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UNBOUNDEDNESS OF THE BALL BILINEAR MULTIPLIER OPERATOR

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Abstract. For all $n > 1$, the characteristic function of the unit ball in $\mathbb{R}^{2n}$ is not the symbol of a bounded bilinear multiplier operator from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ when $1/p + 1/q = 1/r$ and exactly one of p, q, or $r' = r/(r-1)$ is less than 2.

§1. Introduction

We denote the Fourier transform of a function f on $\mathbb{R}^n$ by $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(t)e^{-2\pi i t\cdot\xi}\,dt$ and its inverse Fourier transform by $f^{\vee}(\xi) = \widehat{f}(-\xi)$. Let B be the unit ball in $\mathbb{R}^n$ and χ_A the characteristic function of a set A. The unboundedness of the linear operator

$$T_{\chi_B}(f) = (\widehat{f}\chi_B)^{\vee}$$

on $L^p(\mathbb{R}^n)$ when $p \neq 2$ and $n > 1$ was established by Fefferman [2].

In this article we provide a variant of Fefferman's result in the bilinear setting. Our arguments also work for multilinear operators. Let $1 \leq p_1, \dots, p_k \leq \infty$ and $0 < p < \infty$. We recall that a bounded function $m : (\mathbb{R}^n)^k \mapsto \mathbb{C}$ is called a k-linear multiplier if the k-linear operator

$$(f_1, \dots, f_k) \longrightarrow \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} m(\xi_1, \dots, \xi_k)\widehat{f}_1(\xi_1) \cdots \widehat{f}_k(\xi_k) \times e^{2\pi i(\xi_1+\cdots+\xi_k)\cdot x}\,d\xi_1 \cdots d\xi_k$$

initially defined for Schwartz functions f_j on $\mathbb{R}^n$ admits a bounded extension

$$T_m : L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_k}(\mathbb{R}^n) \longmapsto L^p(\mathbb{R}^n). \tag{1.1}$$

In this case we call m the symbol of T_m. We will denote by $\mathcal{M}_{p_1,p_2,\dots,p_k,p}(\mathbb{R}^n)$ the set of all k-linear multipliers m such that the corresponding operator T_m

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satisfies (1.1). The norm of m in $\mathcal{M}_{p_1,p_2,\ldots,p_k,p}(\mathbb{R}^n)$ is defined as the norm of T_m.

Nontrivial examples of functions in $\mathcal{M}_{p_1,p_2,p}(\mathbb{R})$ are characteristic functions of half-planes (see [7], [8]) when $p_1^{-1}+p_2^{-1}=p^{-1}<3/2$ and characteristic functions of planar ellipses when $p_1^{-1}+p_2^{-1}=p^{-1}$ and $2\le p_1,p_2,p'<\infty$ (see [4]). Here $p'=p/(p-1)$. It is still an open question whether the results of this paper hold if $n=1$. In this work we show that this is not the case for the characteristic function of the ball in $\mathbb{R}^{2n}$ if $1/p+1/q=1/r$ and exactly one of p, q, or r' is less than 2. We will construct a counterexample when $n=2$ and $r>2$. The general result will follow from duality and a multilinear version of de Leeuw's theorem [1].

§2. Bilinearization of Fefferman's counterexample for $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$

For a rectangle R in $\mathbb{R}^2$, let R' be the union of the two copies of R adjacent to R in the direction of its longest side. Hence, $R\cup R'$ is a rectangle three times as long as R with the same center. Key to this argument is the following geometric lemma whose proof can be found in [9], page 435 or [3], page 738.

LEMMA 1. *Let $\delta>0$ be given. Then there exists a measurable subset E of $\mathbb{R}^2$ and a finite collection of rectangles R_j in $\mathbb{R}^2$ such that*

(1) *The R_j are pairwise disjoint.*
(2) *We have $1/2\le|E|\le 3/2$.*
(3) *We have $|E|\le\delta\sum_j|R_j|$.*
(4) *For all j we have $|R_j'\cap E|\ge\frac{1}{12}|R_j|$.*

Let $\delta>0$ and let E and R_j be as in Lemma 1. The proof of Lemma 1 implies that there are 2^k rectangles R_j of dimension $2^{-k}\times 3\log(k+2)$. Here, k is chosen so that $k+2\ge e^{1/\delta}$. Let v_j be the unit vector in $\mathbb{R}^2$ parallel to the longest side of R_j and in the direction of the set E relative to R_j.

PROPOSITION 1. *Let R be a rectangle in $\mathbb{R}^2$ and let v be a unit vector in $\mathbb{R}^2$ parallel to the longest side of R. Let R' be as above. Consider the half space $\mathcal{H}_v$ of $\mathbb{R}^4$ defined by*

$$\mathcal{H}_v=\{(\xi,\eta)\in\mathbb{R}^2\times\mathbb{R}^2:(\xi+\eta)\cdot v\ge 0\}.$$

Then the following estimate is valid for all $x \in \mathbb{R}^2$:

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \chi_{\mathcal{H}_v}(\xi,\eta)\widehat{\chi_R}(\xi)\widehat{\chi_R}(\eta)e^{2\pi i x\cdot(\xi+\eta)}\, d\xi d\eta \right| \geq \frac{1}{10}\chi_{R'}(x). \tag{2.1}$$

Proof. We introduce a rotation (i.e. orthogonal matrix) $\mathcal{O}$ of $\mathbb{R}^2$ such that $\mathcal{O}(v) = (1,0)$. Setting $\xi = (\xi_1,\xi_2)$, $\eta = (\eta_1,\eta_2)$ we can write the expression on the left in (2.1) as

$$\begin{aligned} &\left| \iint_{\mathcal{O}^{-1}(\xi+\eta)\cdot v\geq 0} \widehat{\chi_R}(\mathcal{O}^{-1}\xi)\widehat{\chi_R}(\mathcal{O}^{-1}\eta)e^{2\pi i x\cdot\mathcal{O}^{-1}(\xi+\eta)}\, d\xi d\eta \right| \\ &\quad = \left| \iint_{\xi_1+\eta_1\geq 0} \widehat{\chi_{\mathcal{O}[R]}}(\xi)\widehat{\chi_{\mathcal{O}[R]}}(\eta)e^{2\pi i\mathcal{O}x\cdot(\xi+\eta)}\, d\xi d\eta \right|. \end{aligned}$$

Now the rectangle $\mathcal{O}[R]$ has sides parallel to the axes, say $\mathcal{O}[R] = I_1 \times I_2$. Assume that $|I_1| > |I_2|$, i.e. its longest side is horizontal. Let H be the classical Hilbert transform on the line. Setting $\mathcal{O}x = (y_1,y_2)$ we can write the last displayed expression as

$$\begin{aligned} &\left| \chi_{I_2}(y_2)^2 \int_{\xi_1\in\mathbb{R}} \widehat{\chi_{I_1}}(\xi_1)e^{2\pi i y_1\xi_1} \int_{\eta_1\geq -\xi_1} \widehat{\chi_{I_1}}(\eta_1)e^{2\pi i y_1\eta_1}\, d\eta_1 d\xi_1 \right| \\ &\quad = \chi_{I_2}(y_2) \left| \int_{\xi_1\in\mathbb{R}} \widehat{\chi_{I_1}}(\xi_1)\frac{1}{2}(I+iH)\big[\chi_{I_1}(\,\cdot\,)e^{2\pi i\xi_1(\,\cdot\,)}\big](y_1)\, d\xi_1 \right| \\ &\quad = \chi_{I_2}(y_2)\left| \frac{1}{2}(I+iH)(\chi_{I_1})(y_1) \right| = \left| \big[\chi_{\xi_1\geq 0}\, \widehat{\chi_{I_1\times I_2}}(\xi_1,\xi_2)\big]^{\vee}(y_1,y_2) \right|. \end{aligned}$$

Using the result from [3] (Proposition 10.1.2) or [9] (estimate (33), page 453) we deduce that the previous expression is at least

$$\frac{1}{10}\chi_{(I_1\times I_2)'}(y_1,y_2) = \frac{1}{10}\chi_{(\mathcal{O}[R])'}(\mathcal{O}x) = \frac{1}{10}\chi_{R'}(x).$$

This proves the required conclusion. ∎

Next we have the following result concerning bilinear operators on $\mathbb{R}^2$ of the form

$$\begin{aligned} T_m(f,g)(x) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} & m(\xi_1,\xi_2,\eta_1,\eta_2)\widehat{f}(\xi_1,\xi_2)\widehat{g}(\eta_1,\eta_2) \\ & \times e^{2\pi i x\cdot(\xi_1+\eta_1,\xi_2+\eta_2)}\, d\xi_1 d\xi_2 d\eta_1 d\eta_2. \end{aligned}$$

Lemma 2. *Let $v_1, v_2, \dots, v_j, \dots$ be a sequence of unit vectors in $\mathbb{R}^2$. Define a sequence of half-spaces $\mathcal{H}_{v_j}$ in $\mathbb{R}^4$ as in Proposition 1. Let B, B^{*1}, B^{*2} be the following sets in $\mathbb{R}^4$*

$$\begin{aligned} B &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\eta|^2 \le 1\} \\ B^{*1} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi+\eta|^2 + |\eta|^2 \le 1\} \\ B^{*2} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\xi+\eta|^2 \le 1\}. \end{aligned}$$

Assume that one of T_{χ_B}, $T_{\chi_{B^{*1}}}$, $T_{\chi_{B^{*2}}}$ lies in $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ and has norm $C = C(p,q,r)$. Then we have the following vector-valued inequality

$$\Big\| \Big(\sum_j |T_{\chi_{\mathcal{H}_{v_j}}}(f_j, g_j)|^2 \Big)^{1/2} \Big\|_r \le C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \Big\|_q .$$

for all functions f_j and g_j.

Proof. We begin with the assumption that T_{χ_B} lies in $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ for some $p,q,r > 0$. Set $\xi = (\xi_1, \xi_2)$ and $\eta = (\eta_1, \eta_2) \in \mathbb{R}^2$. For $\rho > 0$ we define sets

$$\begin{aligned} B_\rho &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\eta|^2 \le 2\rho^2\} \\ B_{j,\rho} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi - \rho v_j|^2 + |\eta - \rho v_j|^2 \le 2\rho^2\}. \end{aligned}$$

Note that bilinear multiplier norms are translation and dilation invariant. Easy computations give that

$$\|\chi_{B_{j,\rho}}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^2)} \le \|\chi_{B_\rho}\|_{\mathcal{M}_{p,q,r}(\mathbb{R}^2)} = C.$$

The important observation is that $\chi_{B_{j,\rho}} \to \chi_{\mathcal{H}_{v_j}}$ pointwise as $\rho \to \infty$ and that the multiplier norms of the functions $\chi_{B_{j,\rho}}$ are bounded above by C.

Moreover, by the bilinear version of a theorem of Marcinkiewicz and Zygmund ([5], Section 9), we have the following inequality for all $\rho > 0$.

$$\Big\| \Big(\sum_j |T_{\chi_{B_\rho}}(f_j, g_j)|^2 \Big)^{1/2} \Big\|_r \le C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \Big\|_q .$$

Since $\chi_{B_{j,\rho}} \to \chi_{\mathcal{H}_{v_j}}$ pointwise as $\rho \to \infty$, we can deduce that

$$\lim_{\rho \to \infty} T_{\chi_{B_{j,\rho}}}(f,g)(x) = T_{\chi_{\mathcal{H}_{v_j}}}(f,g)(x)$$

for all $x \in \mathbb{R}^2$ and suitable functions f and g. We note that the curvature of the ball B is used here. By Fatou's lemma we conclude

$$\Big\| \Big(\sum_j \big| T_{\chi_{\mathcal{H}_{v_j}}}(f_j, g_j) \big|^2 \Big)^{1/2} \Big\|_r \le \liminf_{\rho \to \infty} \Big\| \Big(\sum_j \big| T_{\chi_{B_{j,\rho}}}(f_j, g_j) \big|^2 \Big)^{1/2} \Big\|_r . \tag{2.2}$$

Now, observe the following identity:

$$T_{\chi_{B_{j,\rho}}}(f,g)(x) = e^{4\pi i \rho v_j \cdot x} T_{\chi_{B_\rho}}(e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} f, e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} g)(x).$$

Using (2.2) and the previous identity gives

$$\begin{aligned}
&\Big\| \Big(\sum_j \big| T_{\chi_{\mathcal{H}_j}}(f_j, g_j) \big|^2 \Big)^{1/2} \Big\|_r \\
&\le \liminf_{\rho \to \infty} \Big\| \Big(\sum_j \big| e^{4\pi i \rho v_j \cdot (\,\cdot\,)} T_{\chi_{B_\rho}}(e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} f_j, e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} g_j) \big|^2 \Big)^{1/2} \Big\|_r \\
&\le \liminf_{\rho \to \infty} \big\| \chi_{B_\rho} \big\|_{\mathcal{M}_{p,q,r}} \\
&\qquad \times \Big\| \Big(\sum_j \big| e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} f_j \big|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_j \big| e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} g_j \big|^2 \Big)^{1/2} \Big\|_q \\
&= C \Big\| \Big(\sum_j |f_j|^2 \Big)^{1/2} \Big\|_p \Big\| \Big(\sum_j |g_j|^2 \Big)^{1/2} \Big\|_q ,
\end{aligned}$$

where the last equality follows from the dilation invariance of bilinear multiplier norms.

The proof of the analogous statements for $T_{B^{*1}}$ and $T_{B^{*2}}$ is as follows. We introduce sets

$$\begin{aligned}
B_\rho^{*1} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi+\eta|^2 + |\eta|^2 \le \rho^2\} \\
B_{j,\rho}^{*1} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi - \rho v_j + \eta|^2 + |\eta|^2 \le \rho^2\} \\
B_\rho^{*2} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\xi+\eta|^2 \le \rho^2\} \\
B_{j,\rho}^{*2} &= \{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 : |\xi|^2 + |\xi+\eta - \rho v_j|^2 \le \rho^2\}.
\end{aligned}$$

Note that both $B_{j,\rho}^{*1}$ and $B_{j,\rho}^{*2}$ converge to $\mathcal{H}_{v_j}$ as $\rho \to \infty$. Using the identities

$$\begin{aligned}
T_{\chi_{B_{j,\rho}^{*1}}}(f,g)(x) &= e^{2\pi i \rho v_j \cdot x} T_{\chi_{B_\rho^{*1}}}(e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} f, g)(x) \\
T_{\chi_{B_{j,\rho}^{*2}}}(f,g)(x) &= e^{2\pi i \rho v_j \cdot x} T_{\chi_{B_\rho^{*2}}}(f, e^{-2\pi i \rho v_j \cdot (\,\cdot\,)} g)(x),
\end{aligned}$$

we obtain a similar conclusion for the bilinear operators $T_{\chi_{B^{*1}}}$ and $T_{\chi_{B^{*2}}}$. ∎

The next ingredient that we will need is a multilinear version of de Leeuw's theorem. For $1 \le j \le k$ we will consider $\xi_j \in \mathbb{R}^n$, $\eta_j \in \mathbb{R}^m$. Then the pairs $(\xi_j, \eta_j) \in \mathbb{R}^{n+m}$. Also for a function f on $\mathbb{R}^n$ and g on $\mathbb{R}^m$ we introduce another function $f \otimes g$ on $\mathbb{R}^{n+m}$ by setting $(f \otimes g)(\xi, \eta) = f(\xi)g(\eta)$.

Proposition 2. *Suppose that*

$$m(\xi_1, \eta_1, \xi_2, \eta_2, \dots, \xi_k, \eta_k) \in \mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^{n+m})$$

for some $1 < p < \infty$. Then for almost every $(\xi_1, \dots, \xi_k) \in (\mathbb{R}^n)^k$ the function $m(\xi_1, \cdot, \xi_2, \cdot, \dots, \xi_k, \cdot)$ lies in $\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^m)$, with norm

$$\|m(\xi_1, \cdot, \xi_2, \cdot, \dots, \xi_k, \cdot)\|_{\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^m)} \le \|m\|_{\mathcal{M}_{p_1, p_2, \dots, p_k, p}(\mathbb{R}^{n+m})}.$$

Proof. In the proof that follows for simplicity we take $k = 2$. The case of a general k does not present any complications, only notational changes. We also assume that m is continuous. This assumption may be easily removed by considering convolutions of m in each variable with smooth approximate identities.

Fix $f_1, g_1, h_1 \in \mathcal{S}(\mathbb{R}^n)$ and $f_2, g_2, h_2 \in \mathcal{S}(\mathbb{R}^m)$ with $\|f_2\|_{p_1} = \|g_2\|_{p_2} = \|h_2\|_{p'} = 1$. Let

$$M(\xi_1, \xi_2) = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} m(\xi_1, \eta_1, \xi_2, \eta_2) \widehat{f_2}(\eta_1) \widehat{g_2}(\eta_2) \times e^{2\pi i(\eta_1 + \eta_2) \cdot x_2} \, d\eta_1 d\eta_2 \, h_2(x_2) \, dx_2.$$

If we can show that $M \in \mathcal{M}_{p_1, p_2, p}(\mathbb{R}^n)$, then by Proposition 4 (vi) in [6], we can deduce that $\|M\|_\infty \le \|M\|_{\mathcal{M}_{p_1, p_2, p}}$. Then, by duality, it will follow that $\|T_m(f_2, g_2)\|_p \le \|M\|_\infty \le \|M\|_{\mathcal{M}_{p_1, p_2, p}}$. We have

$$\begin{aligned}
&\big|\langle T_M(f_1, g_1), h_1 \rangle\big| \\
&\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} M(\xi_1, \xi_2) \widehat{f_1}(\xi_1) \widehat{g_1}(\xi_2) e^{2\pi i(\xi_1 + \xi_2) \cdot x_1} \, d\xi_1 d\xi_2 \, h_1(x_1) \, dx_1 \\
&\quad = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} m(\xi_1, \eta_1, \xi_2, \eta_2) \widehat{f_2}(\eta_1) \widehat{g_2}(\eta_2) e^{2\pi i(\eta_1 + \eta_2) \cdot x_2} \\
&\qquad\qquad \times d\eta_1 d\eta_2 \, h_2(x_2) \, dx_2 \, \widehat{f_1}(\xi_1) \widehat{g_1}(\xi_2) e^{2\pi i(\xi_1 + \xi_2) \cdot x_1} \, d\xi_1 d\xi_2 \, h_1(x_1) \, dx_1 \\
&\quad = \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} \int_{\mathbb{R}^{n+m}} m(\xi_1, \eta_1, \xi_2, \eta_2) \widehat{f_1}(\xi_1) \widehat{f_2}(\eta_1) \widehat{g_1}(\xi_2) \widehat{g_2}(\eta_2) \\
&\qquad\qquad \times e^{2\pi i((\xi_1, \eta_1) + (\xi_2, \eta_2)) \cdot (x_1, x_2)} \, d(\xi_1, \eta_1) d(\xi_2, \eta_2) \, h_1(x_1) h_2(x_2) \, d(x_1, x_2)
\end{aligned}$$

$$
\begin{aligned}
&= \big|\langle T_m(f_1 \otimes f_2, g_1 \otimes g_2), h_1 \otimes h_2\rangle\big| \\
&\le \|m\|_{\mathcal{M}_{p_1,p_2,p}(\mathbb{R}^{n+m})} \|f_1 \otimes f_2\|_{p_1} \|g_1 \otimes g_2\|_{p_2} \|h_1 \otimes h_2\|_{p'} \\
&= \|m\|_{\mathcal{M}_{p_1,p_2,p}(\mathbb{R}^{n+m})} \|f_1\|_{p_1} \|f_2\|_{p_1} \|g_1\|_{p_2} \|g_2\|_{p_2} \|h_1\|_{p} \|h_2\|_{p'} \\
&= \|m\|_{\mathcal{M}_{p_1,p_2,p}(\mathbb{R}^{n+m})} \|f_1\|_{p_1} \|g_1\|_{p_2} \|h_1\|_{p'},
\end{aligned}
$$

where the inequality follows from the boundedness of T_m. □

The following is the main result of this article.

Theorem 1. *Let $n > 1$ and $1/p + 1/q = 1/r$ with exactly one of p, q, or r' less than 2. Let B be the unit ball in $\mathbb{R}^{2n}$. Then $\chi_B \notin \mathcal{M}_{p,q,r}(\mathbb{R}^n)$.*

Proof. Using Proposition 2 and considering the two dual operators $T_{\chi_{B^{*1}}}$ and $T_{\chi_{B^{*2}}}$ of T_{χ_B}, it suffices to show that all of these operators are not in $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ for $p, q, r > 2$. Therefore, we fix $n = 2$ and p, q, r satisfying $p^{-1} + q^{-1} = r^{-1} < 1/2$. We suppose that χ_B is in $\mathcal{M}_{p,q,r}(\mathbb{R}^2)$ with norm C.

Suppose that $\delta > 0$ is given. Let E and R_j be as in Lemma 1. Let v_j be the the unit vector parallel to the longest side of R_j and pointing in the direction of the set E relative to R_j. In the spirit of Fefferman's argument, we estimate $\sum_j \int_E \big|T_j(\chi_{R_j}, \chi_{R_j})(x)\big|^2\,dx$ from above and below and arrive to a contradiction. We have

$$
\begin{aligned}
&\sum_j \int_E \big|T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})(x)\big|^2\,dx \\
&\quad\le |E|^{\frac{r-2}{r}} \bigg\| \Big(\sum_j \big|T_{\mathcal{H}_{v_j}}(\chi_{R_j}, \chi_{R_j})\big|^2\Big)^{1/2} \bigg\|_r^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(Hölder's inequality with } r > 2\text{)} \\
&\quad\le C\,|E|^{\frac{r-2}{r}} \bigg\| \Big(\sum_j |\chi_{R_j}|^2\Big)^{1/2} \bigg\|_p^2 \bigg\| \Big(\sum_j |\chi_{R_j}|^2\Big)^{1/2} \bigg\|_q^2 \qquad \text{(by Lemma 2)} \\
&\quad= C\,|E|^{\frac{r-2}{r}} \Big(\sum_j |R_j|\Big)^{2/r} \qquad \text{(by the disjointness of the } R_j\text{s)} \\
&\quad\le C\,\delta^{\frac{r-2}{r}} \sum_j |R_j| \qquad \text{(Lemma 1).}
\end{aligned}
$$

For the reverse inequality we argue as follows:

$$
\begin{aligned}
&\sum_j \int_E \big|T_{\mathcal{H}_{v_j}}(\chi_{R_j},\chi_{R_j})(x)\big|^2\,dx \\
&\geq \sum_j \int_E \big(\tfrac{1}{10}\chi_{R_j'}(x)\big)^2\,dx \qquad \text{(Proposition 1)} \\
&= \tfrac{1}{100}\sum_j \big|E\cap R_j'\big| \\
&\geq \tfrac{1}{1200}\sum_j \big|R_j\big| \qquad \text{(Lemma 1).}
\end{aligned}
$$

Putting these two estimates together, we obtain that

$$\tfrac{1}{1200}\sum_j |R_j| \leq C\,\delta^{\frac{r-2}{r}}\sum_j |R_j|$$

and therefore

$$\tfrac{1}{1200} \leq C\,\delta^{\frac{r-2}{r}}$$

for any $\delta > 0$. This is a contradiction since $r > 2$. ∎

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