{S}^n} f(\zeta)\overline{g(\zeta)}\, d\sigma(\zeta).$$

Hence,

$$\|f^*\|_{L^p(w)} = \sup\left\{\left|\int_{\mathbf{S}^n} f^*(\zeta)g^*(\zeta)\, d\sigma(\zeta)\right|, g^* \in \mathcal{C}(\mathbf{S}^n), \|g^*\|_{L^{p'}(w^{-(p'-1)})} \leq 1\right\}.$$

If $g = P[g^*]$, we have (see [AhBrCa] page 131)

$$\begin{aligned}(2.8)\qquad &\frac{n\pi^n}{(n-1)!}\int_{\mathbf{S}^n} f^*(\zeta)g^*(\zeta)\, d\sigma(\zeta)\\ &\quad = n^2\int_{\mathbf{B}^n} f(z)g(z)\, dv(z) + \int_{\mathbf{B}^n} (\nabla_{\mathbf{B}^n} f(z), \nabla_{\mathbf{B}^n} g(z))_{\mathbf{B}^n}\frac{dv(z)}{1-|z|^2},\end{aligned}$$

where $\nabla_{\mathbf{B}^n}$ is the gradient in the Bergman metric (see for instance [St2]), and

$$(F(z), G(z))_{\mathbf{B}^n} = (1-|z|^2)\bigg(\sum_{i,j}(\delta_{i,j} - z_i\overline{z_j})F_i(z)\overline{G}_j(z)\bigg).$$

We then have (see [St2]) that since F is holomorphic

$$\begin{aligned}\|\nabla_{\mathbf{B}^n} F(z)\|^2_{\mathbf{B}^n} &= (\nabla_{\mathbf{B}^n} F(z), \nabla_{\mathbf{B}^n} F(z))_{\mathbf{B}^n} \\ &\simeq (1-|z|^2)\left\{\sum_{i=1}^n \left|\frac{\partial}{\partial z_i}F(z)\right|^2 - \left|\sum_{i=1}^n z_i \frac{\partial}{\partial z_i}F(z)\right|^2\right\}.\end{aligned}$$

In order to estimate $\int_{\mathbf{B}^n} f(z)g(z)\,dv(z)$ we will need to obtain estimates of the values of the functions f, g on compact subsets of $\mathbf{B}^n$ in terms of the norms $\|A_{\alpha,1,2,0}(f)\|_{L^p(w)}$ and $\|A_{\alpha,1,2,0}(g)\|_{L^{p'}(w^{-(p'-1)})}$ respectively.

LEMMA 2.12. *Let $1 < p < +\infty$ and w an A_p-weight. There exists $C > 0$ such that for any holomorphic function f in $\mathbf{B}^n$, and any $z = r\zeta$*

$$|f(z)| \preceq \left(|f(0)| + \int_0^r \frac{dt}{W(B(\zeta, 1-t^2))^{1/p}(1-t^2)}\|A_{\alpha,1,2,0}(f)\|^p_{L^p(w)}\right).$$

In particular, if $K \subset \mathbf{B}^n$ is compact and

$$\|f\|_K = \sup_{z\in K}|f(z)|,$$

then there exists a constant $C > 0$, depending only on w, p and K such that $\|f\|_K \leq C\big(|f(0)| + \|A_{\alpha,1,2,0}(f)\|_{L^p(w)}\big)$.

Proof of Lemma 2.12. Since f is holomorphic, we obtain that if $z = r\zeta \in \mathbf{B}^n$, there exist $C_i > 0$, $i = 1, 2$, such that for any $\eta \in B(\zeta, C_1(1-r^2))$, then

$$\begin{aligned}|\nabla f(z)|^2 &\preceq \frac{1}{(1-|z|^2)^{n+1}}\int_{K(z,C_2(1-|z|^2))}|\nabla f(y)|^2\,dv(y) \\ &\preceq \frac{1}{(1-|z|^2)^2}\int_{K(z,C_2(1-|z|^2))}(1-|y|^2)^{1-n}|\nabla f(y)|^2\,dv(y) \\ &\leq \frac{C}{(1-|z|^2)^2}(A_{\alpha,1,2,0}(f)(\eta))^2.\end{aligned}$$

Consequently

$$\big((1-|z|^2)|\nabla f(z)|\big)^p \preceq (A_{\alpha,1,2,0}(f)(\eta))^p.$$

Then we have

$$\begin{aligned}&\big((1-|z|^2)|\nabla f(z)|\big)^p W(B(\zeta,1-r^2))\\&\quad\preceq \int_{B(\zeta,1-r^2)} (A_{\alpha,1,2,0}(f)(\eta))^p w(\eta)\,d\sigma(\eta) \preceq \|A_{\alpha,1,2,0}(f)\|^p_{L^p(w)}.\end{aligned}$$

In particular, if $0<r<1$ and $\zeta\in\mathbf{S}^n$,

$$\left|\frac{\partial f}{\partial r}(r\zeta)\right| \preceq \frac{1}{W(B(\zeta,1-r^2))^{1/p}(1-r^2)}\|A_{\alpha,1,2,0}(f)\|_{L^p(w)},$$

and integrating, we finally obtain

$$|f(r\zeta)| \preceq \left(|f(0)| + \int_0^r \frac{dt}{W(B(\zeta,1-t^2))^{1/p}(1-t^2)}\|A_{\alpha,1,2,0}(f)\|_{L^p(w)}\right).$$

For the remaining affirmation, let $K\subset\mathbf{B}^n$ be compact. Then there exists $0<\delta<1$ such that for any $z=r\zeta\in K$, $r\le 1-\delta$, and

$$|f(z)| \preceq \left(|f(0)| + \frac{1}{W(B(\zeta,\delta))^{1/p}\delta}\|A_{\alpha,1,2,0}(f)\|_{L^p(w)}\right).$$

Since w is doubling, and there exists $N>0$ (not depending on ζ) such that $\mathbf{S}^n\subset B(\zeta,cN\delta))$, $W(\mathbf{S}^n)\preceq W(B(\zeta,\delta))$, and consequently

$$\|f\|_K \preceq |f(0)| + \|A_{\alpha,1,2,0}(f)\|_{L^p(w)}. \qquad \square$$

Going back to the proof of the Theorem 2.11, let $0<\varepsilon<1$. The above lemma together with the fact that if w is an A_p weight, then $w^{-(p'-1)}$ is an $A_{p'}$-weight, give by (2.8) that

$$\begin{aligned}&\left|\int_{\mathbf{S}^n} f^*(\zeta)g^*(\zeta)\,d\sigma(\zeta)\right| \preceq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)}\|A_{\alpha,1,2,0}(g)\|_{L^{p'}(w^{-(p'-1)})}\\&\quad+\left|\int_{1-\varepsilon\le|z|<1} f(z)g(z)\,dv(z)\right| + \int_{\mathbf{B}^n}\|\nabla_{\mathbf{B}^n}f(z)\|_{\mathbf{B}^n}\|\nabla_{\mathbf{B}^n}g(z)\|_{\mathbf{B}^n}\frac{dv(z)}{1-|z|^2}.\end{aligned}$$

In order to estimate the second integral, we use polar coordinates, and obtain

$$\left|\int_{1-\varepsilon\le|z|<1} f(z)g(z)\,dv(z)\right|,$$

which by Hölder's inequality is bounded by

$$\int_{1-\varepsilon}^{1}\int_{\mathbf{S}^n} |f(r\zeta)||g(r\zeta)|\, d\sigma(\zeta)dr \preceq \int_{1-\varepsilon}^{1} \|f_r\|_{L^p(w)}\|g_r\|_{L^{p'}(w^{-(p'-1)})}\, dr$$
$$\preceq \varepsilon\|f\|_{H^p(w)}\|g\|_{H^{p'}(w^{-(p'-1)})} \preceq \varepsilon\|f^*\|_{L^p(w)}\|g^*\|_{L^{p'}(w^{-(p'-1)})}.$$

For the third integral, we use (5.1) of [CoiMeSt] to estimate it by

$$\int_{\mathbf{S}^n} A_{\alpha,1,2,0}(f)(\zeta)A_{\alpha,1,2,0}(g)(\zeta)\, d\sigma(\zeta)$$
$$\preceq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)}\|A_{\alpha,1,2,0}(g)\|_{L^{p'}(w^{-(p'-1)})}.$$

Since we already know (see [KaKo]) that $\|A_{\alpha,1,2,0}(g)\|_{L^{p'}(w^{-(p'-1)})} \preceq \|g^*\|_{L^{p'}(w^{-(p'-1)})}$, we finally obtain

$$\|f^*\|_{L^p(w)} \preceq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)} + \varepsilon\|f^*\|_{L^p(w)},$$

which gives the result for $f \in H(\overline{\mathbf{B}}^n)$.

So we are left to show that the estimate we have already obtained holds for a general holomorphic function in $\mathbf{B}^n$. If f is an holomorphic function on $\mathbf{B}^n$ such that $\|A_{\alpha,1,2,0}(f)\|_{L^p(w)} < +\infty$, let $f_r(z) = f(rz) \in H(\overline{\mathbf{B}^n})$, for $0 < r < 1$. We then have that

$$\|f_r\|_{H^p(w)} \preceq \|A_{\alpha,1,2,0}(f_r)\|_{L^p(w)}. \tag{2.9}$$

Let us check first that

$$\sup_r \|A_{\alpha,1,2,0}(f_r)\|_{L^p(w)} \leq C\|A_{\alpha,1,2,0}(f)\|_{L^p(w)}.$$

Notice that

$$\|A_{\alpha,1,2,0}(f_r)\|^p_{L^p(w)} = \|J_\alpha((1-|\cdot|^2)(I+R)f_r)\|_{L^p(w)(L^2(\frac{dv(z)}{(1-|z|2)^{n+1}}))}.$$

We will check that there exists $0 \leq G(\zeta,z) \in L^p(w)(L^2(\frac{dv(z)}{(1-|z|^2)^{n+1}}))$ such that for any $0<r<1$, $\zeta\in\mathbf{S}^n$, $z\in\mathbf{B}^n$, $J_\alpha((1-|\cdot|^2)(I+R)f_r)(\zeta,z) \leq G(\zeta,z)$, and $\|G\|_{L^p(w)(L^2(\frac{dv(z)}{(1-|z|^2)^{n+1}}))} \preceq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)}$.

Let us obtain such a function G. Since by hypothesis $A_{\alpha,1,2,0}f \in L^p(w)$, we have that the holomorphic function f satisfies that $A_{\alpha,1,2,0}f \in L^1(d\sigma)$, and consequently that there exists $C>0$ such that for any $z\in\mathbf{B}^n$, $|f(z)| \preceq$

$1/(1-|z|^2)^n$. Hence, the integral representation theorem gives that for $N > 0$ sufficiently large, and $z \in \mathbf{B}^n$,

$$(I+R)f(rz) = C\int_{\mathbf{B}^n} \frac{(1-|y|^2)^N (I+R)f(y)}{(1-rz\overline{y})^{n+1+N}}\, dv(y).$$

Next, there is a constant $C > 0$ such that for any $0 < r < 1$, $z, y \in \mathbf{B}^n$, $|1-rz\overline{y}| \geq C|1-z\overline{y}|$, and the above formula gives that

$$|(I+R)f(rz)| \preceq \int_{\mathbf{B}^n} \frac{(1-|y|^2)^N |(I+R)f(y)|}{|1-z\overline{y}|^{n+1+N}}\, dv(y).$$

Combining the above results we have that

$$\begin{aligned}
&\chi_{D_\alpha(\zeta)}(z)(1-|z|^2)|(I+R)f(rz)| \\
&\preceq \chi_{D_\alpha(\zeta)}(z)\int_{\mathbf{B}^n} \frac{(1-|y|^2)^{N-1}(1-|z|^2)((1-|y|^2)|(I+R)f(y)|)}{|1-z\overline{y}|^{n+1+N}}\, dv(y) \\
&= C\chi_{D_\alpha(\zeta)}(z) P^{N-1,1}((1-|\cdot|^2)(I+R)f)(z) := G(z,\zeta).
\end{aligned}$$

Theorem 2.8 shows that provided N is chosen sufficiently large, $P^{N-1,1}$ maps $F^{\alpha,p,2}(w)$ to itself, and in particular that

$$\begin{aligned}
\|G\|_{L^p(w)(L^2(\frac{dv(z)}{(1-|z|^2)^{n+1}}))} &= \|P^{N-1,1}((1-|\cdot|^2)(I+R)f)\|_{\alpha,p,2,w} \\
&\preceq \|(1-|\cdot|^2)(I+R)f\|_{\alpha,p,2,w} = C\|A_{\alpha,1,2,0}(f)\|_{L^p(w)} < +\infty.
\end{aligned}$$

Consequently

$$\|f_r\|_{H^p(w)} \preceq \|A_{\alpha,1,2,0}(f)\|_{L^p(w)},$$

and therefore $f \in H^p(w)$. □

We will now remark on some facts about weighted Hardy-Sobolev spaces. Let us recall, that if $1 < p < +\infty$, $0 < s < n$, and w is an A_p-weight, we denote by $H^p_s(w)$ the space of holomorphic functions f on $\mathbf{B}^n$ satisfying that

$$\|f\|_{H^p_s(w)} = \|(I+R)^s f\|_{H^p(w)} < +\infty.$$

The results obtained in the previous theorems give alternative equivalent definitions of the spaces $H^p_s(w)$ in terms of admissible maximal or radial functions and admissible area functions.

On the other hand, when $w \equiv 1$, and $0 < s < n$, it is well known, see for instance [CaOr1], that the space H^p_s admits a representation in terms of a fractional Cauchy-type kernel C_s defined by

$$C_s(z,\zeta) = \frac{1}{(1 - z\overline{\zeta})^{n-s}}.$$

The same lines of the proof of the unweighted case can be used to obtain a similar characterization in the weighted case. We just have to use that the Hardy-Littlewood maximal operator is bounded in $L^p(w)$, if w is an A_p-weight and Lemma 2.1.

THEOREM 2.13. *Let $1 < p < +\infty$, $0 < s < n$, and w be an A_p-weight. We then have that the map*

$$C_s(f)(z) = \int_{\mathbf{B}^n} \frac{f(\zeta)}{(1 - z\overline{\zeta})^{n-s}}\, d\sigma(\zeta),$$

is a bounded map of $L^p(w)$ onto $H^p_s(w)$.

§3. Holomorphic potentials and Carleson measures

In this section we will study Carleson measures for $H^p_s(w)$, $1 < p < +\infty$ and $0 < s < n$, that is, the positive finite Borel measures μ on $\mathbf{B}^n$ satisfying

$$\|f\|_{L^p(d\mu)} \le C\|f\|_{H^p_s(w)}, \quad f \in L^p(w). \tag{3.1}$$

In what follows we will write

$$⨍_E w\, d\sigma = \frac{1}{|E|}\int_E w,$$

where E is a measurable set in $\mathbf{S}^n$ and $|E|$ denotes its Lebesgue measure.

By Theorem 2.13, this inequality can be rewritten as follows:

$$\|C_s(f)\|_{L^p(d\mu)} \le C\|f\|_{L^p(w)}, \quad f \in L^p(w). \tag{3.2}$$

We recall that we have defined the non-isotropic potential of a positive Borel function f on $\mathbf{S}^n$ by

$$K_s(f)(z) = \int_{\mathbf{S}^n} K_s(z,\zeta) f(\zeta)\, d\sigma(\zeta) = \int_{\mathbf{S}^n} \frac{f(\zeta)}{|1 - z\overline{\zeta}|^{n-s}}\, d\sigma(\zeta), \tag{3.3}$$

for $z \in \overline{\mathbf{B}}^n$.

Analogously to what happens for isotropic potentials (see [Ad]), in the nonisotropic case it can be proved that if w is an A_p weight and $\zeta_0 \in \mathbf{S}^n$ satisfies that

$$\int_{\mathbf{S}^n} \frac{1}{|1-\zeta_0\overline{\zeta}|^{(n-s)p'}} w^{-(p'-1)}(\zeta)\, d\sigma(\zeta) < +\infty, \tag{3.4}$$

then for any $f \in L^p(w)$, $K_s(f)$ is continuous in ζ_0. Observe that when $w \equiv 1$, (3.4) holds if and only if $n - sp < 0$. In the general weighted case, if w satisfies a doubling condition of order τ, and $\tau - sp < 0$, we also have that (3.4) holds, and consequently the Carleson measures in this case for weighted Hardy Sobolev spaces are just the finite ones. Indeed, assume that $\tau - sp < 0$. We then have

$$\begin{aligned}
&\int_{\mathbf{S}^n} \frac{1}{|1-\zeta_0\overline{\zeta}|^{(n-s)p'}} w^{-(p'-1)}(\zeta)\, d\sigma(\zeta) \\
&= \int_{\mathbf{S}^n} w^{-(p'-1)}(\zeta) \int_{|1-\zeta_0\overline{\zeta}|<t} \frac{dt}{t^{(n-s)p'}}\, d\sigma(\zeta) \le \int_0^K \frac{\int_{B(\zeta_0,t)} w^{-(p'-1)}}{t^{(n-s)p'}} \\
&\simeq \int_0^K \frac{t^n\, dt}{\left(⨍_{B(\zeta_0,t)} w\right)^{p'-1} t^{(n-s)p'}} \preceq \sum_k \frac{2^{-ksp'}}{W(B(\zeta_0, 2^{-k}))}.
\end{aligned}$$

The fact that w satisfies condition D_τ gives that $W(\mathbf{S}^n) \preceq 2^{k\tau} W(B(\zeta_0, 2^{-k}))$, and consequently the above sum is bounded, up to constants, by

$$\sum_k 2^{k(\tau(p'-1)-sp')}.$$

Since $\tau - sp < 0$ we also have that $\tau(p'-1) - sp' < 0$, and we are done.

From now on we will assume that $\tau - sp \ge 0$.

The problem of characterizing the positive finite Borel measures μ on $\mathbf{B}^n$ for which the following inequality holds

$$\|K_s(f)\|_{L^p(d\mu)} \le C \|f\|_{L^p(w)}, \tag{3.5}$$

has been thoroughly studied, and there are, among others, characterizations in terms of weighted nonisotropic Riesz capacities that are defined as follows: if $E \subset \mathbf{S}^n$, $1 < p < +\infty$ and $0 < s < n$,

$$C_{sp}^w(E) = \inf\{\|f\|_{L^p(w)}^p\,;\, f \ge 0,\ K_s(f) \ge 1 \text{ on } E\}.$$

It is well known, that when $w \equiv 1$, $C_{sp}(B(\zeta,r)) \simeq r^{n-sp}$, $\zeta \in \mathbf{S}^n$, $r<1$. See [Ad] for expressions of weighted capacities of balls in $\mathbf{R}^n$.

As it happens in $\mathbf{R}^n$ (see [Ad]), we have that if $0 \le n-sp$, (3.5) holds if and only if there exists $C>0$ such that for any open set $G \subset \mathbf{S}^n$,

$$\mu(T(G)) \le C C_{sp}^w(G). \tag{3.6}$$

Here $T(G)$ is the admissible tent over G, defined by

$$T(G) = T_\alpha(G) = \left(\bigcup_{\zeta \notin G} D_\alpha(\zeta) \right)^c .$$

The problem of characterizing the Carleson measures μ for the holomorphic case (3.2) is much more complicated, even in the nonweighted case. Since $|C_s(z,\zeta)| \le K_s(z,\zeta)$, it follows from Theorem 2.13, that (3.5) implies (3.2), and consequently that if condition (3.6) is satisfied, then μ is a Carleson measure for $H_s^p(w)$. Of course, when $n-s<1$ both problems are equivalent, even in the weighted case, simply because if $f \ge 0$, $|C_s(f)| \simeq K_s(f)$, but when $n>1$ (see [Ah] and [CaOr2]), condition (3.5) for the unweighted case is not, in general, equivalent to condition (3.2). Observe that when $n-sp \le 0$, H_s^p consists of regular functions, and consequently any finite measure is a Carleson measure for the holomorphic and the real case. It is proved in [CohVe1] that this equivalence still remains true if we are not too far from the regular case, namely, if $0 \le n-sp < 1$. The main purpose of this section is to obtain a result in this line for a wide class of A_p-weights.

In [Ah] it is also shown that if (3.2) holds for $w \equiv 1$, then the capacity condition on balls is satisfied, i.e. there exists $C>0$ such that $\mu(T(B(\zeta,r))) \le Cr^{n-sp}$, for any $\zeta \in \mathbf{S}^n$ and any $0<r<1$. The following proposition obtains a necessary condition in this line for the weighted holomorphic trace inequality.

Proposition 3.1. *Let $1<p<+\infty$, $0<s<n$. Let μ be a positive finite Borel measure on $\mathbf{B}^n$, and w be an A_p-weight. Assume that there exists $C>0$ such that*

$$\|f\|_{L^p(d\mu)} \le C\|f\|_{H_s^p(w)},$$

for any $f \in H_s^p(w)$. We then have that there exists $C>0$ such that for any $\zeta \in \mathbf{S}^n$, $r>0$,

$$\mu(T(B(\zeta,r)) \le C\frac{W(B(\zeta,r))}{r^{sp}}.$$

Proof of Proposition 3.1. Let $\zeta \in \mathbf{S}^n$, $0 < r < 1$ be fixed. If $z \in \overline{\mathbf{B}}^n$, let

$$F(z) = \frac{1}{(1-(1-r)z\overline{\zeta})^N},$$

with $N > 0$ to be chosen later. If $z \in T(B(\zeta, r))$, and $z_0 = z/|z|$, $(1-|z|) \preceq r$ and $|1 - z_0\overline{\zeta}| \preceq r$. Hence $|1-(1-r)z\overline{\zeta}| \preceq r$, and consequently,

$$\frac{\mu(T(B(\zeta,r)))}{r^{Np}} \leq C \int_{T(B(\zeta,r))} |F(z)|^p \, d\mu(z).$$

On the other hand,

$$\begin{aligned}
\|F\|^p_{H^p_s(w)} &\leq C \int_{\mathbf{S}^n} \frac{1}{|1-(1-r)\eta\overline{\zeta}|^{(N+s)p}} w(\eta)\, d\sigma(\eta) \\
&= \int_{B(\zeta,r)} \frac{1}{|1-(1-r)\eta\overline{\zeta}|^{(N+s)p}} w(\eta)\, d\sigma(\eta) \\
&\qquad + \sum_{k\geq 1} \int_{B(\zeta,2^{k+1}r)\setminus B(\zeta,2^k r)} \frac{1}{|1-(1-r)\eta\overline{\zeta}|^{(N+s)p}} w(\eta)\, d\sigma(\eta).
\end{aligned}$$

If $k \geq 1$, and $\eta \in B(\zeta, 2^{k+1}r) \setminus B(\zeta, 2^k r)$, $|1-(1-r)\eta\overline{\zeta}| \simeq 2^k r$. This estimates together with the fact that w is doubling, give that the above is bounded by

$$\sum_{k\geq 0} \frac{W(B(\zeta, 2^{k+1}r))}{(2^k r)^{(N+s)p}} \preceq \frac{W(B(\zeta,r))}{r^{(N+s)p}} \sum_{k\geq 0} \left(\frac{C}{2^{(N+s)p}} \right)^k,$$

which gives the desired estimate, provided N is chosen big enough. ∎

We observe that for some special weights besides the case $w \equiv 1$, the expression that appears in the above proposition $W(B(\zeta,r))/r^{sp}$ coincide with the weighted capacity of a ball (see [Ad]).

If ν is a positive Borel measure on $\mathbf{S}^n$, $1 < p < +\infty$, $0 < s < n$ and w is an A_p-weight, it is introduced in [Ad] the (s,p)-energy of ν with weight w, which is defined by

$$\mathcal{E}^w_{sp}(\nu) = \int_{\mathbf{S}^n} (K_s(\nu)(\zeta))^{p'} w(\zeta)^{-(p'-1)}\, d\sigma(\zeta). \tag{3.7}$$

If we write $(K_s(\nu))^{p'} = (K_s(\nu))^{p'-1} K_s(\nu)$, Fubini's theorem gives that

$$\mathcal{E}^w_{sp}(\nu) = \int_{\mathbf{S}^n} \mathcal{U}^w_{sp}(\nu)(\zeta)\, d\nu(\zeta),$$

where

$$\mathcal{U}^w_{sp}(\zeta) = K_s(w^{-1}K_s(\nu))^{p'-1}(\zeta)$$

is the weighted nonlinear potential of the measure ν. When $w \equiv 1$, Wolff's theorem (see [HeWo]) gives another representation of the energy, in terms of the so-called Wolff's potential.

In the general case, it is introduced in [Ad] a weighted Wolff-type potential of a measure ν as

$$\begin{aligned}(3.8)\qquad &\mathcal{W}^w_{sp}(\nu)(\zeta)\\ &= \int_0^1 \left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-sp}}\right)^{p'-1} \fint_{B(\zeta,1-r)} w^{-(p'-1)}(\eta)\, d\sigma(\eta) \frac{dr}{1-r}.\end{aligned}$$

In the same paper, it is shown that provided w is an A_p-weight, the following weighted Wolff-type theorem holds:

$$\mathcal{E}^w_{sp}(\nu) \simeq \int_{\mathbf{S}^n} \mathcal{W}^w_{sp}(\nu)(\zeta)\, d\nu(\zeta). \tag{3.9}$$

In fact, we have the pointwise estimate $\mathcal{W}^w_{sp}(\nu)(\zeta) \le C\mathcal{U}^w_{sp}(\nu)(\zeta)$, and Wolff's theorem gives that the converse is true, provided we integrate with respect to ν.

In [Ad] a weighted extremal theorem for the weighted Riesz capacities it is also shown, namely, if $G \subset \mathbf{S}^n$ is open, there exists a positive capacitary measure ν_G such that

(i) $\operatorname{supp} \nu_G \subset G$.
(ii) $\nu_G(G) = C^w_{sp}(G) = \mathcal{E}^w_{sp}(\nu_G)$.
(iii) $\mathcal{W}^w_{sp}(\nu_G)(\zeta) \ge C$, for C^w_{sp}-a.e. $\zeta \in G$.
(iv) $\mathcal{W}^w_{sp}(\nu_G)(\zeta) \le C$, for any $\zeta \in \operatorname{supp} \nu_G$.

We now introduce two holomorphic weighted Wolff-type potentials, which generalize the ones defined in [CohVe1]. These potentials will be used in the proof of the characterization of the Carleson measures for $H^p_s(w)$, for the case $0 \le \tau - sp < 1$. Let $1 < p < +\infty$, $0 < s < n/p$, and ν be a positive Borel measure on $\mathbf{S}^n$. For any $\lambda > 0$, and $z \in \mathbf{B}^n$, we set

$$\begin{aligned}(3.10)\qquad \mathcal{U}^{w\lambda}_{sp}(\nu)(z) = \int_0^1 \int_{\mathbf{S}^n} &\left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-sp}}\right)^{p'-1} \frac{(1-r)^{\lambda-n}}{(1-rz\overline{\zeta})^{\lambda}}\\ &\times \left(\fint_{B(\zeta,1-r)} w^{-(p'-1)}\right) d\sigma(\zeta) \frac{dr}{1-r},\end{aligned}$$

and

$$\mathcal{V}^{w\lambda}_{sp}(\nu)(z) = \int_0^1 \left(\int_{\mathbf{S}^n} \frac{(1-r)^{\lambda+sp-n}}{(1-rz\overline{\zeta})^{\lambda}} \times \left(\fint_{B(\zeta,1-r)} w^{-(p'-1)} \right)^{\frac{1}{p'-1}} d\nu(\zeta) \right)^{p'-1} \frac{dr}{1-r}. \tag{3.11}$$

Obviously, both potentials are holomorphic functions in the unit ball. We will see, that if $p \leq 2$ the first one is bounded from below by the weighted Wolff-type potential we have just introduced, whereas if $p \geq 2$, the second one is bounded from below by the same potential.

In the unweighted case, [CohVe1] the proof of the estimates of the holomorphic potentials, rely on an extension of Wolff's theorem. This extension gives that if $1 < p < +\infty$, $s > 0$, $0 < q < +\infty$, and ν is a positive Borel measure on $\mathbf{S}^n$, then

$$\int_{\mathbf{S}^n} \left(\int_0^1 \left(\frac{\nu(B(\zeta,t))}{t^{n-s}} \right)^q \frac{dt}{t} \right)^{p'/q} d\sigma(\zeta) \preceq \int_{\mathbf{S}^n} \mathcal{W}^w_{sp}(\nu)(\zeta)\, d\nu(\zeta).$$

Observe that if the above estimate holds for one q_0, it also holds for any $q \geq q_0$. The case $q = 1$ is the integral estimate in Wolff's theorem, since we have that

$$\mathcal{E}_{sp}(\nu) \simeq \int_{\mathbf{S}^n} \left(\int_0^1 \frac{\nu(B(\zeta,t))}{t^{n-s}} \frac{dt}{t} \right)^{p'} d\sigma(\zeta).$$

The arguments in [CohVe1] can easily be used to show the following weighted version of the above theorem. We omit the details of the proof.

Theorem 3.2. *Let $1 < p < +\infty$, w an A_p weight, $s > 0$, $K > 0$, $0 < q < +\infty$, and ν be a positive Borel measure on $\mathbf{S}^n$. Then*

(3.12)
$$\int_{\mathbf{S}^n} \left(\int_0^K \left(\frac{\nu(B(\zeta,t))}{t^{n-s}} \left(\fint_{B(\zeta,t)} w^{-(p'-1)}(\eta)\, d\sigma(\eta) \right)^{\frac{1}{p'-1}} \right)^q \frac{dt}{t} \right)^{p'/q} w(\zeta)\, d\sigma(\zeta) \preceq \int_{\mathbf{S}^n} \mathcal{W}^w_{sp}(\nu)(\zeta)\, d\nu(\zeta).$$

Before we obtain estimates of the $H^p_s(w)$-norm of the weighted holomorphic potentials already introduced, we will give a characterization for weights satisfying a doubling condition

LEMMA 3.3. *Let $1 < p < +\infty$ and w be an A_p weight on $\mathbf{S}^n$, and assume that $w \in D_\tau$, for some $\tau > 0$. We then have*:

(i) For any $t \in \mathbf{R}$ satisfying that $t > \tau - n$, there exists $C > 0$ such that

$$\int_r^{+\infty} \frac{1}{x^t} \fint_{B(\zeta,x)} w \frac{dx}{x} \leq C \frac{1}{r^t} \fint_{B(\zeta,r)} w, \tag{3.13}$$

$r < 1$, $\zeta \in \mathbf{S}^n$.

(ii) For any $t \in \mathbf{R}$ satisfying that $t > \tau - n$, there exists $C > 0$ such that

$$\int_0^r x^t \left(\fint_{B(\zeta,x)} w^{-(p'-1)} \right)^{p-1} \frac{dx}{x} \leq C r^t \left(\fint_{B(\zeta,r)} w^{-(p'-1)} \right)^{p-1}, \tag{3.14}$$

$r < 1$, $\zeta \in \mathbf{S}^n$.

Proof of Lemma 3.3. We begin with the proof of part (i). Let $t > \tau - n$. Then

$$\begin{aligned}
\int_r^{+\infty} \frac{1}{x^t} \fint_{B(\zeta,x)} w \frac{dx}{x} &= \sum_{k \geq 0} \int_{2^k r}^{2^{k+1} r} \frac{1}{x^t} \fint_{B(\zeta,x)} w \frac{dx}{x} \\
&\preceq \sum_{k \geq 0} \frac{1}{2^{k(t+n)} r^{t+n}} W(B(\zeta, 2^{k+1} r)) \preceq \sum_{k \geq 0} \frac{1}{2^{k(t+n)} r^{t+n}} 2^{k\tau} W(B(\zeta, r)) \\
&= C \frac{1}{r^\delta} \fint_{B(\zeta,r)} w,
\end{aligned}$$

since w is in D_τ, and $t + n > \tau$.

Next we show that (ii) holds. If $\zeta \in \mathbf{S}^n$ and $r > 0$, the fact that $w \in A_p$ gives that $\left(\fint_{B(\zeta,x)} w^{-(p'-1)} \right)^{p-1} \simeq \left(\fint_{B(\zeta,x)} w \right)^{-1}$, and consequently,

$$\begin{aligned}
\int_0^r x^t \left(\fint_{B(\zeta,x)} w^{-(p'-1)} \right)^{p-1} \frac{dx}{x} &= \sum_{k \geq 0} \int_{2^{-k} r}^{2^{-k+1} r} x^t \left(\fint_{B(\zeta,x)} w^{-(p'-1)} \right)^{p-1} \frac{dx}{x} \\
&\preceq \sum_{k \geq 0} 2^{-kt} r^t \frac{1}{\fint_{B(\zeta, 2^{-k} r)} w} \preceq \sum_{k \geq 0} \frac{1}{2^{k(t+n)}} r^{t-n} 2^{k\tau} W(B(\zeta, r)) \\
&\simeq r^t \left(\fint_{B(\zeta,r)} w^{-(p'-1)} \right)^{p-1}. \qquad \square
\end{aligned}$$

Remark. In fact, it can be proved that both conditions (i) and (ii) are in turn equivalent to the fact that the A_p weight is in D_τ.

We can now obtain the estimates on the weighted holomorphic potentials defined in (3.10) and (3.11).

THEOREM 3.4. *Let $1 < p < +\infty$, $0 < \alpha < n$, w an A_p-weight. Assume that w is in D_τ for some $0 \le \tau - sp < 1$. We then have*:

(1) *If $1 < p < 2$, there exists $0 < \lambda < 1$ and $C > 0$ such that for any finite positive Borel measure ν on $\mathbf{S}^n$ the following assertions hold*:

a) *For any $\eta \in \mathbf{S}^n$,*

$$\lim_{\rho\to 1} \operatorname{Re} \mathcal{U}^{w\lambda}_{sp}(\nu)(\rho\eta) \ge C\mathcal{W}^{w\lambda}_{sp}(\nu)(\eta).$$

b) $\|\mathcal{U}^{w\lambda}_{sp}(\nu)\|^p_{H^p_s(w)} \le C\mathcal{E}^w_{sp}(\nu)$.

(2) *If $p \ge 2$, there exists $0 < \lambda < 1$ and $C > 0$ such that for any finite positive Borel measure ν on $\mathbf{S}^n$ the following assertions hold*:

a) *For any $\eta \in \mathbf{S}^n$,*

$$\lim_{\rho\to 1} \operatorname{Re} \mathcal{V}^{w\lambda}_{sp}(\nu)(\rho\eta) \ge C\mathcal{W}^{w\lambda}_{sp}(\nu)(\eta).$$

b) $\|\mathcal{V}^{w\lambda}_{sp}(\nu)\|^p_{H^p_s(w)} \le C\mathcal{E}^w_{sp}(\nu)$.

Proof of Theorem 3.4. We will follow the scheme of [CohVe1] where it is proved for the unweighted case. The weights introduce new technical difficulties that require a careful use of the hypothesis A_p and D_τ that we assume on the weight w. In order to make the proof easier to follow we sketch some of the arguments in [CohVe1], emphasizing the necessary changes we need to make in the weighted case.

Let us prove (1). We choose λ such that $\tau - sp < \lambda < 1$ and define $\mathcal{U}^{w\lambda}_{sp}$ as in 3.10. Then $\tau - s < \frac{\lambda+s-\tau(2-p)}{p-1}$. Consequently there exists t such that $\tau - s < t < \frac{\lambda+s-\tau(2-p)}{p-1}$. Observe that $t + s - n > \tau - n$ and $\frac{\lambda+s-t(p-1)}{2-p} - n > \tau - n$.

We begin now the proof of a). The fact that $\lambda < 1$ gives that if $\rho < 1$, $\eta \in \mathbf{S}^n$, and $C > 0$,

$$\operatorname{Re}\mathcal{U}_{sp}^{w\lambda}(\rho\eta) \succeq \int_0^1 \int_{B(\eta,C(1-r))} \left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-sp}}\right)^{p'-1} \frac{(1-r)^{\lambda-n}}{|1-r\rho\eta\overline{\zeta}|^\lambda} \times \left(\fint_{B(\zeta,1-r)} w^{-(p'-1)}\right) d\sigma(\zeta)\frac{dr}{1-r}.$$

If $C > 0$ has been chosen small enough, we have that for any $\zeta \in B(\eta, C(1-r))$, $B(\eta, C(1-r)) \subset B(\zeta, 1-r)$. In addition, $|1 - r\rho\eta\overline{\zeta}| \preceq |1 - r\rho|$. These estimates, together with the fact that $w^{-(p'-1)}$ satisfies a doubling condition, give that the above integral is bounded from below by

$$\begin{aligned} &C\int_0^1 \int_{B(\eta,C(1-r))} \left(\frac{\nu(B(\eta,C(1-r)))}{(1-r)^{n-sp}}\right)^{p'-1} \frac{(1-r)^{\lambda-n}}{|1-r\rho|^\lambda} \times \left(\fint_{B(\eta,1-r)} w^{-(p'-1)}\right) d\sigma(\zeta)\frac{dr}{1-r} \\ &\geq C\int_0^\rho \left(\frac{\nu(B(\eta,C(1-r)))}{(1-r)^{n-sp}}\right)^{p'-1} \frac{(1-r)^{\lambda}}{|1-r\rho|^\lambda}\left(\fint_{B(\eta,1-r)} w^{-(p'-1)}\right)\frac{dr}{1-r} \\ &\geq C\int_0^\rho \left(\frac{\nu(B(\eta,C(1-r)))}{(1-r)^{n-sp}}\right)^{p'-1} \left(\fint_{B(\eta,1-r)} w^{-(p'-1)}\right)\frac{dr}{1-r}, \end{aligned}$$

where in last estimate we have used that since $r < \rho$, $1 - r\rho \simeq 1 - r$.

We have proved then

$$\int_0^\rho \left(\frac{\nu(B(\eta,C(1-r)))}{(1-r)^{n-sp}}\right)^{p'-1} \left(\fint_{B(\eta,1-r)} w^{-(p'-1)}\right)\frac{dr}{1-r} \leq C\operatorname{Re}\mathcal{U}_{sp}^{w\lambda}(\nu)(\rho\eta),$$

and letting $\rho \to 1$, we obtain a).

In order to obtain the norm estimate, lets us simply write $\mathcal{U}(z) = \mathcal{U}_{sp}^{w\lambda}(\nu)(z)$, and prove that for $k > s$,

$$\begin{aligned} &\|\mathcal{U}\|^p_{HF_s^{p1}(w)} \\ &= |\mathcal{U}(0)|^p + \int_{\mathbf{S}^n}\left(\int_0^1 (1-\rho)^{k-s}|(I+R)^k\mathcal{U}(\rho\eta)|\frac{d\rho}{1-\rho}\right)^p w(\eta)\,d\sigma(\eta) \\ &\leq C\mathcal{E}_{sp}^w(\nu). \end{aligned}$$

But

$$\int_0^1 (1-\rho)^{k-s}|(I+R)^k\mathcal{U}(\rho\eta)|\frac{d\rho}{1-\rho}$$
$$\preceq \int_0^1 (1-\rho)^{k-s}\int_0^1\int_{\mathbf{S}^n}\left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-sp}}\right)^{p'-1}\frac{(1-r)^{\lambda-n}}{|1-\rho r\eta\overline{\zeta}|^{\lambda+k}}$$
$$\times\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)d\sigma(\zeta)\frac{dr}{1-r}\frac{d\rho}{1-\rho}\preceq \Upsilon(\eta),$$

where

$$\Upsilon(\eta)=\int_0^1\int_{\mathbf{S}^n}\left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-sp}}\right)^{p'-1}\frac{(1-r)^{\lambda-n}}{|1-r\eta\overline{\zeta}|^{\lambda+s}}$$
$$\times\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)d\sigma(\zeta)\frac{dr}{1-r}.$$

Observe that $|\mathcal{U}(0)|^p \leq C\|\Upsilon\|^p_{L^p(w)}$. Consequently, in order to finish the proof of the theorem, we just need to show that

$$\|\Upsilon\|^p_{L^p(w)}\leq C\mathcal{E}^w_{sp}(\nu). \tag{3.15}$$

Hölder's inequality with exponent $\frac{1}{p-1}>1$ gives that

$$\Upsilon(\eta)\leq \Upsilon_1(\eta)^{p-1}\Upsilon_2(\eta)^{2-p}, \tag{3.16}$$

where

$$\Upsilon_1(\eta)=\int_0^1\int_{\mathbf{S}^n}\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-s}}\frac{(1-r)^{t-n}}{|1-r\eta\overline{\zeta}|^{t}}$$
$$\times\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^{p-1}d\sigma(\zeta)\frac{dr}{1-r},$$

and

$$\Upsilon_2(\eta)=\int_0^1\int_{\mathbf{S}^n}\left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-s}}\right)^{p'}\frac{(1-r)^{\frac{\lambda+s-t(p-1)}{2-p}-n}}{|1-r\eta\overline{\zeta}|^{\frac{\lambda+s-t(p-1)}{2-p}}}$$
$$\times\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^{p}\frac{d\sigma(\zeta)dr}{1-r}.$$

We begin estimating the function Υ_1. If $\zeta \in B(\tau, 1-r)$, we have that $B(\zeta, 1-r) \subset B(\tau, C(1-r))$, and since $w^{-(p'-1)}$ satisfies a doubling condition,

$$\Upsilon_1(\eta) \preceq \int_0^1 (1-r)^{t-2n+s} \int_{\mathbf{S}^n} \int_{B(\tau,C(1-r))} \frac{d\sigma(\zeta)}{|1-r\eta\overline{\zeta}|^t} \times \left(\fint_{B(\tau,1-r)} w^{-(p'-1)} \right)^{p-1} \frac{d\nu(\tau)dr}{1-r}. \tag{3.17}$$

Next, we observe that if $\zeta \in B(\tau, C(1-r))$, $|1-r\eta\overline{\tau}| \preceq |1-r\eta\overline{\zeta}|$. Hence, the above is bounded by

$$C \int_0^1 (1-r)^{t-n+s} \int_{\mathbf{S}^n} \frac{\left(\fint_{B(\tau,1-r)} w^{-(p'-1)} \right)^{p-1}}{|1-r\eta\overline{\tau}|^t} d\nu(\tau) \frac{dr}{1-r}.$$

Since

$$\int_{\mathbf{S}^n} \frac{\left(\fint_{B(\tau,1-r)} w^{-(p'-1)} \right)^{p-1}}{|1-r\eta\overline{\tau}|^t} d\nu(\tau) \preceq \int_{\mathbf{S}^n} \left(\fint_{B(\tau,1-r)} w^{-(p'-1)} \right)^{p-1} \int_{|1-r\eta\overline{\tau}|\leq\delta} \frac{d\delta}{\delta^{t+1}} d\nu(\tau),$$

the above estimate, together with Fubini's theorem and the fact that $t-n+s > \tau - n$ give that $\Upsilon_1(\eta)$ is bounded by

$$C \int_0^1 \int_{B(\eta,\delta)} \delta^{t-n+s} \left(\fint_{B(\tau,\delta)} w^{-(p'-1)} \right)^{p-1} d\nu(\tau) \frac{d\delta}{\delta^{t+1}} \preceq \int_0^1 \left(\fint_{B(\eta,\delta)} w^{-(p'-1)} \right)^{p-1} \frac{\nu(B(\eta,\delta))}{\delta^{n-s}} \frac{d\delta}{\delta},$$

where we have used the fact that if $\tau \in B(\eta, \delta)$, then $B(\tau,\delta) \subset B(\eta, C\delta)$, for some $C > 0$ and that $w^{-(p'-1)}$ satisfies a doubling condition.

Applying Hölder's inequality with exponent $\frac{1}{(p-1)^2} > 1$, we deduce that

$$\|\Upsilon\|_{L^p(w)} \preceq \left(\int_{\mathbf{S}^n} \left(\int_0^1 \left(\fint_{B(\eta,1-r)} w^{-(p'-1)} \right)^{p-1} \times \frac{\nu(B(\eta,\delta))}{\delta^{n-s}} \frac{d\delta}{\delta} \right)^{p'} w \, d\sigma \right)^{(p-1)^2} \left(\int_{\mathbf{S}^n} \Upsilon_2 w \right)^{p(2-p)}. \tag{3.18}$$

Theorem 3.2 with $q = 1$ gives that the first factor on the right is bounded by $C\mathcal{E}_{sp}^w(\nu)^{(p-1)^2}$.

Next we deal with the integral involving Υ_2. We recall that $l = \frac{\lambda+s-t(p-1)}{2-p} - n > \tau - n$. Fubini's theorem gives that

$$\int_{\mathbf{S}^n} \Upsilon_2 w = \int_{\mathbf{S}^n} \int_0^1 \left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-s}}\right)^{p'} (1-r)^l \left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^p \times \int_{\mathbf{S}^n} \frac{w(\eta)\,d\sigma(\eta)}{|1-r\eta\overline{\zeta}|^{l+n}} \frac{d\sigma(\zeta)dr}{1-r}.$$

But, as before, since $l > \tau - n$,

$$\int_{\mathbf{S}^n} \frac{w(\eta)\,d\sigma(\eta)}{|1-r\eta\overline{\zeta}|^{l+n}} \le \frac{C}{(1-r)^l} ⨍_{B(\zeta,1-r)} w.$$

The above, together with Fubini's theorem gives that

$$\int_{\mathbf{S}^n} \Upsilon_2 w \preceq \int_0^1 \int_{\mathbf{S}^n} ⨍_{B(\eta,1-r)} \left(\frac{\nu(B(\zeta,1-r))}{(1-r)^{n-s}}\right)^{p'} \times \left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^p d\sigma(\zeta) w(\eta) \frac{d\sigma(\eta)dr}{1-r}.$$

But if $\zeta \in B(\eta, 1-r)$, $B(\zeta, 1-r) \subset B(\eta, C(1-r))$, for some $C > 0$, and in consequence the above is bounded by

$$C \int_{\mathbf{S}^n} \int_0^1 \left(\frac{\nu(B(\eta, C(1-r)))}{(1-r)^{n-s}}\right)^{p'} \left(⨍_{B(\eta,1-r)} w^{-(p'-1)}\right)^p \frac{dr}{1-r} w(\eta)\,d\sigma(\eta).$$

The change of variables $C(1-r) = y - 1$ gives that we can estimate the previous expression by

$$C \int_{\mathbf{S}^n} \int_0^1 \left(\frac{\nu(B(\eta,(1-y)))}{(1-y)^{n-s}}\right)^{p'} \left(⨍_{B(\eta,1-y)} w^{-(p'-1)}\right)^p \frac{dy}{1-y} w(\eta)\,d\sigma(\eta) + \nu(\mathbf{S}^n)^{p'} \left(\int_{\mathbf{S}^n} w^{-\frac{1}{p-1}}\right)^p = I + II.$$

Theorem 3.2 gives that $II \preceq C\mathcal{E}_{sp}^w(\nu)$, and Theorem 3.2 with $q = p'$ gives that $I \le C\mathcal{E}_{sp}^w(\nu)$. Consequently, $\int_{\mathbf{S}^n} \Upsilon_2 w \le C\mathcal{E}_{sp}^w(\nu)$, and plugging this estimate in (3.18), we deduce that

$$\|\Upsilon\|_{L^p(w)}^p \preceq C\mathcal{E}_{sp}^w(\nu)^{(p-1)^2} \mathcal{E}_{sp}^w(\nu)^{p(2-p)} \simeq \mathcal{E}_{sp}^w(\nu).$$

We now sketch the proof of part (2). We choose $\lambda > 0$ such that $\tau - sp < \lambda < 1$, and define $\mathcal{V}_{sp}^{w\lambda}(\nu)(z)$ as in (3.11). Let us simplify the notation and just write $\mathcal{V}(z) = \mathcal{V}_{sp}^{w\lambda}(\nu)(z)$. Let $\varepsilon \in \mathbf{R}$ such that $\tau < \varepsilon + n < \lambda + sp$.

The proof of a) is analogous to the one in case $1 < p < 2$.

For the proof of b), let us consider $k > s$. It will be enough to prove the following:

$$\begin{aligned}(3.19)\quad &\|\mathcal{V}\|^p_{HF_s^{p1}(w)} \\ &= |\mathcal{V}(0)|^p + \int_{\mathbf{S}^n}\left(\int_0^1 (1-\rho)^{k-s}|(I+R)^k\mathcal{V}(\rho\zeta)|\frac{d\rho}{1-\rho}\right)^p w(\zeta)\,d\sigma(\zeta) \\ &\leq C\mathcal{E}_{sp}^w(\nu).\end{aligned}$$

Let us begin with the estimate $|\mathcal{V}(0)|^p \preceq \mathcal{E}_{sp}^w(\nu)$. If $p > 2$, Hölder's inequality with exponent $\frac{1}{p'-1} > 1$, gives that

$$\begin{aligned}|\mathcal{V}(0)| &\leq \left(\int_0^1\int_{\mathbf{S}^n}(1-r)^{\varepsilon}\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^{\frac{1}{p'-1}} d\nu(\zeta)\frac{dr}{1-r}\right)^{p'-1} \\ &\quad\times\left(\int_0^1\left((1-r)^{(p'-1)(\lambda+sp-n-\varepsilon)}\right)^{\frac{1}{2-p'}}\frac{dr}{1-r}\right)^{2-p'} \\ &\preceq \nu(\mathbf{S}^n)^{p'-1}⨍_{\mathbf{S}^n} w^{-(p'-1)}.\end{aligned}$$

The case $p = 2$ is proved similarly. Consequently, for any $p \geq 2$,

$$|\mathcal{V}(0)|^p \preceq \nu(\mathbf{S}^n)^{p'}\left(⨍_{\mathbf{S}^n} w^{-(p'-1)}\right)^p \leq C\mathcal{E}_{sp}^w(\nu),$$

where the constant C may depend on w.

Following with the estimate of $\|\mathcal{V}\|_{HF_s^{p1}(w)}$, we recall (for example see [CohVe2], Proposition 1.4) that if $k > 0$, $0 < \lambda < 1$, and $z \in \mathbf{B}^n$,

$$\left|(I+R)^k\left(\int_{\mathbf{S}^n}\frac{d\nu(\zeta)}{(1-z\overline{\zeta})^{\lambda}}\right)^{p'-1}\right| \leq C\left(\int_{\mathbf{S}^n}\frac{d\nu(\zeta)}{|1-z\overline{\zeta}|^{\lambda}}\right)^{p'-2}\int_{\mathbf{S}^n}\frac{d\nu(\zeta)}{|1-z\overline{\zeta}|^{\lambda+k}}.$$

Plugging this estimate in (3.19) and using that $p' - 2 \leq 0$, we get

$$\begin{aligned}&|(I+R)^k\mathcal{V}(\rho\eta)| \preceq \int_0^1\int_{1-r<\delta,\,1-\rho<\delta<3} \\ &\frac{(1-r)^{(p'-1)(\lambda+sp-n)}\left(\int_{B(\eta,\delta)}\left(⨍_{B(\zeta,1-r)}w^{-(p'-1)}\right)^{\frac{1}{p'-1}}d\nu(\zeta)\right)^{p'-1}}{\delta^{\lambda+k+1+(p'-2)\lambda}}\frac{d\delta dr}{1-r}.\end{aligned}$$

Assume first that $p > 2$. Fubini's theorem and Hölder's inequality with exponent $\frac{1}{p'-1} > 1$, gives that the above is bounded by

(3.20)
$$\int_{1-\rho}^{3}\left(\int_{1-r<\delta<3}(1-r)^{\varepsilon}\int_{B(\eta,\delta)}\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^{\frac{1}{p'-1}} d\nu(\zeta)\frac{dr}{1-r}\right)^{p'-1}$$
$$\times\left(\int_{1-r<\delta<3}\left(\frac{(1-r)^{(\lambda+sp-n)(p'-1)-\varepsilon(p'-1)}}{\delta^{\lambda+k+1+(p'-2)\lambda}}\right)^{\frac{1}{2-p'}}\frac{dr}{1-r}\right)^{2-p'} d\delta.$$

Next, Fubini's theorem and the fact that $\varepsilon > \tau - n$ give that

$$\int_{1-r<\delta}(1-r)^{\varepsilon}\int_{B(\eta,\delta)}\left(⨍_{B(\zeta,1-r)} w^{-(p'-1)}\right)^{\frac{1}{p'-1}} d\nu(\zeta)\frac{dr}{1-r}$$
$$\preceq \int_{B(\eta,\delta)}\delta^{\varepsilon}\left(⨍_{B(\zeta,\delta)} w^{-(p'-1)}\right)^{\frac{1}{p'-1}} d\nu(\zeta).$$

We also have that since $\lambda + sp - n - \varepsilon > 0$, (3.20) is bounded by

$$\int_{1-\rho}^{3}\left(\int_{B(\eta,\delta)}\left(⨍_{B(\zeta,\delta)} w^{-(p'-1)}\right)^{\frac{1}{p'-1}} d\nu(\zeta)\right)^{p'-1}\frac{d\delta}{\delta^{(n-sp)(p'-1)+k+1}}.$$

For the case $p = 2$, we obtain the same estimate, applying directly condition (3.14) on (3.20).

Integrating with respect to ρ, and applying Fubini's theorem we get

$$\int_0^1(1-\rho)^{k-s}|(I+R)^k\mathcal{V}(\rho\eta)|\frac{d\rho}{1-\rho}$$
$$\preceq \int_0^3\left(\int_{B(\eta,\delta)}\left(⨍_{B(\zeta,\delta)} w^{-(p'-1)}\right)^{\frac{1}{p'-1}} d\nu(\zeta)\right)^{p'-1}\frac{d\delta}{\delta^{(n-s)(p'-1)+1}},$$

since $(n-sp)(p'-1)+s = (n-s)(p'-1)$. If $\tau \in B(\zeta,\delta)$, and $\zeta \in B(\eta,\delta)$, we have that $\tau \in B(\eta, C\delta)$. The fact that $w^{-(p'-1)}$ satisfies a doubling condition, gives that the last integral is bounded by

$$C\int_0^3\left(\frac{\nu(B(\eta,\delta))}{\delta^{n-s}}\right)^{p'-1}⨍_{B(\eta,\delta)} w^{-(p'-1)}\frac{d\delta}{\delta}.$$

Applying Theorem 3.2 with exponent $q = p'-1$, we finally obtain that

$$\int_{\mathbf{S}^n}\left(\int_0^1(1-\rho)^{k-s}|(I+R)^k\mathcal{V}(\rho\eta)|\frac{d\rho}{\rho}\right)^p w(\eta)\,d\sigma(\eta) \preceq \int_{\mathbf{S}^n}\mathcal{W}_{sp}^{w}(\nu)(\zeta)\,d\nu(\zeta).$$

□

We can now state the characterization of the weighted Carleson measures.

Theorem 3.5. *Let* $1 < p < +\infty$, $0 < n - sp < 1$, w *an* A_p*-weight, and* μ *a finite positive Borel measure on* $\mathbf{B}^n$. *Assume that* w *is in* D_τ *for some* $0 \le \tau - sp < 1$. *We then have that the following statements are equivalent*:

(*i*) $\|K_\alpha(f)\|_{L^p(d\mu)} \le C\|f\|_{L^p(w)}$.

(*ii*) $\|f\|_{L^p(d\mu)} \le C\|f\|_{H^p_s(w)}$.

Proof of Theorem 3.5. Let us show first that (i) $\Rightarrow$ (ii). Theorem 2.13 gives that condition (ii) can be rewritten as

$$\|C_s(g)\|_{L^p(d\mu)} \le C\|g\|_{L^p(w)}.$$

This fact together with the estimate $|C_s(f)| \le CK_s(|f|)$ finishes the proof of the implication.

Assume now that (ii) holds. Since a measure μ on $\mathbf{B}^n$ satisfies (i) if and only if (see (3.6)) there exists $C > 0$ such that for any open set $G \subset \mathbf{S}^n$, $\mu(T(G)) \le CC^w_{sp}(G)$, we will check that this estimate holds. Let $G \subset \mathbf{S}^n$ be an open set, and let ν be the extremal measure for $C^w_{sp}(G)$. We then have that $\mathcal{W}^w_{sp}(\nu) \ge 1$ except on a set of C^w_{sp}-capacity zero, and $\int_{\mathbf{S}^n} \mathcal{W}^w_{sp}(\nu)\, d\nu \le CC^w_{sp}(G)$. Let us check that the first estimate also holds for a.e. $x \in G$ (with respect to Lebesgue measure on $\mathbf{S}^n$). Indeed, if $A \subset \mathbf{S}^n$ satisfies that $C^w_{sp}(A) = 0$, and $\varepsilon > 0$, let $f \ge 0$ be a function such that $K_s(f) \ge 1$ on A and $\int_{\mathbf{S}^n} f^p w \le \varepsilon$. Since $L^p(w) \subset L^{p_1}(d\sigma)$, for some $1 < p_1 < p$, (see Lemma 2.1) we then have $\|f\|_{L^{p_1}(d\sigma)} \le C\|f\|_{L^p(w)} \le C\varepsilon^{1/p}$. Thus $C_{sp_1}(A) = 0$, and in particular $|A| = 0$.

Following with the proof of the implication consider the holomorphic function on $\mathbf{B}^n$ defined by $F(z) = \mathcal{U}^{w\lambda}_{sp}(\nu)(z)$ if $1 < p < 2$, $F(z) = \mathcal{V}^{w\lambda}_{sp}(\nu)(z)$, if $p \ge 2$ where λ is as in Theorem 3.4. Theorem 3.4 and the fact that ν is extremal give that

$$\lim_{r\to 1} \operatorname{Re} F(r\zeta) \ge C\mathcal{W}^w_{sp}(\nu)(\zeta) \ge C,$$

for a.e. $x \in G$ with respect to C^w_{sp}, and in consequence, for a.e. $x \in G$ with respect to Lebesgue measure on G. Hence, if P is the Poisson-Szegö kernel

$$|F(z)| = |P[\lim_{r\to 1} F(r\,\cdot)](z)| \ge |P[\operatorname{Re} \lim_{r\to 1} F(r\,\cdot)](z)| \ge C,$$

for any $z \in T(G)$, and since we are assuming that (ii) holds, we obtain

$$\mu(T(E)) \leq \int_{T(E)} |F(z)|^p \, d\mu(z) \leq C\|F\|^p_{H^p_s(w)} \leq C\mathcal{E}^w_{sp}(\nu) \leq CC^w_{sp}(G). \qquad \square$$

We finish with an example which shows that, simmilarly to what happens if $w \equiv 1$, if $w \in D_\tau$ and $\tau - sp > 1$, then the equivalence between (i) and (ii) in the previous theorem need not to be true.

Proposition 3.6. *Let $n \geq 3$, $p = 2$, and $\tau \geq 0$, $0 < s$ such that $1 + 2s < \tau < 2s + n - 1$. Assume also that $n < \tau < n + 1$. Then there exists $w \in A_2 \cap D_\tau$ and a positive Borel measure μ on $\mathbf{S}^n$ such that μ is a Carleson measure for $H^2_s(w)$, but it is not Carleson for $K_s[L^2(w)]$.*

Proof of Proposition 3.6. If $\varepsilon = \tau - n$, and $\zeta = (\zeta', \zeta_n) \in \mathbf{S}^n$, we consider the weight on $\mathbf{S}^n$ defined by $w(\zeta) = (1 - |\zeta'|^2)^\varepsilon$. A calculation gives that $w(z) = (1 - |z|^2)^\varepsilon \in A_2$ if and only if $-1 < \varepsilon < 1$, which is our case. We also have that if $\zeta \in \mathbf{S}^n$, $R > 0$ and $j \geq 0$, then $W(B(\zeta, 2^j R)) \simeq 2^{j\tau} W(B(\zeta, R))$, i.e. $w \in D_\tau$.

Next, any function in $H^2_s(w)$ can be written as $\int_{\mathbf{S}^n} \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-s}} \, d\sigma(\zeta)$, $f \in L^2(w)$. It is then immediate to check that the restriction to B^{n-1} of any such function can be written as

$$\int_{\mathbf{B}^{n-1}} \frac{g(\zeta')(1 - |\zeta'|^2)^{-\varepsilon/2}}{(1 - z'\overline{\zeta'})^{n-s}} \, dv(\zeta'),$$

with $g \in L^2(dv)$. This last space coincides (see for instance [Pe]) with the Besov space $B^2_{s-\frac{1}{2}-\frac{\varepsilon}{2}}(\mathbf{B}^{n-1}) = H^2_{s-\frac{1}{2}-\frac{\varepsilon}{2}}(\mathbf{B}^{n-1})$.

Next, $n - 1 - (s - \frac{1}{2} - \frac{\varepsilon}{2})2 = \tau - 2s > 1$, and Proposition 3.1 in [CaOr2] gives that there exists a positive Borel measure μ on $\mathbf{B}^n$ which is Carleson for $H^2_{s-\frac{1}{2}-\frac{\varepsilon}{2}}(\mathbf{S}^{n-1})$, but it fails to be Carleson for the space $K_{s-\frac{1}{2}-\frac{\varepsilon}{2}}[L^2(d\sigma)]$. Thus the operator

$$f \longrightarrow \int_{\mathbf{S}^{n-1}} \frac{f(\zeta)}{|1 - z\bar{\zeta}|^{n-1-(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \, d\sigma(\zeta),$$

is not bounded from $L^2(d\sigma)$ to $L^2(d\mu)$. Duality gives that the operator

$$g \longrightarrow \int_{\mathbf{B}^{n-1}} \frac{g(z)}{|1 - z\bar{\zeta}|^{n-1-(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \, d\mu(z)$$

is also not bounded from $L^2(d\mu)$ to $L^2(d\sigma)$. But if $g \geq 0$, $g \in L^2(d\mu)$, Fubini's theorem gives

$$\begin{aligned}
&\left\| \int_{\mathbf{B}^{n-1}} \frac{g(z)}{|1 - z\overline{\zeta}|^{n-1-(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \right\|^2_{L^2(d\sigma)} \\
&= \int_{\mathbf{S}^{n-1}} \left(\int_{\mathbf{B}^{n-1}} \frac{g(z)}{|1 - z\overline{\zeta}|^{n-1-(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \, d\mu(z) \right)^2 d\sigma(\zeta) \\
&= \int_{\mathbf{S}^{n-1}} \int_{\mathbf{B}^{n-1}} \frac{g(z)}{|1 - z\overline{\zeta}|^{n-1-(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \, d\mu(z) \\
&\qquad \times \int_{\mathbf{B}^{n-1}} \frac{g(w)}{|1 - w\overline{\zeta}|^{n-1-(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \, d\mu(w) d\sigma(\zeta) \\
&\simeq \int_{\mathbf{B}^{n-1}} \int_{\mathbf{B}^{n-1}} \frac{g(z)g(w)}{|1 - z\overline{w}|^{n-1-2(s-\frac{1}{2}-\frac{\varepsilon}{2})}} \, d\mu(z) d\mu(w),
\end{aligned}$$

where the last estimate holds since $n - 1 - 2(s - \frac{1}{2} - \frac{\varepsilon}{2}) = \tau - 2s > 0$. Consequently, we have that for the measure μ, it does not hold that for any $g \in L^2(d\mu)$

$$\int_{\mathbf{B}^{n-1}} \int_{\mathbf{B}^{n-1}} \frac{g(z)g(w)}{|1 - z\overline{w}|^{n-2(s-\frac{\varepsilon}{2})}} \, d\mu(z) d\mu(w) \leq C \|g\|_{L^2(d\mu)}. \tag{3.21}$$

We next check that the failure of being a Carleson measure for $K_s[L^2(w)]$ can be also rewritten in the same terms. An argument similar to the previous one, gives that μ is not Carleson for $K_s[L^2(w)]$ if and only if the operator

$$f \longrightarrow \int_{\mathbf{B}^{n-1}} \frac{f(z)}{|1 - y\overline{z}|^{n-s}} \, dv(z)$$

is not bounded from $L^2(wdv)$ to $L^2(d\mu)$. Equivalently, writing $f(z) = h(z)(1 - |z|^2)^{\varepsilon/2}$, this last assertion holds if and only if the operator

$$f \longrightarrow \int_{\mathbf{B}^{n-1}} \frac{f(z)(1 - |z|^2)^{-\varepsilon/2}}{|1 - y\overline{z}|^{n-s}} \, dv(z)$$

is not bounded from $L^2(dv)$ to $L^2(d\mu)$. But an argument as before, using duality and Fubini's theorem, gives that the fact that of the unboundedness of the operator can be rewritten in terms of (3.21). ∎

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