T. Ohsawa
Nagoya Math. J.
Vol. 185 (2007), 161–169

ON THE COMPLEMENT OF LEVI-FLATS IN KÄHLER MANIFOLDS OF DIMENSION ≥ 3

TAKEO OHSAWA

Abstract. Applying the L^2 method of solving the $\bar{\partial}$-equation, it is shown that compact Kähler manifolds of dimension ≥ 3 admit no Levi flat real analytic hypersurfaces whose complements are Stein.

Introduction

Let M be a compact complex manifold and let X be a (smooth and closed) real hypersurface of M. X is called a Levi-flat in M if X locally separates M into two Stein domains, or in other words if X is pseudoconvex from both sides.

It is known by Siu [S-1,2] that the complex projective space $\mathbb{CP}^n$ admits no Levi-flats of class C^8 if $n \geq 2$ (see also [I] and [C-S-W]). On the other hand, besides the classical example of Grauert [G], there exist various kinds of Levi-flats in some classes of complex surfaces (cf. [O-2, 3, 4], [D-O], [Ne]).

Moreover, a construction of Nemirovski [Ne] shows that there exist compact complex manifolds of any dimension which admit Levi-flats with Stein complements (see §3).

Since such manifolds are typically the Hopf manifolds if the dimension is ≥ 3, it becomes a natural question whether or not there exist a Kähler manifold of dimension ≥ 3 which admits a Levi-flat with Stein complement.

The purpose of the present article is to answer this for the real analytic Levi-flats by establishing the following.

Theorem 0.1. *Let M be a compact Kähler manifold of dimension $n \geq 3$ and let X be a real analytic Levi-flat in M. Then, $M \setminus X$ does not admit any C^2 plurisubharmonic exhaustion function whose Levi form has at least 3 positive eigenvalues outside a compact subset of $M \setminus X$. In particular, $M \setminus X$ is not a Stein manifold.*

Received July 8, 2005.
2000 Mathematics Subject Classification: Primary 32V40, Secondary 53C40.

When the assumption of real analyticity of X is relaxed to C^∞-smoothness, our method confronts a serious technical problem in obtaining a similar conclusion in full generality.

In spite of this, we shall prove the following.

Theorem 0.2. *Let M be a compact Kähler manifold of dimension $n \geq 3$ and let $X \subset M$ be a Levi-flat of class C^∞. Then, $M \setminus X$ does not admit any C^2 plurisubharmonic function of logarithmic growth near X whose Levi form has at least 3 positve eigenvalues outside a compact subset of $M \setminus X$ and at least 2 positive eigenvalues everywhere on $M \setminus X$.*

Based on Theorem 0.2, we shall proceed in a forthcoming paper to generalize the results in [O-3] to higher dimensions.

§1. Tools from cohomology theory

Let (M_0, g) be a Kähler manifold of dimension n and let φ be a real valued function of class C^2 on M_0.

For each point $x \in M_0$, let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be the eigenvalues of the Levi form (= the complex Hessian) $\partial\bar{\partial}\varphi$ of φ at x, with respect to the metric g, and put

$$\Gamma_q(\varphi, x)_g = \min\left\{ \sum_{\alpha=1}^{q} \gamma_{i_\alpha} \;\middle|\; 1 \leq i_1 < \cdots < i_q \leq n \right\}.$$

M_0 is called a hyper-q-convex manifold if M_0 admits a C^2 exhaustion function φ such that $\Gamma_q(\varphi, x)_g > 0$ outside a compact subset of M_0.

Let us recall first a cohomology vanishing theorem of Grauert–Riemenschneider type in a somewhat generalized form.

Let $E \to M_0$ be a holomorphic vector bundle. By $H^{p,q}(M_0, E)$ (resp. $H_0^{p,q}(M_0, E)$) we benote the E-valued (resp. the E-valued and compactly supported) Dolbeault cohomology group of type (p, q). E is said to be Nakano semipositive if it admits a fiber metric h whose curvature form Θ satisfies the semipositivity condition $h \circ \Theta \geq 0$. Here $h \circ \Theta$ is naturally identified with a Hermitian form along the fibers of $T^{1,0}M_0 \otimes E$, where $T^{1,0}M_0$ denotes the holomorphic tangent bundle of M_0. We shall also say that Θ is Nakano semipositive if $h \circ \Theta \geq 0$.

Theorem 1.1. *Let $q \in \mathbf{N}$ and let (M_0, g) be a noncompact connected Kähler manifold of dimension n which admits a C^2 exhaustion function φ*

such that $\Gamma_q(\varphi, x)_g > 0$ *outside a compact subset of* M_0. *Then, for any Nakano semipositive vector bundle* $E \to M_0$,

$$H^{n,k}(M_0, E) = 0 \quad \textit{for} \quad k \geq q \tag{1.1}$$

and

$$H_0^{0,k}(M_0, E^*) = 0 \quad \textit{for} \quad k \leq n - q. \tag{1.2}$$

Here E^* *denotes the dual bundle of* E.

Proof. Let h be a fiber metric of E whose curvature form is Nakano semipositive. For any C^2 function $\psi \colon M_0 \to \mathbb{R}$, let $L^{p,q}(M_0, E, \psi)$ denote the space of suqare integrable E-valued (p, q)-forms on M_0 with respect to g and $he^{-\psi}$. The L^2 norm on the space $L^{p,q}(M_0, E, \psi)$ will be denoted by $\| \cdot \|_\psi$ or $\| \cdot \|$.

Let φ be a C^2 exhaustion function on M_0 satisfying $\Gamma_q(\varphi, x)_q > 0$ outside a compact subset K. Enlarging K and replacing φ by $\lambda(\varphi)$ for some convex increasing function λ if necessary, we may assume that $\Gamma_q(\varphi, x)_g > 1$ outside K.

Let us fix a C^2 exhaustion function $\Phi \colon M_0 \to [0, \infty)$ in such a way that $\Phi | K = 0$ and that, for any C^2 function $\psi \colon M_0 \to \mathbb{R}$, $C_0^{p,q}(M_0, E)$ is dense in the domain of the maximal closed extension of $\bar{\partial}$ from $L^{p,q}(M_0, E, \psi)$ to $L^{p,q+1}(M_0, E, \psi + \Phi)$ with respect to the graph norm (Hörmander's trick; see [H-2]).

Let $\bar{\partial}^*$ denote the adjoint of $\bar{\partial} \colon L^{p,q}(M_0, E, \varphi - \Phi) \to L^{p,q+1}(M_0, E, \varphi)$. Here p and q run through nonnegative integers and we abbreviate φ and Φ for simplicity.

Then, in virtue of the Nakano formula for the $\bar{\partial}$-Laplacian (see [O-T] for instance), one can find a C^2 convex increasing function $\lambda \colon \mathbb{R} \to \mathbb{R}$ such that the estimate for the L^2 norms

$$\begin{aligned} \|\chi_{M_0 \setminus K} u\|_{\mu(\varphi)} &\leq C(\|\bar{\partial} u\|_{\mu(\varphi)+\Phi} + \|\bar{\partial}^* u\|_{\mu(\varphi)-\Phi}) \\ &\text{for} \quad u \in \operatorname{Dom} \bar{\partial} \cap \operatorname{Dom} \bar{\partial}^* \cap L^{n,k}(M_0, E, \mu(\varphi)), k \geq q \end{aligned} \tag{1.3}$$

holds for any $\mu \colon \mathbb{R} \to \mathbb{R}$ satisfying $\mu > \lambda$, $\mu' > \lambda'$ and $\mu'' > \lambda''$. Here $\chi_{M_0 \setminus K}$ denotes the characteristic function of $M_0 \setminus K$ and C is a constant which is independent of μ.

By the strong ellipticity of the Laplace operator and by Rellich's lemma, it follows from (1.3) easily that $H^{n,k}(M_0, E)$ is finite dimensional for $k \geq$

q (see [H-1] for the detail of the argument). Therefore, the elements of $H^{n,k}(M_0, E)$ can be represented, for all $k \geq q$, by L^2 harmonic forms with respect to

$$\bar{\partial} : L^{n,k}(M_0, E, \mu(\varphi)) \longrightarrow L^{n,k+1}(M_0, E, \mu(\varphi) + \Phi)$$

and

$$\bar{\partial}^* : L^{n,k}(M_0, E, \mu(\varphi)) \longrightarrow L^{n,k-1}(M_0, E, \mu(\varphi) - \Phi)$$

for some convex increasing function μ.

Hence, since M_0 is noncompact, (1.3) implies that the L^2 harmonic forms must vanish on $M_0 \setminus K$, and hence vanish on M_0, too, by the unique continuation theorem of Aronszajn [A]. Therefore $H^{n,k}(M_0, E) = 0$ if $k \geq q$. (1.2) follows from the Serre duality theorem applied to (1.1).

We need also the followaing variant of Grauert–Riemenschneider's vanishing theorem.

THEOREM 1.2. (cf. [O-1, Addendum], [D], or [O-T]) *Let (M_0, g) be a connected Kähler manifold of dimension n which admits a C^∞ plurisubharmonic exhaustion function φ such that $\Gamma_q(\varphi, x)_g > 0$ outside a compact subset of M_0. Then, for any unitary flat vector bundle $E \to M_0$, the restriction homomorphisms*

$$\rho^{s,t} : H^{s,t}(M_0, E) \longrightarrow \varinjlim_{K \Subset M_0} H^{s,t}(M_0 \setminus K, E)$$

are surjective if $s + t < n - q$.

Next we shall refine Theorem 1.1 to the case of bounded domains in M_0 with C^2-smooth boundary.

Let D be a relatively compact domain in M_0 whose boundary ∂D is a real hypersurface of class C^2. Given any Kähler metric g' on D, any Hermitian holomorphic vector bundle (E, h) over D and any C^2 function $\varphi\colon D \to \mathbb{R}$, let $L^{p,q}(D, E, g', \varphi)$ be the space of square integrable E-valued (p, q)-forms on D with respect to g' and $he^{-\varphi}$.

Let $\bar{\partial}_{p,q}\colon L^{p,q}(D, E, g', \varphi) \to L^{p,q+1}(D, E, g', \varphi)$ be the maximal closed extension of $\bar{\partial}$, and put

$$H^{p,q}_{(2)}(D, E, g', \varphi) = \operatorname{Ker} \bar{\partial}_{p,q} / \operatorname{Im} \bar{\partial}_{p,q-1}.$$

Let $\delta(x)$ denote the distance from x to ∂D measured by g.

Definition. *A* complete Kähler metric g' on D is said to be admissible if there exist a positive constant C and a compact subset $K \subset D$ such that

$$Cg' > g + \delta^{-2}\partial\delta \otimes \bar{\partial}\delta$$

holds on $D \setminus K$.

Then, as a refinement of Theorem 1.1, we have the following vanishing theorem of Donnelly-Fefferman type.

Theorem 1.3. *Let (M_0, g) and D be as above. Suppose moreover that there exists a C^2 plurisubharmonic function φ such that $\Gamma_q(\varphi, x)_g > 1$ holds everywhere on D. Then, for any holomorphic Hermitian vector bundle (E, h) over M_0, and for any admissible metric g' on D, there exists a constant $a_0 \in \mathbb{R}$ such that*

$$H^{n,k}_{(2)}(D, E, g', a\varphi) = 0 \quad \textit{for} \quad k \geq q \tag{1.4}$$

and

$$H^{0,n-k}_{(2)}(D, E, g', -a\varphi) = 0 \quad \textit{for} \quad k \geq q \tag{1.5}$$

hold for any $a > a_0$. Here the dual bundle E^* of E is equipped with the dual metric of h.

Proof. Similarly as in the proof of Theorem 1.1, by the trick of Kodaira-Nakano-Donnelly-Fefferman, we obtain, with respect to any admissible metric g', the estimates

$$\|\chi_{D\setminus K^u}\| \leq C(\|\bar{\partial}u\| + \|\bar{\partial}^* u\|) \qquad \text{for} \quad u \in \operatorname{Dom}\bar{\partial}_{n,k} \cap \operatorname{Dom}\bar{\partial}^*_{n,k-1} \cap L^{n,k}(D, E, g', a\varphi) \tag{1.6}$$

for $k \geq q$. Here K and C are respectively a compact set in D and a constant independent of u (see [O-T] for the technical detail). (1.6) implies that $H^{n,k}_{(2)}(D, E, g', a\varphi)$ is finite dimensional for $k \geq q$, so that (1.4) follows from Aronszajn's theorem. The proof of (1.5) is completely similar.

We say that a continuous function $\varphi \colon D \to \mathbb{R}$ is of logarithmic growth near ∂D if there exists a constant $C > 0$ such that

$$C^{-2}(\varphi(x) + C) < -\log\delta(x) + C < C^2(\varphi(x) + C)$$

holds for any $x \in D$.

Later we shall apply Theorem 1.3 when $q = n - 2$ and φ is a plurisubharmonic function of logarithmic growth.

§2. Proof of Theorem 0.1

Let M and X be as in the assumption, and suppose that there exists a C^2 plurisubharmonic exhaustion function on $M \setminus X$ satisfying $\Gamma_{n-2}(\varphi, x)_g > 0$ outside a compact subset say K of $M \setminus X$.

Then, by the real analyticity of the Levi-flat X, there exists a neighbourhood $U \supset X$ and a holomorphic subbundle $\mathcal{L}$ of $T^{1,0}M|U$ which is of corank one and closed under the Lie bracket.

We put

$$\mathcal{N} = T^{1,0}M|U/\mathcal{L}$$

Note that there exists a system of nowhere vanishing local holomorphic sections $\{\omega_\alpha\}_{\alpha\in A}$ of $\mathcal{N}^* \subset (T^{1,0}U)^*$ with respect to an open covering $\{U_\alpha\}_{\alpha\in A}$ of U such that ω_α are holomorphic 1-forms on U satisfying Ker $\omega_\alpha = \mathcal{L}|U_\alpha$. ($\omega_\alpha$ can be chosen to be exact).

Let $\omega_\alpha = e_{\alpha\beta}\omega_\beta$ on $U_\alpha \cap U_\beta$.

Then $\{e_{\alpha\beta}\}$ is a system of transition functions of $\mathcal{N}$, so that $\{\omega_\alpha\}_{\alpha\in A}$ is naturally identified with an $\mathcal{N}$-valued 1-form say ω.

By taking a double cover of M and by shrinking U if necessary, we may assume that $\mathcal{N}$ is a topologically trivial line bundle over U.

Then $\mathcal{N}$ lies in the image of the exponential map

$$\exp\colon H^1(U, \mathcal{O}) \longrightarrow H^1(U, \mathcal{O}^*).$$

Let us choose a $\xi \in H^1(U, \mathcal{O})$ such that $\exp \xi = \mathcal{N}$.

Then, by the assumed hyper-$(n-2)$-convexity of $M \setminus X$, ξ can be extended to M as an element of $H^1(M, \mathcal{O})$. In fact, the restriction homomorphism $H^1(M, \mathcal{O}) \to H^1(U, \mathcal{O})$ is surjective because $H^2_0(M \setminus X, \mathcal{O})(\simeq H^{0,2}_0(M \setminus X)) = 0$ by Theorem 1.1.

Therefore $\mathcal{N}$ is extendable to M as a topologically trivial holomorphic line bundle, say $\widetilde{\mathcal{N}}$.

Since $\widetilde{\mathcal{N}}$ is topologically trivial and M is Kählerian, one may choose a system of local trivialization of $\mathcal{N}$ in such a way that the transition functions are all constants of modulus 1. In particular $\widetilde{\mathcal{N}}$ is Nakano semipositive.

Now we apply Theorem 1.2 to extend the holomorphic $\mathcal{N}$-valued 1-form ω to a holomorphic $\widetilde{\mathcal{N}}$-valued 1-form $\tilde{\omega}$ on M.

Since M is Kählerian, one has $d\tilde{\omega} = 0$. Therefore, as local representations of $\tilde{\omega}$, we hava a system of closed holomorphic 1-forms $\tilde{\omega}_\alpha$ on U_α such that $\tilde{\omega}_\alpha = e^{i\theta_{\alpha\beta}}\tilde{\omega}_\beta$ holds on $U_\alpha \cap U_\beta$ for some $\theta_{\alpha\beta} \in \mathbb{R}$. (we choose $U_\alpha \cap U_\beta$ to be connected in advance).

Let f_α be holomorphic functions on U_α such that $\tilde{\omega}_\alpha = df_\alpha$. Then we have

$$f_\alpha = e^{i\theta_{\alpha\beta}} f_\beta + \eta_{\alpha\beta} \tag{2.1}$$

on $U_\alpha \cap U_\beta$, for some constant $\eta_{\alpha\beta} \in \mathbb{C}$.

For any point $x \in U$ we put

$$d(x) = \inf_{\alpha,c} |f_\alpha(x) + c|. \tag{2.2}$$

Here α is chosen so that $x \in U_\alpha$ and, for each α, c runs through complex numbers satisfying $\inf\limits_{y \in X \cap U_\alpha} |f_\alpha(y) + c| = 0$.

Then it is clear that $d(x)$ measures the distance from x to X with respect to the semipositive form $df_\alpha \otimes d\bar{f}_\alpha$ and that

$$X = \{x \in U \mid d(x) = 0\}$$

holds.

Moreover, we may assume that d is constant on $f_\alpha^{-1}(\zeta)$ for any $\zeta \in f_\alpha(U_\alpha)$, by shrinking U if necessary.

Hence $-\log d(x)$ is a plurisubharmonic function on $U \setminus X$ which tends to infinity as x approaches X.

Since the level sets of $-\log d(x)$ are compact for $0 < d(x) \ll 1$, and they are foliated by $(n-1)$-dimensional complex submanifolds of M, there cannot exist on $U \setminus X$ any C^2 function whose Levi form has everywhere at least 2 positive eigenvalues, because otherwise it would contradict the maximum principle for subharmonic functions.

Hence the assumed condition that $\Gamma_{n-2}(\varphi, x) > 0$ cannot hold. $\square$

§3. Proof of Theorem 0.2 and a remark

Suppose that there existed *a* C^∞ Levi-flat X in a compact Kähler manifold M such that $M \setminus X$ admits a C^2 plurisubharmonic function of logarithmic growth, say φ, whose Levi form satisfies the conditions as stated.

Then, by applying Theorem 1.3 instead of Theorems 1.1 and 1.2, to extend the CR line bundle

$$\mathcal{N}_X = (T^{1,0}M|X)/(T^{1,0}M|X) \cap (TX \otimes \mathbb{C})$$

to a holomorphic line bundle $\widetilde{\mathcal{N}}$ over M, and to obtain an $\widetilde{\mathcal{N}}$-valued holomorphic 1-form on M which annihilates $(T^{1,0}M|X) \cap (TX \otimes \mathbb{C})$ on X, we arrive at the same contradiction as before.

Finally we show by an example that the Kähler condition is necessary for the validity of Theorem 0.1.

The following is essentially contained in [Ne].

Example. Let Y be a compact complex manifold, let $L \to Y$ be a holomorphic line bundle, and let s be a meromorphic section of L whose sets of zeros and poles are reduced, smooth, and mutually disjoint.

Let N and P be respectively the set of zeros and that of poles.

Then we put

$$\widetilde{X} = \{s(x) \mid x \in Y \setminus (N \cup P)\}.$$

Let us take the quotient of $L \setminus \{0\text{-section}\}$ by a standard free action of the infinite cyclic group and put

$$M = (L \setminus \{0\text{-section}\})/\mathbb{Z}$$
$$X_0 = \widetilde{X}/\mathbb{Z}.$$

Let X be the closure of X_0 in M. Then X is a real analytic Levi-flat in M.

If $Y \setminus (N \cup P)$ is Stein, which is the case if either $\dim Y = 1$ and $N \cup P \neq \emptyset$, or $P \neq \emptyset$ and L is ample, then $M \setminus X$ is Stein in virtue of Mok's theorem (cf. [M]), since $M \setminus X$ is an annulus bundle over $Y \setminus (N \cup P)$.

Added in proof. After this paper was accepted for publication, it turned out that there is a serious gap in [S-2], which affects the results of [I] and [O-3] (see [O-3, Erratum]). [C-S-W] also contains some flaw. What remains valid there was recently reproved in "Cao, J. and Shaw, M.-C., The $\bar{\partial}$-Cauchy problem and nonexistence of Lipschitz. Levi-flat hypersurfaces in $\mathbb{CP}^n$ with $n \geq 3$ to appear in Math. Z."

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Graduate School of Mathematics
Nagoya University
Chikusa-ku Furo-cho 464-8602
Nagoya
Japan

ohsawa@math.nagoya-u.ac.jp