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THE FOURTH-ORDER Q-CURVATURE FLOW ON CLOSED 3-MANIFOLDS

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Abstract. Let the Paneitz operator P_0 be strictly positive on a closed 3-manifold M with a fixed conformal class. It is proved that the solution of a fourth-order Q-curvature flow exists on M for all time and converges smoothly to a metric of constant Q-curvature.

§1. Introduction

Let $(M, [g_0])$ be a closed smooth Riemannian n-manifold with a fixed conformal class $[g_0]$. Let P_0 be the conformal Paneitz operator which was introduced by Paneitz ([P]) in a 4-manifold with fixed conformal class and was generalized to general dimensions $n \neq 4$ by Branson ([B]). The operator is defined by

$$P_0^n = \Delta_0^2 + \mathrm{div}_0(a_n R_0 g_0 + b_n Ric_0)d + \frac{n-4}{2}Q_0^n \tag{1.1}$$

where

$$Q_0^n = -\frac{1}{2(n-1)}\Delta_0 R_0 - \frac{2}{(n-2)^2}|Ric_0|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R_0^2,$$

and

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \quad b_n = -\frac{4}{n-2}.$$

Here $\Delta_0 = \mathrm{div}_0 \nabla$ is the Laplacian w.r.t. g_0 and R_0, Ric_0 are the scalar curvature and Ricci curvature tensor w.r.t. g_0 respectively.

In dimension four,

$$P_0^4 = \Delta_0^2 + \mathrm{div}_0\Big(\frac{2}{3}R_0 g_0 - 2Ric_0\Big)d$$

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and is conformally invariant in the sense that if $g = e^{2\lambda} g_0$, then

$$P_g^4 = e^{-4\lambda} P_0^4.$$

On the other hand, it is well known that in dimension two if $g = e^{2\lambda} g_0$, the associated Laplacians are related by

$$\Delta_g = e^{-2\lambda} \Delta_0$$

and the associated scalar curvatures R are related by the equation

$$\Delta_0 \lambda - \frac{1}{2} R_0 = -\frac{1}{2} e^{2\lambda} R,$$

where $R = 2K$ and K is the Gaussian curvature. If the dimension is four and $g = e^{2\lambda} g_0$, we get

$$P_0^4 \lambda + Q_0^4 = e^{4\lambda} Q_g^4$$

where

$$Q_0^4 = -\frac{1}{6} \Delta_0 R_0 - \frac{1}{2} |Ric_0|^2 + \frac{1}{6} R_0^2.$$

Moreover, the fourth order Q-curvature equation is closely related to the Gauss-Bonnet-Chern formula

$$8\pi^2 \chi(M^4) = \int_M \Big(Q_g^4 + \frac{1}{4} W \Big) d\mu_g$$

which is the 2-dimensional analogue of Euler-Poincaré characteristic

$$4\pi \chi(M^2) = \int_M R_g \, d\mu_g$$

where W denotes the Weyl tensor w.r.t. g.

From the previous point of view, we may call attention to the generalized notion of the Gaussian curvature — the fourth order Q-curvature equation. More precisely, the well known uniformization theorem on closed surfaces says that every smooth metric is pointwise conformal to a constant scalar one. Branson, Chang and Yang studied the problem of prescribed constant Q-curvature metrics ([BCY]) on 4-manifolds with the positive Paneitz operator. However, it is quite different for the Q-curvature equations between 3-manifolds and 4-manifolds, part of reasons was due to the presence of a possibly negative term of the Q-curvature in the conformal Paneitz operator as in (1.1). In this paper, we deal with existence problems of constant Q-curvature metrics on 3-manifolds.

From now on we consider a closed smooth Riemannian 3-manifold $(M, [g_0])$ with a fixed conformal class $[g_0]$. The Paneitz operator with respect to g_0 is

$$P_0 = P_0^3 = (-\Delta_0)^2 + \mathrm{div}_0\Big(\frac{5}{4}R_0 g_0 - 4Ric_0\Big)d - \frac{1}{2}Q_0,$$

and

$$Q_0 = Q_0^3 = -2|Rc_0|^2 + \frac{23}{32}R_0^2 - \frac{1}{4}\Delta_0 R_0 = -\frac{1}{4}\Big(\Delta_0 R_0 + 8|Z_0|^2 - \frac{5}{24}R_0^2\Big)$$

where Z_0 is the traceless Ricci curvature tensor with respect to the metric g_0.

Let $g = e^{2\lambda}g_0 = u^{-4}g_0$, for $g \in [g_0]$. The Paneitz operator P with respect to g has the following conformal covariance property:

$$Pw = u^7 P_0(uw).$$

Moreover, the Q-curvature is related by the nonlinear equation

$$P_0 u = -\frac{1}{2}Qu^{-7} \tag{1.2}$$

where

$$Q = -\frac{1}{4}\Big(\Delta R + 8|Z|^2 - \frac{5}{24}R^2\Big).$$

In the present paper, we consider the functional $\mathcal{F}$ on a given conformal class $[g_0]$:

$$\mathcal{F}(g) = \Big(\int_M u^{-6}\, d\mu_0\Big)^{\frac{1}{3}} \int_M P_0 u \cdot u \, d\mu_0. \tag{1.3}$$

Since for $g = e^{2\lambda}g_0 = u^{-4}g_0$, we have $\int_M u^{-6}\, d\mu_0 = \int_M e^{3\lambda}\, d\mu_0 = \int_M d\mu$. Then it reduces to:

$$\mathcal{F}(g) = \frac{1}{8}\Big(\int_M d\mu\Big)^{\frac{1}{3}}\Big[8\int_M |Z|^2\, d\mu - \frac{5}{24}\int_M R^2\, d\mu\Big].$$

Note that $\mathcal{F}(g)$ is neither bounded above nor below on $(M^3, [g_0])$. Furthermore, the critical points of the functional $\mathcal{F}(g)$ satisfy the equation (1.2) with Q given by a constant. Then, for minimizing $\mathcal{F}(g)$ in $[g_0]$, it is natural to consider the following fourth order parabolic equation:

$$\frac{\partial g}{\partial t} = -8(Q - \overline{Q})g, \tag{1.4}$$

where $\overline{Q}=\frac{\int_M Q\,d\mu}{\int_M d\mu}=\frac{\frac{1}{4}[-8\int_M |Z|^2\,d\mu+\frac{5}{24}\int_M R^2\,d\mu]}{\int_M d\mu}$.

Let $\lambda : M\times[0,\infty)\to\mathbf{R}$ be a smooth function and $g(p,t)=e^{2\lambda(p,t)}g_0(p)$ $=u(p,t)^{-4}g_0(p)$, $p\in M$. Then the equation (1.4) reduce to the following initial value problem of fourth order parabolic scalar equation:

$$\begin{cases}\dfrac{\partial\lambda}{\partial t}=-4(Q-\overline{Q})=\Delta R+8|Z|^2-\dfrac{5}{24}R^2+4\overline{Q},\\ g=e^{2\lambda}g_0;\ \lambda(p,0)=\lambda_0(p),\\ \displaystyle\int_{M^3}e^{3\lambda_0}\,d\mu_0=\int_{M^3}d\mu_0,\end{cases}\tag{1.5}$$

where $d\mu_0$ is the volume element of g_0. Note that the volume V with respect to g is fixed under the flow (1.5).

In the paper [Br], the author proved the longtime existence and convergence of solutions of the Q-curvature flow on 4-manifolds with positive Paneitz operator. In this paper, we prove the same result on 3-manifolds with positive Paneitz operator. Our approach is inspired by earlier works of the authors ([CW1], [CW3]). The crucial step is how to obtain the so-called Bondi-mass loss formula on 3-manifolds.

Let us compare the functional $\mathcal{F}$ with the following nonnegative quadratic Riemannian functional $\mathcal{E}$ of the scalar curvature on $(M,[g_0])$ with fixed volume

$$\mathcal{E}(g)=\int_M R^2\,d\mu.$$

A critical point of $\mathcal{E}$ is called an extremal metric. On closed surfaces, it is due to Calabi that the extremal metric always has constant scalar curvature if it exists ([Ca]). Thus one may consider the following gradient flow of $\mathcal{E}$:

$$\begin{cases}\dfrac{\partial\lambda}{\partial t}=\triangle R-\dfrac{n-4}{4(n-1)}R^2+\dfrac{n-4}{4(n-1)}r,\\ g=e^{2\lambda}g_0;\ \lambda(p,0)=\lambda_0(p),\\ \displaystyle\int_{M^n}e^{n\lambda_0}\,d\mu_0=\int_{M^n}d\mu_0,\ r=\frac{\int_M R^2\,d\mu}{\int_M d\mu}.\end{cases}\tag{1.6}$$

For closed surfaces with constant Gaussian curvature background metric g_0, Chruściel ([Chru])[1] proved the longtime existence and convergence of

[1]In fact, Chruściel considered the Robinson-Trautman equation $\frac{\partial\lambda}{\partial t}=\triangle R$, in which decreases the functional on closed surfaces.

solutions of (1.6). Later, the first author generalized the results to any arbitrary background metric on closed surfaces as well as on 3-manifolds ([Ch], [CW1], [CW2], [CW3]) due to Bondi-mass type estimates.

We first show a Harnack-type estimate.

THEOREM 1.1. *Let P_0 be the Paneitz operator on $(M,[g_0])$. Suppose that under the flow* (1.5)

$$\int_M e^{-\lambda}\, d\mu_0 \leq H, \tag{$*$}$$

for a positive constant H which is independent of t. Then the solution of (1.5) *exists on $M\times[0,\infty)$ and converges smoothly to a metric of constant Q-curvature.*

Condition $(*)$ is satisfied if P_0 is a strictly positive Paneitz operator on $(M,[g_0])$. Then we have our main Theorem as follows:

THEOREM 1.2. *Let the Paneitz operator P_0 on $(M,[g_0])$ be strictly positive. Then the solution of* (1.5) *exists on $M\times[0,\infty)$ and converges smoothly to a metric of constant Q-curvature.*

Let $A_0 = Rc_0 - \frac{R_0}{4}g_0$ be the Schouten tensor on $(M^3,[g_0])$ and $\sigma_2(A_0)$ denote the second elementary symmetric function of the eigenvalues of the Schouten tensor A_0. A simple calculation gives

$$\sigma_2(A_0) = -\frac{1}{2}|Z_0|^2 + \frac{1}{48}R_0^2.$$

Then

$$Q_0 = 4\sigma_2(A_0) - \frac{1}{32}R_0^2 - \frac{1}{4}\Delta_0 R_0$$

and

$$\int_M Q_0\, d\mu_0 = 4\int_M \sigma_2(A_0)\, d\mu_0 - \frac{1}{32}\int_M R_0^2\, d\mu_0.$$

Suppose that $R_0 > 0$, $\sigma_2(A_0) > 0$ and $Q_0 \leq 0$. Then based on the Bochner type estimates, the Paneitz operator P_0 is strictly positive if Q_0 is not identically zero ([HY]). As a consequence of Theorem 1.2, it follows that

THEOREM 1.3. *Let* $(M, [g_0])$ *be a closed* 3*-manifold with* $R_0 > 0$, $\sigma_2(A_0) > 0$, $Q_0 \leq 0$ *and suppose* Q_0 *is not identically zero. Then the solution of* (1.5) *exists on* $M \times [0, \infty)$ *and converges smoothly to a metric of constant* Q*-curvature.*

Remark 1.1. (i) The method of [Br] does not apply to 4-manifolds for which the Paneitz operator has negative first eigenvalue. By using the Bondi-mass type estimate, it is our goal in a forthcoming paper to prove the longtime existence and convergence of solutions of (1.5) on 3-manifolds for which the Paneitz operator has negative first eigenvalue. Moreover, we would like to find further geometric aspects of condition $(*)$ when $Q_0 > 0$. In fact, we conjecture that $(*)$ holds if under the flow (1.5)

$$(**) \qquad \int_M Q \, d\mu > 0$$

for $t = 0$. Moreover when M is a CR 3-manifold, the Q-curvature is defined as in [H] and [FH]. However condition $(**)$ fails on a CR 3-manifold. We will discuss it elsewhere ([CCC], see Remark 4.1 in Section 4).

(ii) In the papers [XY], [YZ] and [HY], the existence of minimizers for a variational functional was proved for more general situations.

Due to the lack of a maximum principle for the fourth order parabolic equation (1.5), we will apply the integral method as in [CW3] to obtain the C^0-estimate. However, we point out that the Bondi-mass type estimate (Theorem 2.2) is the starting point for applying the integral method.

We briefly describe the methods used in our proofs. In Section 2, we will derive the key estimate of equation (1.5) which is based on the energy bound (Lemma 2.1).

In Section 3, we are able to control the L^2-norm of curvatures and $W_{2,2}$-estimates for the conformal factor λ under the flow (1.5) with condition $(*)$. Finally based on our previous work ([Chru], [CW3]), we obtain higher-order $W_{k,2}$-estimates. Then the long-time existence and asymptotic convergence of solutions of (1.5) follows easily.

In view of previous sections, we reduce the proof of Theorem 1.2 to find a uniformly bound $(*)$ under the flow (1.5) as in Section 4.

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§2. The Bondi-mass type estimate

In this section, we will prove a kind of the Bondi-mass type estimate of equation (1.5) as in [CW3].

For $g = e^{2\lambda} g_0$, $R_0 = R_{g_0}$, we have the following formulae for (1.5):

$$R = R_g = e^{-2\lambda}(R_0 - 4\Delta_0\lambda - 2|\nabla_0\lambda|^2), \tag{2.1}$$

$$\Delta R = e^{-2\lambda}(\Delta_0 R + \langle \nabla_0 R, \nabla_0 \lambda\rangle), \text{ where } \Delta_0 = \Delta_{g_0},\ \Delta = \Delta_g, \tag{2.2}$$

$$Z = Z_0 - e^{-\lambda}(\nabla^2 e^{\lambda}) + \frac{1}{3}(e^{-\lambda}\Delta e^{\lambda})g, \tag{2.3}$$

$$Z = Z_0 + e^{\lambda}(\nabla_0^2 e^{-\lambda}) - \frac{1}{3}(e^{\lambda}\Delta_0 e^{-\lambda})g_0, \tag{2.4}$$

$$\frac{\partial R}{\partial t} = -8R(-Q + \overline{Q}) - 16\Delta(-Q), \tag{2.5}$$

$$\int_M d\mu = \int_M e^{3\lambda}\, d\mu_0 = \int_M e^{3\lambda_0}\, d\mu_0 = \int_M d\mu_0 = V_0. \tag{2.6}$$

From now on, C denotes a generic constant which may vary from line to line. Then we have

LEMMA 2.1. *Under the flow* (1.5)*, there exists a positive constant $\beta = \beta(R_0, \lambda_0)$ such that*

$$\mathcal{F}(g) = \Big[8\int_M |Z|^2\, d\mu - \frac{5}{24}\int_M R^2\, d\mu\Big] \le \beta^2, \tag{2.7}$$

for $0 \le t \le T \le \infty$.

Proof. This is because (1.5) is the negative gradient flow of $\mathcal{F}(g)$. That is

$$\frac{d}{dt}\Big[8\int_M |Z|^2\, d\mu - \frac{5}{24}\int_M R^2\, d\mu\Big] = -\int_M \Big(\Delta R + 8|Z|^2 - \frac{5}{24}R^2 + 4\overline{Q}\Big)^2\, d\mu.$$

□

THEOREM 2.2. *Let P_0 be the Paneitz operator on $(M, [g_0])$. Suppose that condition $(*)$ is satisfied under the flow* (1.5)*. Then there exist a positive constant $C(H, \beta, \alpha)$ such that*

$$\lambda \ge -C(H, \beta, \alpha),$$

for $0 \le t \le T$*. Furthermore for all* $\alpha > 0$

$$\int_M e^{\alpha\lambda}\, d\mu_0 \le C(H, \beta, \alpha).$$

In particular we have

$$\int_M e^{6\lambda}\, d\mu_0 \le C(H, \beta), \tag{2.8}$$

for $0 \le t \le T$*.*

Remark 2.1. In Theorem 3.1, we obtain the C^0-estimate for λ which is based on the so called Bondi-mass type estimate as in (2.8). We refer to our previous works [CW1], [CW2] and [CW3].

Proof. From $(*)$ we have

$$\begin{aligned}
&\int_M P_0 u \cdot u\, d\mu_0 \\
&= \int_M (\Delta_0 u)^2\, d\mu_0 - \frac{5}{4}\int_M R_0 |\nabla_0 u|^2\, d\mu_0 \\
&\qquad + 4\int_M Rc_0(\nabla_0 u, \nabla_0 u)\, d\mu_0 - \frac{1}{2}\int_M Q_0 u^2\, d\mu_0 \\
&\ge \int_M (\Delta_0 u)^2\, d\mu_0 - C(Rc_0)\int_M |\nabla_0 u|^2\, d\mu_0 - C(Q_0)\int_M u^2\, d\mu_0 \\
&\ge \frac{1}{2}\int_M (\Delta_0 u)^2\, d\mu_0 - C(Q_0)\int_M u^2\, d\mu_0 - C(Rc_0, Q_0) \\
&\ge \frac{1}{2}\int_M (\Delta_0 u)^2\, d\mu_0 - C(Rc_0, Q_0, H).
\end{aligned} \tag{2.9}$$

Then from (2.7) and (2.9), we have

$$\int_M (\Delta_0 u)^2\, d\mu_0 \le C(\beta). \tag{2.10}$$

Furthermore from (2.10), one obtains

$$\|u\|_{W_{2,2}(d\mu_0)} \le C(H, \beta)$$

and then

$$\|u\|_{W_{1,6}(d\mu_0)} \le C(H, \beta) \quad \text{and} \quad \lambda \ge -C(H, \beta). \tag{2.11}$$

On the other hand from (2.11) one can compute

$$\begin{aligned}
\int_M |\nabla_0\lambda|^3\, d\mu_0 &= 8\int_M e^{\frac{3}{2}\lambda}|\nabla_0 e^{-\frac{1}{2}\lambda}|^3\, d\mu_0 \\
&\le C\int_M e^{3\lambda}\, d\mu_0 + C\int_M |\nabla_0 e^{-\frac{1}{2}\lambda}|^6\, d\mu_0 \\
&= C\int_M e^{3\lambda}\, d\mu_0 + C\int_M |\nabla_0 u|^6\, d\mu_0 \\
&\le C.
\end{aligned} \tag{2.12}$$

Now since $\int_M e^{3\lambda}\, d\mu_0$ is fixed and $\lambda \ge -C(H,\beta)$, one has

$$|\overline{\lambda}| \le C,$$

for $\overline{\lambda} = \frac{\int_M \lambda\, d\mu_0}{\int_M d\mu_0}$. Then from (2.12), it follows that

$$\|\lambda - \overline{\lambda}\|_{W_{1,3}} \le C.$$

Also it follows from Moser's inequality ([A]) that

$$\int_M e^{\alpha(\lambda-\overline{\lambda})}\, d\mu_0 \le C(\alpha)$$

and

$$\int_M e^{\alpha\lambda}\, d\mu_0 \le C(H,\beta,\alpha),$$

for all $\alpha > 0$. ∎

§3. Harnack-type estimates and asymptotic convergence

In this section, we will prove a kind of the Harnack-type estimate of equation (1.5) as in [CW3].

THEOREM 3.1. *Let P_0 be any Paneitz operator on $(M, [g_0])$. Suppose that condition $(*)$ is satisfied under the flow (1.5). Then there exist positive constants $C_0 = C_0(H,\beta)$, $C_1 = C_1(H,\beta)$ such that*

$$\|\lambda\|_{W_{2,2}} \le C_0$$

and

$$\|\lambda\|_{L_\infty} \le C_1$$

for all $0 \le t \le T$.

Remark 3.1. We will show later that condition $(*)$ is satisfied if P_0 is a positive Paneitz operator on $(M,[g_0])$.

Proof. From (2.11) and (2.8), we have

$$\begin{aligned}\int_M |\nabla_0\lambda|^4\,d\mu_0 &= \frac{1}{8}\int_M e^{2\lambda}|\nabla_0 e^{-\frac{1}{2}\lambda}|^4\,d\mu_0\\ &\le C\Big(\int_M e^{6\lambda}\,d\mu_0\Big)^{\frac{1}{3}} + C\Big(\int_M |\nabla_0 e^{-\frac{1}{2}\lambda}|^6\,d\mu_0\Big)^{\frac{2}{3}}\\ &= C\Big(\int_M e^{6\lambda}\,d\mu_0\Big)^{\frac{1}{3}} + C\Big(\int_M |\nabla_0 u|^6\,d\mu_0\Big)^{\frac{2}{3}}\\ &\le C.\end{aligned}$$

Then

$$\lambda \in W_{1,4}(d\mu_0) \quad \text{and} \quad \|\lambda\|_{L_\infty} \le C_1(H,\beta). \tag{3.1}$$

Moreover, since

$$\int_M e^{-\lambda}\Big(-\frac{1}{2}\Delta_0\lambda + \frac{1}{4}|\nabla_0\lambda|^2\Big)^2\,d\mu_0 = \int_M (\Delta_0 u)^2\,d\mu_0 \le C(\beta),$$

it follows from (3.1) that

$$\int_M (\Delta_0\lambda)^2\,d\mu_0 \le C_0.$$

This completes the proof. ∎

For higher order estimates ([Chru], [CW3]), we have

Theorem 3.2. *Suppose the same assumptions of the previous Theorem* 3.1 *hold. There exists a constant* $C = C(\|\lambda-\overline{\lambda}\|_{W_{2,2}}, H, \beta)$*,* $l \ge 2$ *such that*

$$\|\nabla_0^l\lambda(p,t)\|_{L_2} \le C,$$

for $0 \le t \le T$*.*

Proof. First it follows easily from Theorem 3.1 that

$$\int_M R^2\,d\mu = \int_M e^{-\lambda}(R_0 - 4\Delta_0\lambda - 2|\nabla_0\lambda|^2)^2\,d\mu_0 \le \overline{\beta}^2.$$

This implies that

$$\int_M |Z|^2\, d\mu \le C(\beta) + C\int_M R^2\, d\mu \le C(\beta, \overline{\beta}).$$

Then it is straightforward that

$$\frac{d}{dt}\|e^{-4\lambda}{\nabla_0}^l\lambda(p,t)\|_{L_2}^2 \le -C\|{\nabla_0}^{l+2}\lambda(p,t)\|_{L_2}^2 + C\|\lambda - \overline{\lambda}\|_{W_{2,2}}^2.$$

Therefore the Theorem follows easily. We refer to [Chru] and [CW3] for details. ∎

THEOREM 3.3. *Suppose the same assumptions of the previous Theorem* 3.1 *hold. Then*

$$R \overset{C^\infty}{\longrightarrow} R_\infty;\ Z \overset{C^\infty}{\longrightarrow} Z_\infty;\ \overline{Q} \overset{C^\infty}{\longrightarrow} \overline{Q}_\infty$$

where

$$\Delta_\infty R_\infty + 8|Z_\infty|^2 - \frac{5}{24}R_\infty^2 + 4\overline{Q}_\infty = 0$$

over M.

Proof. Since

$$-C \le 8\int_M |Z|^2\, d\mu - \frac{5}{24}\int_M R^2\, d\mu \le \beta^2$$

and

$$-\frac{d}{dt}\Big[8\int_M |Z|^2\, d\mu - \frac{5}{24}\int_M R^2\, d\mu\Big] = \int_M \Big(\Delta R + 8|Z|^2 - \frac{5}{24}R^2 + 4\overline{Q}\Big)^2 d\mu,$$

it follows that

$$\int_0^\infty \int_M \Big(\Delta R + 8|Z|^2 - \frac{5}{24}R^2 + 4\overline{Q}\Big)^2 d\mu dt < \infty.$$

On the other hand from Theorem 3.1 and Theorem 3.2, it follows that

$$\|\lambda\|_{W_{k,2}(d\mu_0)} \le C,$$

for all $0 \le t < T \le \infty$. Then based on the work [S]

$$\int_M \Big(\Delta R + 8|Z|^2 - \frac{5}{24}R^2 + 4\overline{Q}\Big)^2 d\mu_0 \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore from elliptic estimates and interpolation inequalities

$$R \xrightarrow{C^\infty} R_\infty;\ Z \xrightarrow{C^\infty} Z_\infty;\ \overline{Q} \xrightarrow{C^\infty} \overline{Q}_\infty$$

where

$$\Delta_\infty R_\infty + 8|Z_\infty|^2 - \frac{5}{24} R_\infty^2 + 4\overline{Q}_\infty = 0$$

over M. ∎

§4. The positive Paneitz operator

In this section, we prove that condition $(*)$ is satisfied if P_0 is a strictly positive Paneitz operator on $(M, [g_0])$ and then the main Theorem 1.2 follows easily.

Lemma 4.1. *Let the Paneitz operator P_0 on $(M, [g_0])$ be strictly positive. Then under the flow* (1.5)*, there exists a positive constant $H = H(\beta)$ such that*

$$\int_M e^{-\lambda}\, d\mu_0 \leq H,$$

for $0 \leq t \leq T \leq \infty$.

Proof. Since P_0 is a positive operator, there exists a positive constant $\lambda_1 > 0$ such that

$$\mathcal{F}(g) = \Big(\int_M u^{-6}\, d\mu_0\Big)^{\frac{1}{3}} \int_M P_0 u \cdot u\, d\mu_0 \geq V_0 \lambda_1 \int_M u^2\, d\mu_0. \tag{4.1}$$

But from Lemma 2.1, one has

$$\mathcal{F}(g) \leq \beta^2,$$

for all $0 \leq t \leq T \leq \infty$. It follows from (4.1) that

$$\int_M e^{-\lambda}\, d\mu_0 = \int_M u^2\, d\mu_0 \leq H,$$

for $0 \leq t \leq T \leq \infty$. ∎

Combining Theorem 3.3 and Lemma 4.1, we have convergence of solutions of the Q-curvature flow (1.5) on $(M, [g_0])$ as follows:

THEOREM 4.2. *Let the Paneitz operator P_0 on $(M,[g_0])$ be strictly positive. Then the solution of* (1.5) *exists on $M\times[0,\infty)$ and converges smoothly to a metric of constant Q-curvature*

$$Q_\infty = \overline{Q}_\infty.$$

Now if $R_0 > 0$, $\sigma_2(A_0) > 0$, $Q_0 \le 0$ and Q_0 is not identically zero. Then P_0 is strictly positive. As a consequence of Theorem 1.2, it follows that

THEOREM 4.3. *Let $(M,[g_0])$ be a closed 3-manifold with $R_0 > 0$, $\sigma_2(A_0) > 0$, $Q_0 \le 0$ and Q_0 is not identically zero. Then the solution of* (1.5) *exists on $M\times[0,\infty)$ and converges smoothly to a metric of constant Q-curvature.*

Remark 4.1. Next we consider Paneitz operator P_0 with negative first eigenvalue, for example when $Q_0 > 0$. Following the methods as in [CW2] or [CW3], one can compute (from (1.2))

$$\begin{aligned}\frac{d}{dt}\int_M e^{-\lambda}\,d\mu_0 &= 4\int_M e^{-\lambda}Q\,d\mu_0 - 4\overline{Q}\int_M e^{-\lambda}\,d\mu_0\\ &= -8\int_M e^{-\frac{9}{2}\lambda}P_0e^{-\frac{1}{2}\lambda}\,d\mu_0 - 4\overline{Q}\int_M e^{-\lambda}\,d\mu_0.\end{aligned}$$

Then

$$\frac{d}{dt}\int_M e^{-\lambda}\,d\mu_0 \le C - 4\overline{Q}\int_M e^{-\lambda}\,d\mu_0$$

provided we have

$$-8\int_M e^{-\frac{9}{2}\lambda}P_0e^{-\frac{1}{2}\lambda}\,d\mu_0 \le C. \tag{4.2}$$

Moreover, if we assume that

$$\int_M Q\,d\mu > 0$$

for $t=0$. Thus

$$-4\overline{Q} < 0$$

for all $t > 0$ and

$$\frac{d}{dt}\int_M e^{-\lambda}\,d\mu_0 \le C - C\int_M e^{-\lambda}\,d\mu_0.$$

It follows that condition $(*)$ holds if we can have the estimate (4.2) when the operator P_0 has negative first eigenvalue. We will deal this case in a forthcoming paper.

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