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SOME GROUPS OF TYPE E_7

T. A. SPRINGER

Dedicated to George Lusztig

Abstract. An algebraic group of type E_7 over an algebraically closed field has an irreducible representation in a vector space of dimension 56 and is, in fact, the identity component of the automorphism group of a quartic form on the space. This paper describes the construction of the quartic form if the characteristic is $\neq 2, 3$, taking into account a field of definition F. Certain F-forms of E_7 appear in the automorphism groups of quartic forms over F, as well as forms of E_6. Many of the results of the paper are known, but are perhaps not easily accessible in the literature.

§1. Introduction

1.1. A simply connected algebraic group of type E_7 over $\mathbb{C}$ has an irreducible representation of dimension 56 and is, in fact, the identity component of the isotropy group of a quartic form in 56 variables. These facts are already contained in É. Cartan's thesis of 1894 (see [Ca, pp. 273–274]).

One encounters the representation in other places: in the theory of prehomogeneous vector spaces due to M. Sato (see [SK, (29), p. 147]) and in the Dynkin-Kostant analysis of nilpotent elements of simple Lie algebras (as a representation of a Levi group on a graded piece of a simple Lie algebra of type E_8, the ingredients being associated to a nilpotent element of type A_1, see [Car, p. 401, p. 405]).

The present paper is oriented towards the use of this particular representation for obtaining information about groups of type E_7 over arbitrary ground fields. For groups of type E_6 the irreducible representation of dimension 27 has been used for a similar purpose (see, for example, [SV]). The algebraic machinery of [loc. cit.] can also be exploited to deal with E_7 over fields of characteristic $\neq 2, 3$. Some ingredients can be found in the

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literature (for example in [F], [Br], [G1], [G2]). But as far as I know there is no treatment of these matters in the context of the theory of algebraic groups, also taking care of fields of definition. A large part of the present paper is devoted to an exposition of relevant material (some of it very old). Thus the paper is of a somewhat expository character. I hope it will be useful in further studies of groups of type E_7.

We follow Freudenthal's construction [F, Section 4] of a 56-dimensional quartic form over $\mathbb{R}$. This requires some material about the cubic forms in 27 variables whose isotropy group is of type E_6. These forms appear in Albert algebras (exceptional simple Jordan algebras). What we need is mainly contained in [SV].

It is convenient to build in a duality into the discussion of the cubic forms, which leads to the notion of an E_6-structure (see 1.2). In 1.8 the twisted version of a Hermitian E_6-structure is introduced.

In Section 3, starting from an E_6-structure, a quartic form is introduced and it is shown that the identity component of its isotropy group is a simply connected group of type E_7 (see Cor. 2.6).

One then introduces a ternary product on the vector space underlying the quartic form. It satisfies certain identities, viz. those of a "Freudenthal triple system". We call the triple systems occurring here *E_7-structures*. Section 4 contains some basic results about these. It does not exploit too much the formalism of Freudenthal triple systems, but instead uses geometric arguments.

Section 5 discusses questions involving ground fields. For example, the automorphism group of an E_7-structure over the field F is a "strong" form over F of the simply connected group of type E_7 and any such form can be so obtained (see Prop. 5.3). The isotropy group in G of a rational point where the quartic form does not vanish is a (possibly outer) F-form of the simply connected group of type E_6 (see Prop. 5.5).

A further study of E_7-structures and of the closely related Hermitian E_6-structures should be useful for understanding some forms of groups of type E_7 (respectively, some outer forms of groups of type E_6). Section 6 contains some indications about what a further study might lead to.

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1.2. Recollections

In the sequel k is an algebraically closed field of characteristic $\neq 2, 3$.

Let A and B be vector spaces over k. Assume given a (non-degenerate) pairing $\langle\ ,\ \rangle$ between A and B.

Let f, g be cubic forms on A, respectively, B. We denote by $f(\ ,\ ,\ )$ the symmetric trilinear form on A with $f(a,a,a) = f(a)$ $(a \in A)$ and by $g(\ ,\ ,\ )$ the analogous trilinear form on B.

We introduce symmetric bilinear product maps $A \times A \to B$, $B \times B \to A$, denoted by $(a_1, a_2) \mapsto a_1 a_2$, $(b_1, b_2) \mapsto b_1 b_2$, by

$$3f(a, a_1, a_2) = \langle a, a_1 a_2 \rangle, \quad 3g(b, b_1, b_2) = \langle b_1 b_2, b \rangle. \tag{1}$$

The crucial conditions are

$$(aa)(aa) = f(a)a, \quad (bb)(bb) = g(b)b \quad (a \in A,\ b \in B). \tag{2}$$

If, moreover,

(a) $\dim A = 27$,

(b) the cubic forms f, g are irreducible,

we say that $S = (A, B, \langle\ ,\ \rangle, f, g)$ is an E_6*-structure* (the name is explained by Prop. 1.6).

Let F be subfield of k. We say that S is (defined) over F if A and B have F-structures (in the sense of [Sp3, 11.1]) and if $\langle\ ,\ \rangle$, f and g are defined over F (relative to these F-structures).

Let S' be another E_6-structure over F. The notion of F-isomorphism of S and S' is clear. S' is *equivalent* to S if it is F-isomorphic to an E_6-structure of the form

$$S'' = (A, B, \gamma\langle\ ,\ \rangle, \alpha f, \beta g)$$

with $\alpha, \beta, \gamma \in F^*$. It is readily seen that for (2) to hold in S'' we must have $\gamma^3 = \alpha\beta$ and that S'' is F-isomorphic to S if $\alpha, \beta \in (F^*)^3$.

1.3. Example

Let $\mathcal{A}$ be an Albert algebra over F, see [SV, p. 118]. Take $A = B = k \otimes_F \mathcal{A}$. On $\mathcal{A}$ we have a non-degenerate symmetric bilinear form, which defines a pairing $\langle\ ,\ \rangle$ between A and B. Let $f = g$ be the cubic form on A defined by the determinant form det of the Albert algebra and define the products by (1). Then (2) holds, see [loc. cit., Lemma 5.2.1]. We obtain an E_6-structure $S(\mathcal{A})$ over F.

It follows from the Theorem of [Sp1, p. 260] and [SV, 5.4.5, 5.4.6] that any E_6-structure over F is equivalent to such an $S(\mathcal{A})$.

1.4. The standard E_6-structure

There is a particular case of the construction of 1.3 which goes back to Freudenthal [F, Section 26].

Let M be the vector space of 3×3-matrices over k. Let d be the determinant function on M. Put

$$A_0 = M \oplus M \oplus M.$$

Define a symmetric bilinear form on A_0 by

$$\langle (a, b, c), (a', b', c') \rangle = \mathrm{tr}(aa' + bb' + cc')$$

and a cubic form f_0 by

$$f_0((a, b, c)) = \mathrm{d}(a) + \mathrm{d}(b) + \mathrm{d}(c) - \mathrm{tr}(abc).$$

Then $S_0 = (A_0, A_0, \langle\ ,\ \rangle, f_0, f_0)$ is the *standard* E_6-structure (which, in fact, comes from an Albert algebra structure on A_0, as in 1.3). A proof of (2) for this case is given in [Sp2, 5.12].

S_0 is defined over any subfield F of k. In fact, it is a specialization of a "universal" E_6-structure.

Let $\mathcal{R} = \mathbb{Z}[\frac{1}{6}]$. Let $\mathcal{A}_0$ be the direct sum of three copies of the 3×3-matrices over $\mathcal{R}$, provided with a symmetric bilinear form and a cubic form defined as above. We define in an obvious manner the notion of E_6-structure $\mathcal{S}_0$ over $\mathcal{R}$, such that A_0 is obtained by specialization: $A_0 = k \otimes_{\mathcal{R}} \mathcal{A}_0$.

1.5. Some algebraic groups

Let S, as before, be an E_6-structure over F. Let H (the automorphism group of S) be the subgroup of of $GL(A) \times GL(B)$ formed by the pairs $(t, \tilde{t})$ such that

$$t(a)t(a) = \tilde{t}(aa), \quad \tilde{t}(b)\tilde{t}(b) = t(bb), \tag{3}$$

$$\langle t(a), \tilde{t}(b) \rangle = \langle a, b \rangle \quad (a \in A,\, b \in B). \tag{4}$$

Then t ($\tilde{t}$) leaves invariant f (respectively, g).

Assume that $t \in GL(A)$ leaves invariant f and define $\tilde{t}$ by (4). Then the first formula (3) holds by (1). The second relation (3) also holds, cf. [SV, proof of Prop. 7.3.1], hence $\tilde{t}$ leaves g invariant.

These facts imply that the first (second) projection defines an isomorphism of H onto the invariance group of f in $GL(A)$ (respectively, of g in $GL(B)$). Hence equivalent E_6-structures have isomorphic automorphism groups.

Proposition 1.6. *H is a connected, quasi-simple, simply connected group of type E_6 which is defined over F.*

Proof. We saw in 1.3 that S is F-equivalent to an E_6-structure of the form $S(\mathcal{A})$ where $\mathcal{A}$ is an Albert algebra over F. If $S = S(\mathcal{A})$ the form f is the cubic form det of the Albert algebra $\mathcal{A}$. By [SV, 7.3.2] the invariance group of det has the asserted properties.

Corollary 1.7. *Equivalent F-structures over F have F-isomorphic automorphism groups.*

A group of type E_6 has outer automorphisms of order 2. It follows from Prop. 1.6 that $(t, \tilde{t}) \mapsto (\tilde{t}, t)$ defines an automorphism of H of order 2. By [loc. cit.] it is an outer automorphism.

Let i be the imbedding of $\mathbb{G}_m$ in $GL(A) \times GL(B)$ with $i(\alpha)(a, b) = (\alpha a, \alpha^{-1} b)$ ($a \in A$, $b \in B$, $\alpha \in k^*$). Define $H_1 = i(\mathbb{G}_m).H$, a closed subgroup of $GL(A) \times GL(B)$.

1.8. Hermitian E_6-structures

Let $S = (A, B, \langle\ ,\ \rangle, f, g)$ be an E_6-structure, as before. Put $R = k \oplus k$ and let σ be the permutation isomorphism of R. Then $W = A \oplus B$ is a free R-module. For $w = (a, b)$, $w' = (a', b') \in W$ define a non-degenerate σ-Hermitian form H on the R-module W by

$$H(w, w') = (\langle a, b' \rangle, \langle a', b \rangle) \in R.$$

Moreover define

$$(a, b) \star (a', b') = (bb', aa'), \quad G((a, b)) = (f(a), g(b)).$$

Then G is a cubic form on the R-module W, let $G(\ ,\ ,\ )$ be the associated symmetric trilinear form. $(w, w') \mapsto w \star w'$ is a σ-bilinear map and the properties (1) and (2) are equivalent to

$$3G(w, w_1, w_2) = H(w, w_1 \star w_2), \quad (w \star w) \star (w \star w) = G(w)w. \tag{5}$$

Now let E/F be a separable quadratic field extension and let σ be its non-trivial automorphism. Then

$$a \otimes b \longmapsto (ab, a\sigma(b)) \quad (a \in k,\ b \in E)$$

defines an isomorphism $k \otimes_F E \mapsto R$, via which R obtains an F-structure with $R(F) = E$. The permutation automorphism of R is defined over F and induces σ on E.

Assume that W has an F-structure such that H, $\star$ and G are defined over F. We then say that $\Sigma = (E/F, W, H, G)$ is a *Hermitian E_6-structure over E and F* or briefly, *over E/F*.

Then $W(F)$ is a vector space over E, H induces a σ-Hermitian form on F and G a cubic form. $(a_1, a_2) \mapsto a_1 \star a_2$ defines a σ-bilinear product on $W(F)$.

More generally, in the situation considered in the beginning of this section we shall also speak of an Hermitian E_6-structure over E/F, in which case E is the étale algebra $F \oplus F$.

Let E be a quadratic étale algebra over F and denote its non-trivial automorphism by σ. Let Σ, as above, be a Hermitian E_6-structure over E/F. A similar Σ' is *equivalent* to Σ if it is E/F-isomorphic (defined in the obvious way) to a Hermitian E_6-structure of the form

$$(E/F, W, \beta H, \alpha G),$$

where $\beta \in F^*$, $\alpha \in E^*$. It is easy to see that we then must have $\beta^3 = \alpha\alpha^\sigma$ and that Σ' is E/F-isomorphic to Σ if $\alpha \in (E^*)^3$.

§2. The quartic form

2.1. Let S be an E_6-structure. Notations are as before. Put

$$V = A \oplus B \oplus k \oplus k.$$

For $v = (a, b, \xi, \eta)$, $v' = (a', b', \xi', \eta') \in V$ define

$$[v, v'] = \langle a, b' \rangle - \langle a', b \rangle + \xi\eta' - \xi'\eta. \tag{6}$$

$$h(v) = \langle bb, aa \rangle - \xi f(a) - \eta g(b) - \frac{1}{4}(\langle a, b \rangle - \xi\eta)^2. \tag{7}$$

Then $[\ ,\ ]$ is a non-degenerate alternating bilinear form on the 56-dimensional vector space V and h is a quartic form on V. We denote by $[\ ,\ ,\ ,\ ]$ the symmetric quadrilinear form on V such that $h(v) = [v, v, v, v]$. A straightforward computation (v and v' being as before) shows that

$$\begin{aligned} 4[v, v, v, v'] = {} & 2\langle bb', aa \rangle + 2\langle bb, aa' \rangle - \xi\langle a', aa \rangle - \eta\langle bb, b' \rangle - \xi' f(a) \\ & - \eta' g(b) - \frac{1}{2}(\langle a, b \rangle - \xi\eta)(\langle a', b \rangle + \langle a, b' \rangle - \xi\eta' - \xi'\eta). \end{aligned} \tag{8}$$

Let G be the subgroup of $GL(V)$ whose elements leave invariant h and $[\ ,\ ]$.

THEOREM 2.2. *G is a connected, quasi-simple, simply connected linear algebraic group of type E_7.*

We first establish some lemmas, to be used in the proof. For $x \in A$, $y \in B$ and $v = (a, b, \xi, \eta) \in V$ define

$$X_x(v) = (\eta x, 2xa, \langle x, b\rangle, 0), \quad Y_y(v) = (2yb, \xi y, 0, \langle a, y\rangle).$$

LEMMA 2.3. (i) $[X_x(v), v, v, v] = 0$;
(ii) $X_x^4 = 0$;
(iii) *The* X_x $(x \in V)$ *commute mutually*;
(iv) $[X_x(v), v'] + [v, X_x(v')] = 0$;
(v) (i), (ii), (iii) *and* (iv) *hold with* X_x *replaced by* Y_y.

Proof. To prove (i) use (8) with $v' = X_x(v)$. In the right-hand side several terms cancel. To deal with the remaining ones one uses (1) and the formulas

$$4(xa)(aa) = f(a)x + \langle x, aa\rangle a, \quad 4(yb)(bb) = g(b)y + \langle bb, y\rangle b,$$

which follow from (2).

The proofs of (ii), (iii) and (iv) are straightforward and can be omitted. (v) follows by symmetry.

For $x \in A$ put

$$t_x = 1 + X_x + \frac{1}{2}X_x^2 + \frac{1}{6}X_x^3.$$

We write $t_x = \exp(X_x)$.

LEMMA 2.4. (i) $t_x \in G$;
(ii) X_x *and* Y_y *lie in the Lie algebra* $\mathfrak{g}$ *of* G;
(iii) *The* t_x $(x \in A)$ *form a connected, commutative, unipotent subgroup of* G.

Proof. Parts (i) and (iv) of Lemma 2.3 shows that X_x lies in the Lie subalgebra of $\mathrm{End}(A)$ whose elements annihilate h and $[\ ,\ ]$. If $\mathrm{char}(k) = 0$ this Lie algebra is $\mathfrak{g}$ and if t is any nilpotent element of that Lie algebra, $\exp(t)$ lies in G, where now exp is the usual exponential map. (i) then follows from the previous Lemma.

If $p = \mathrm{char}(k) > 0$ it is prime to 6. To prove (i) in that case we use a reduction argument. Let $\mathcal{R}$ and $\mathcal{A}_0$ be as in 1.4. Since k is algebraically closed, A is isomorphic to $k \otimes_{\mathcal{R}} \mathcal{A}_0$. This follows from the fact that over an algebraically closed field all Albert algebras are isomorphic (see [SV, p. 153]), together with the observations about the connection between E_6-structures and Albert algebras made in 1.3.

Put $\mathcal{V} = \mathcal{A}_0 \oplus \mathcal{A}_0 \oplus \mathcal{R} \oplus \mathcal{R}$ and define on it an alternating form and a quartic form by (6) and (7). Passing to $\mathbb{C} \otimes_{\mathcal{R}} \mathcal{V}$, one sees that for $a \in \mathcal{A}_0$, $\exp(X_a)$ stabilizes the alternating and the quartic form. It induces a linear map of $V = k \otimes_{\mathcal{R}} \mathcal{V}$ of the form $t_x = \exp(X_x)$ which lies in G. Any t_x may be so obtained. (i) follows.

To prove (ii) for X_x observe that it is an image under the tangent map of the homomorphism $k \to G$ sending ξ to $t_{\xi x}$. The assertion for Y_y follows by symmetry. (iii) follows from the previous lemma.

Let H_1 be as above. For $h = (t, \tilde{t}) \in H_1$ there is $\nu(t) \in k^*$ with

$$f(t(a)) = \nu(t) f(a), \quad g(\tilde{t}(b)) = \nu(t)^{-1} g(b) \quad (a \in A,\, b \in B).$$

Define $\phi(h) \in GL(V)$ by

$$\phi(h)(a, b, \xi, \eta) = (t(a), \tilde{t}(b), \nu(t)^{-1}\xi, \nu(t)\eta).$$

It is straightforward to check that $\phi(h) \in G$ and that ϕ is an injective homomorphism of algebraic groups $H_1 \to G$. To simplify notations we view in the sequel H_1 as a subgroup of G, so we omit ϕ's.

Lemma 2.5. (i) *H_1 is the subgroup of G stabilizing the decomposition $V = A \oplus B \oplus k \oplus k$;*

(ii) *The identity component G° acts irreducibly in V.*

Proof. The proof of (i) is straightforward.

We claim that the four pieces of the decomposition of V afford distinct irreducible representations of H_1. The representations of the group H in the two 27-dimensional parts are dual to each other. These representations are irreducible. It is well-known that H, being a simply connected group of type E_6 has two (classes of) 27-dimensional irreducible representations, related by duality. It follows that the representations of H_1 in the two 27-dimensional pieces of V are inequivalent. The representations in the 1-dimensional pieces are obviously inequivalent, too. Our claim follows.

A G-stable subspace W of V must be a sum of some of the pieces of the decomposition of V. Also, W must be stabilized by $\mathfrak{g}$, in particular by the maps X_x and Y_y (by Lemma 2.4 (ii)). If W contains, say, $(0,0,0,1)$ then applying the X_x one sees that it contains the first 27-dimensional subspace. Continuing in this fashion one concludes that W must coincide with V. Similarly if W contains one of the 27-dimensional subspaces. (ii) follows.

Proof of Theorem 2.2. From Lemma 2.5 (ii) it follows that G is reductive (see [Sp3, Ex. 2.4.15]). Also, by Schur's Lemma the center of G is $\{\pm 1\}$. Consequently, the identity component G° is semi-simple.

Let T be a maximal torus of H. Then $T_1 = i(\mathbb{G}_m).T$ is a maximal torus of H_1. We claim that it is a maximal torus of G. Now the weights of T_1 in V are all distinct, as follows from the fact (which can be read off, for example, from the description of weights in [Sp2, 14.21]) that the weights of T in $A \oplus B \oplus \{0\} \oplus \{0\}$ are distinct. It follows that the centralizer of T_1 in G stabilizes the decomposition. Using Lemma 2.5 (i) the claim follows.

So G° is semi-simple of rank 7. It contains the group H, which is quasi-simple of type E_6. The Lie algebra $L(G)$ contains all X_x $(x \in A)$, which span a subspace of dimension 27 intersecting $L(H_1)$ in 0. Hence $\dim G \geq \dim H_1 + 27 = 106$. The classification of semi-simple group shows that G° is either of type E_7 or of type $A_1 + E_6$. In the latter case $\dim G^\circ$ would be 81, which is impossible. We conclude that G° is quasi-simple of type E_7. Since $-1 \in T_1 \subset G^\circ$ the center of G° has order 2, which implies that G° is simply connected.

To finish the proof we have to show that $G = G^\circ$. Assume that $G \neq G^\circ$ and take $g \in G - G^\circ$. Conjugation by g defines an automorphism of G°. Since an automorphism of a group of type E_7 is inner, there is $h \in G^\circ$ such that gh centralizes G°. By Lemma 2.5 (ii) and Schur's Lemma, gh is a scalar. The definition of G shows that the scalar must be -1. Since $-1 \subset G^\circ$ we arrive at the contradiction $g \in G^\circ$. This implies that G is connected.

Let G_1 be the subgroup of $GL(V)$ stabilizing the quartic form h.

Corollary 2.6. (i) *$G_1 = \mu_4 G$, where μ_4 is the group of* 4^{th} *roots of unity*;

(ii) *G is the identity component of G_1.*

Proof. The proof of (i) is based on the observation that Lemma 2.5 (i) holds with G_1 and $\mu_4 H_1$ instead of G and H. Using this one proceeds as in the proof of the Theorem. (ii) is a consequence of (i).

Let $\mathfrak{h}_1 \subset \mathfrak{g}$ be the Lie algebra of H_1 and denote by $\mathfrak{x}$, $\mathfrak{y}$ the subspaces of $\mathfrak{g}$ (actually, commutative subalgebras) spanned by the X_x, respectively, the Y_y ($x \in A$, $y \in B$). Let $e = (0,0,1,1) \in V$. We denote by Z_e the isotropy group of e in the subgroup Z of $GL(V)$ and by $\mathfrak{z}_e$ the annihilator of e in the Lie algebra $\mathfrak{z}$ of Z.

Let $\tilde{G}$ the subgroup $\mathbb{G}_m.G$ of $GL(V)$ generated by G and the homotheties and let $\tilde{\mathfrak{g}}$ be its Lie algebra.

Corollary 2.7. (i) $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{x} \oplus \mathfrak{y}$;

(ii) *H is the identity component $(G_e)^\circ$ and $\mathfrak{h} = \mathfrak{g}_e$;*

(iii) *G_e is connected and equals H; $\tilde{G}_e$ is a semi-direct product of $(\tilde{G}_e)^\circ$ and a group of order* 2, *whose generator induces an outer automorphism of $H = (\tilde{G}_e)^\circ$;*

(iv) $\tilde{\mathfrak{g}}.e = V$;

(v) *$g \mapsto g.e$ defines a smooth morphism $\tilde{G} \to V$.*

Proof. The sum of the dimension of the spaces in the right-hand of (i) is $133 = \dim \mathfrak{g}$. So it suffices to prove that if $X \in \mathfrak{h}_1$, $x \in A$, $y \in B$ and $X + X_x + Y_y = 0$ we have $X = 0$ and $x = 0$, $y = 0$. This follows by applying $X + X_x + Y_y$ to e, as

$$(X + X_x + Y_y).e = (x, y, \star, \star).$$

To prove (ii), first observe that $H \subset G_e$.

Write (by (i)) an element of $\mathfrak{g}_e$ in the form $X + X_x + Y_y$, as before. Again, $x = 0$, $y = 0$, so X lies in the annihilator of e in $\mathfrak{h}_1$.

There is a cocharacter λ of H_1 with $H_1 = \mathrm{Im}(\lambda).H$ such that

$$\lambda(\xi).e = (0, 0, \xi^{-3}, \xi^3) \quad (\xi \in k^*).$$

We can conclude that the annihilators of e in $\mathfrak{h}$ and $\mathfrak{h}_1$ coincide (here one uses that $\mathrm{char}(k) \neq 3$). We see that

$$\mathfrak{h} \subset L(G_e) \subset \mathfrak{g}_e = \mathfrak{h}.$$

The inclusions must be equalities, which implies (ii).

For the proof of (iii) we introduce $\sigma \in GL(V)$ defined by

$$\sigma(a, b, \xi, \eta) = i(-b, a, -\eta, \xi),$$

where i is a primitive 4^{th} root of unity. One then checks that the normalizer $N_{GL(V)}(H)$ is generated by H, σ and the transformations

$$(a, b, \xi, \eta) \longmapsto (\alpha a, \beta b, p\xi + q\eta, r\xi + s\eta)$$

with $\alpha, \beta, ps - gr \in k^*$. Now it is straightforward to see that $N_G(H)$ is generated by H, σ and the transformations

$$(a, b, \xi, \eta) \longmapsto (\alpha a, \alpha^{-1} b, \alpha^{-3}\xi, \alpha^3 \eta) \quad (\alpha \in k^*),$$

while $N_{\tilde{G}}(H) = \mathbb{G}_m.N_G(H)$. These facts imply (iii).

We have

$$\dim \tilde{\mathfrak{g}}.e = 1 + \dim \mathfrak{g} - \dim \mathfrak{h} = 56 = \dim V,$$

which proves (iv). (v) is another formulation of (iv).

Corollary 2.8. $\tilde{G}.e = \{v \in V \mid h(v) \neq 0\}$.

Proof. Results of this kind are familiar in the theory of prehomogeneous vector spaces. We sketch a proof.

It follows from Cor. 2.7 (iv) that $U = \tilde{G}.e$ is dense in V. Since an orbit is open in its closure, U is open in V. Using [Sp3, Cor. 5.5.4, Th. 5.5.5] it also follows that U is isomorphic to the homogeneous space $\tilde{G}/\tilde{G}_e$. As $\tilde{G}_e$ is reductive by Cor. 2.7 (ii) this space is an affine variety (see e.g. [R]). Assume that the closed set $V - U$ has a component C of codimension > 1. Let $c \in C$ be a point which does not lie on any other component of $V - U$. Then all regular functions on U are regular in c and the evaluation map at c defines a k-algebra homomorphism $k[U] \to k$ which is not an evaluation map at any point of U. This is impossible. Hence $V - U$ is purely of codimension 1.

The components C_i of $V - U$ are irreducible hypersurfaces. Let h_i be a defining irreducible polynomial of C_i. Since the connected algebraic group $\tilde{G}$ acts on U, it stabilizes the C_i. For each i we have a character $\chi_i \in X = \mathrm{Hom}(\tilde{G}, \mathbb{G}_m)$ such that

$$h_i(g.v) = \chi_i(g) h_i(v) \quad (g \in \tilde{G},\ v \in V).$$

The characters χ_i are linearly independent in $\mathbb{Q} \otimes X$. In fact, if the integers e_i are such that $\sum_i e_i \chi_i = 0$, the function $\prod_i h_i^{e_i}$ is constant on U, which is only possible if all e_i vanish. X being of rank one, there is only one component, which is an irreducible hypersurface. Therefore the $\tilde{G}$-invariant hypersurface $h^{-1}(0)$ must coincide with $V - U$.

§3. The ternary product

3.1. We maintain the notations of the previous section. For $v_i \in V$ $(i = 0, 1, 2, 3)$ define $\{v_1 v_2 v_3\} \in V$ by

$$8[v_0, v_1, v_2, v_3] = [\{v_1 v_2 v_3\}, v_0].$$

Then $\{\ \}$ is a ternary product (or triple system) on V, symmetric in its three arguments. We have

$$[\{vvv\}, v'] = 8[v, v, v, v'] \quad (v, v' \in V). \tag{9}$$

The notion of an automorphism of our ternary product is defined in the obvious manner, as is the notion of a derivation of that product.

Notice that if $g \in GL(V)$ stabilizes both h and $[\ ,\ ]$ we have

$$g.\{v_1 v_2 v_3\} = \{g.v_1, g.v_2, g.v_3\}, \tag{10}$$

so g is an automorphism of $\{\ \}$.

With the notations of 2.1 we obtain from (8)

$$\begin{aligned} \frac{1}{2}\{vvv\} = \Big(&2b(aa) - \eta(bb) - \frac{1}{2}(\langle a, b\rangle - \xi\eta)a,\ -2a(bb) + \xi(aa) \\ &+ \frac{1}{2}(\langle a, b\rangle - \xi\eta)b,\ -g(b) + \frac{1}{2}(\langle a, b\rangle - \xi\eta)\xi,\ f(a) - \frac{1}{2}(\langle a, b\rangle - \xi\eta)\eta\Big). \end{aligned} \tag{11}$$

As before, $e = (0, 0, 1, 1)$. Put $f = (0, 0, -1, 1)$.

Lemma 3.2. (i) $\{eee\} = f$, $\{fff\} = -e$;
(ii) $6\{efv\} = [v, f]e + [v, e]f$.

Proof. (i) follows from (11). To prove (ii) observe that $6\{efv\}$ is the coefficient of $\alpha\beta$ in $\{(v+\alpha e+\beta f)(v+\alpha e+\beta f)(v+\alpha e+\beta f)\}$. A computation of this coefficient in the right-hand side of (11) gives (ii).

Proposition 3.3. *For $v, w \in V$ we have*

$$6\{\{vvv\}vw\} = [w, \{vvv\}]v + [w, v]\{vvv\}. \tag{12}$$

Proof. Let $w = (a, b, \xi, \eta)$. For $v = e$ the left-hand side equals $6\{few\} = (\xi + \eta)e + (\xi - \eta)f$, by Lemma 3.2. Observing that

$$[w, e] = \xi - \eta, \quad [w, \{eee\}] = \xi + \eta$$

we see that

$$6\{\{eee\}, e, w\} = [w, \{eee\}]e + [w, e]\{eee\}.$$

To prove the Proposition we may assume k to be algebraically closed. Let G be as in 2.1. We already observed that the elements of G are automorphisms of our ternary product. This implies that the asserted formula holds for arbitrary w and all elements v in the orbit $G.e$ and, by homogeneity, for the elements in the orbit $\tilde{G}.e$. But by Cor. 2.8 this orbit is dense in V. This implies that the formula holds for all v and w.

Corollary 3.4. (i) *G is the automorphism group of* $\{\ \}$;
(ii) *A derivation of* $\{\ \}$ *annihilates* $[\ ,\ ]$ *and* h.

Proof. Choose $v \in V$ with $h(v) \neq 0$. Using Cor. 2.8 and Lemma 3.2 (i) we see that v and $\{vvv\}$ are linearly independent. Let g be an automorphism of $\{\ \}$. (12), used for v, w and $g.v$, $g.w$ gives that

$$[g.w, \{g.v, g.v, g.v\}] = [w, \{vvv\}], \quad [g.w, g.v] = [w, v],$$

if $h(v) \neq 0$, hence for all v. This implies that $g \in G$, proving (i).

View $[\ ,\ ]$ and h as elements of appropriate vector spaces on which $\mathrm{End}(V)$ acts. The proof of (ii) is similar to that of (i), using that D is a derivation if and only if $1+\epsilon D$ lies in the group $G(k[\epsilon])$ of $k[\epsilon]$-valued points of G, where $k[\epsilon]$ is the algebra of dual numbers over k.

We next give some formulas, in which $v = (a, b, \xi, \eta)$.

Lemma 3.5. (i) $3\{eev\} = (a, -b, -2\xi - \eta, \xi + 2\eta)$, $3\{ffv\} = (-a, b, 2\xi - \eta, \xi - 2\eta)$;
(ii) $3\{evv\} = (-2bb + (\xi + \eta)a, 2aa - (\xi + \eta)b, -\xi^2 - 2\xi\eta, \eta^2 + 2\xi\eta)$, $3\{fvv\} = (-2bb + (\xi - \eta)a, -2aa - (\xi - \eta)b, -\xi^2 + 2\xi\eta, -\eta^2 + 2\xi\eta)$.

Proof. $3\{eev\}$ and $3\{evv\}$ are the coefficients of α^2, respectively, α in $\{(v+\alpha e)(v+\alpha e)(v+\alpha e)\}$. Similarly for $3\{ffv\}$ and $3\{fvv\}$. Then (i) and (ii) follow from (11).

3.6. We now show how the ingredients of the E_6-structure S can be recovered from the ternary product and e. For $v \in V$ define a linear map t_v and a quadratic map u_v of V by

$$t_v(w) = 3\{vvw\}, \quad u_v(w) = 3\{vww\}.$$

Let V_v be the subspace of V spanned by v and $\{vvv\}$. If $h(v) \neq 0$ it is two-dimensional and non-singular. Let W_v be its orthogonal complement (relative to our alternating form). The following facts are consequences of Lemmas 3.2 and 3.5.

(a) V_e is spanned by e and f and $W_e = \{(a, b, 0, 0) \mid a \in A,\, b \in B\}$. Identify W_e with $A \oplus B$.

(b) t_e stabilizes V_e and W_e, as follows from

$$V_e = \mathrm{Im}(t_e^2 - 1) = \mathrm{Ker}(t_e^2 - 3), \quad W_e = \mathrm{Im}(t_e^2 - 3) = \mathrm{Ker}(t_e^2 - 1).$$

The restriction $t_e|_{W_e}$ has eigenvalues 1, -1, with respective eigenspaces A, B. Let $w = (a, b) \in W_e$.

(c) $(bb, aa) = -\frac{1}{2}u_f(w)$.

(d) $f(a) = \frac{1}{4}[e - f, \{www\}]$, $g(b) = \frac{1}{4}[e + f, \{www\}]$.

The next result is a complement to Cor. 3.4. It uses the facts from 3.6.

Proposition 3.7. *The Lie algebra of derivations of* $\{\ \}$ *coincides with the Lie algebra* $\mathfrak{g}$ *of* G.

Proof. Let $\mathfrak{d}$ be the derivation algebra. It is clear that it contains $\mathfrak{g}$. Using Cor. 2.7 (iv) one sees that an element of $\mathfrak{d}/\mathfrak{g}$ can be represented by a derivation D such that $D.e = \alpha e$ with some $\alpha \in k$. From Lemma 3.2 (i) we see that then $Df = D\{eee\} = 3\{ee(De)\} = 3\alpha f$ and $\alpha e = De = -D\{fff\} = -3\{ff(Df)\} = 9\alpha e$. So $\alpha = 0$ and $De = Df = 0$.

Since $D.e = 0$, D commutes with t_e. It stabilizes the eigenspaces of t_e. Using Lemma 3.5 (ii) and 3.6 (c) we see that

$$D(aa) = 2a(D.a), \quad D(bb) = 2b(D.b) \quad (a \in A,\, b \in B).$$

Then by (2)

$$f(a)(D.a) = D((aa)(aa)) = 2(D(aa))(aa)) = 4((D.a)a)(aa)).$$

It follows from (2) that for $a, a_1 \in A$

$$4(aa_1)(aa) = 3f(a, a, a_1)a + f(a)a_1.$$

The last two equations imply that $f(D.a, a, a) = 0$. So the restriction of D to A annihilates the cubic form f. But then by [SV, p. 182] this restriction lies in the Lie algebra of the invariance group of f, i.e. in the restriction to A of the Lie algebra of $\mathfrak{h}$. Modifying D by an element of $\mathfrak{h}$ we may assume that $D.a = 0$ for all $a \in A$. Applying this to bb we see that $b(D.b) = 0$ for all $b \in B$. The counterpart for B of the last formula (with b and Db instead of a and a_1) shows that the restriction of D to B is also 0. It follows that $D = 0$. We can now conclude that $\mathfrak{d} = \mathfrak{g}$, as asserted.

§4. E_7-structures

4.1. A k-vector space V equipped with a non-degenerate alternating bilinear form $[\ ,\ ]$, a quartic form h with associated symmetric quadrilinear form $[\ ,\ ,\ ,\ ]$ and a symmetric ternary product $\{\ \}$ such that (9) and (12) hold is a *Freudenthal triple system*[1]. Clearly, it is uniquely determined by $[\ ,\ ]$ and $\{\ \}$. We write $(V, [\ ,\ ], \{\ \})$ for the triple system or simply V if there is no danger of confusion.

We have constructed above a Freudenthal triple system $V(S)$ out of an E_6-structure S over our algebraically closed fields k. For $S = S_0$, the standard E_6-structure, we write $V(S) = V_0$. We call *E_7-structure* a Freudenthal triple system V isomorphic to V_0 over k. Classification results show that an E_7-structure could also be defined as a Freudenthal triple system of dimension 56 satisfying a non-degeneracy condition, but we will not go into this here (cf. [G1], [M]).

V is (defined) over the subfield F of k if V has an F-structure such that the data are defined over F. The definitions show that $V(S)$ is defined over F if this holds for S. We call V_0 the *standard* E_7-structure. It is defined over any subfield F of k.

Let $(V, [\ ,\ ], \{\ \})$ be an E_7-structure over F, with quartic form h. For $\alpha \in F^*$, $V_\alpha = (V, \alpha[\ ,\ ], \alpha\{\ \})$ is also an E_7-structure over F, with quartic form $\alpha^2 h$. If α is a square in F, V_α is F-isomorphic to V. An E_7-structure V' over F is *equivalent* to V if it is F-isomorphic to some V_α.

Let V be an E_7-structure over F and let $v \in V(F)$. We define the maps t_v and u_v as in 3.6. Put

$$E_v = k[T]/(T^2 + 4h(v)).$$

This is an algebra with an F-structure, viz. $F[T]/(T^2 + 4h(v))$. E_v is an étale quadratic algebra if $h(v) \neq 0$. Let $\tau_v \in E_v(F)$ be the image of T.

[1] I extracted this kind of algebraic structure from Freudenthal's work in [F] around 1962 and I established some of its properties. But I did not publish this work. The first publication about Freudenthal triple systems was by K. Meyberg [M], to whom I had communicated my results. He also coined the name.

Denote by σ the non-trivial k-automorphism of E_v, sending τ_v to $-\tau_v$. Choose $\lambda \in k$ with $\lambda^2 = -4h(v)$.

We establish some properties involving the maps t_v. Since V is isomorphic to V_0 over k we may identify V with V_0 in questions not involving a field of definition.

The next four Lemmas are true for $v = e$ by Lemma 3.2, using that $h(e) = -\frac{1}{4}$. An application of Cor. 2.8 proves Lemmas 4.2 and 4.3. Also, Lemmas 4.4 and 4.5 hold if $h(v) \neq 0$. By continuity they hold for all v.

Lemma 4.2. *Assume that* $h(v) \neq 0$.

(i) t_v *is semi-simple with minimum polynomial* $(T^2 + 4h(v))(T^2 + 12h(v))$;

(ii) $V_v = \mathrm{Im}(t_v^2 + 4h(v))$ *and* $W_v = \mathrm{Im}(t_v^2 + 12h(v))$ *are* t_v*-stable and* V *is their orthogonal direct sum*;

(iii) W_v *has an* E_v*-module structure which is defined over* F, *with* $\tau_v w = t_v.w$ $(w \in W_v)$. *The eigenvalues of* $t_v|_{W_v}$ *are* λ *and* $-\lambda$, *their eigenspaces have dimension* 27;

(iv) V_v *is spanned by* v *and* $\{vvv\}$. *The eigenvalues of* $t_v|_{V_v}$ *are* $\lambda\sqrt{3}$ *and* $-\lambda\sqrt{3}$, *their eigenspaces have dimension* 1.

Lemma 4.3. *Let* $h(v) \neq 0$. *Then* $u_v(W_v) \subset W_v$.

Lemma 4.4. *For all* $v, v' \in V$ *we have*

$$t_{\{vvv\}}(v') = -4h(v)t_v(v') + 4h(v)[v, v']v + [\{vvv\}, v']\{vvv\}.$$

Lemma 4.5. *Let* $v \in V$ *and put* $z = \xi v + \eta\{vvv\}$ $(\xi, \eta \in k)$. *Then*

$$\{zzz\} = (\xi^2 + 4h(v)\eta^2)(4h(v)\eta v + \xi\{vvv\}).$$

4.6. Assume that $h(v) \neq 0$. For $w \in W_v$ define a quadratic map $w \mapsto w \star_v w$ of W_v by

$$w \star_v w = -\frac{1}{2}u_v(w) \tag{13}$$

and let $(w, w') \mapsto w \star_v w'$ be the associated symmetric bilinear map. Furthermore define a bilinear map H_v of W_v to E_v by

$$-2H_v(w, w') = [\tau_v w, w'] + [w, w']\tau_v. \tag{14}$$

Then for $\mu \in E_v$

$$H_v(\mu w, w') = \mu H_v(w, w')$$

and

$$H_v(w', w) = H_v(w, w')^\sigma.$$

So H_v is a Hermitian form on the E_v-module W_v. It is defined over F.

Next define a function $F_v : W_v \to E_v$ by

$$F_v(w) = -\frac{1}{4}([\{vvv\}, \{www\}] + [v, \{www\}]\tau_v). \tag{15}$$

Then F_v is a cubic map (over k). Let $F_v(\ ,\ ,\ )$ be the associated symmetric trilinear map with $F_v(w, w, w) = F_v(w)$.

Proposition 4.7. *$(E_v/F, W_v, H_v, F_v)$ is a Hermitian E_6-structure over E_v/F.*

Proof. We have to prove the following facts:
(i) the product $w \star_v w'$ is σ-bilinear for the E_v-action;
(ii) $(w \star_v w) \star_v (w \star_v w) = F_v(w)w$ $(w \in W_v)$;
(iii) $H_v(w_1, w_2 \star_v w_3) = 3F_v(w_1, w_2, w_3)$.

The quadratic map $u_v : V \to V$ induces a map $W_v \to W_v$. Let $\tilde{u}_v(\ ,\ )$ be the symmetric bilinear map with $\tilde{u}_v(w, w) = u_v(w)$. By Lemma 4.2 (ii) the assertion (i) is then equivalent with

$$\tilde{u}_v(t_v(w), w') = -t_v(\tilde{u}_v(w, w')),$$

if $w, w' \in W_v$. Using Cor. 2.8 one sees that it suffices to prove this if v is a multiple of e. We prefer to work with f instead of e, which we can do (the proof of Cor. 2.8 also works for f, mutatis mutandis). Similarly, the proof of (ii) and (iii) can be reduced to the case that v is a multiple of f. So assume that $v = \alpha f$.

Then $\tau_v^2 = \alpha^4$. Choose $\lambda = \alpha^2$. We identify E_v with $k \oplus k$, via the isomorphism

$$\xi + \eta\tau_v \longmapsto (\xi - \lambda\eta, \xi + \lambda\eta).$$

With the notations of 2.1 we have $W = (A, B, 0, 0)$, which we view as the direct sum $A \oplus B$.

Then for $\xi + \eta\tau_v \in E_v$

$$(\xi + \eta\tau_v).(a, b) = ((\xi - \lambda\eta)a, (\xi + \lambda\eta)b).$$

From the results of 3.6 we find that for $w = (a, b)$, $w' = (a', b')$

$$w \star_v w' = \alpha(bb', aa'),$$

whence

$$(\tau_v w) \star_v w' = -\alpha(-\lambda bb', \lambda aa') = -\tau_v(w \star_v w'),$$

which implies (i). Then

$$(w \star_v w) \star_v (w \star_v w) = \alpha^3((aa)(aa), (bb)(bb)) = \alpha^3(f(a), g(b))w.$$

By 3.6 (d) and Lemma 3.2 (i) we have

$$\begin{aligned}
\text{(16)} \qquad \alpha^3(f(a), g(b)) &= \frac{1}{2}\alpha^3(f(a) + g(b) - \lambda^{-1}(f(a) - g(b)\tau_v) \\
&= -\frac{1}{4}\alpha^3([\{fff\}, \{www\}] + \lambda^{-1}[f, \{www\}]\tau_v) \\
&= -\frac{1}{4}([\{vvv\}, \{www\}] + [v, \{www\}]\tau_v) = F_v(w),
\end{aligned}$$

proving (ii).

Finally, if $v = \alpha f$ we have for $w = (a, b)$, $w' = (a', b')$

$$H_v(w, w') = \alpha^2(\langle a, b'\rangle, \langle a', b\rangle)$$

whence (with obvious notations) using (16)

$$\begin{aligned}
H_v(w_1, w_2 \star_v w_3) &= \alpha^3(\langle a_1, a_2a_3\rangle, \langle b_1, b_2b_3\rangle) \\
&= 3\alpha^3(f(a_1, a_2, a_3), g(b_1, b_2, b_3)) \\
&= 3F_v(w_1, w_2, w_3),
\end{aligned}$$

proving (iii).

Let tr and n be the trace and norm maps $E_v \to k$. They are defined over F. According to Lemma 4.2 we can write the elements of V in the form $z = w + \xi v + \eta\{vvv\}$ ($w \in W_v$, $\xi, \eta \in k$). We then have

Corollary 4.8. (i) $-4h(v)h(z) = H_v(w \star_v w, w \star_v w) + \operatorname{tr}(\zeta F_v(w)) - \frac{1}{4}(H_v(w, w) - \mathrm{n}(\zeta))^2$, *where* $\zeta = -4h(v)\eta + \xi\tau_v \in E_v$;

(ii) (*with obvious notations*) $-4h(v)[z, z'] = -\operatorname{tr}((H_v(w, w') + \zeta\sigma(\zeta'))\tau_v)$.

Proof. It suffices to prove (i) if $v = \alpha f$, in which case the formula follows from (7) by a straightforward calculation.

The proof of (ii) is also straightforward.

In the situation of Prop. 4.7 assume that $v \in V(F)$ and that $h(v) \in -(F^*)^2$. Then $E_v \simeq k \oplus k$ over F and W_v is the direct sum of the eigenspaces A_v and B_v of t_v for the respective eigenvalues λ, $-\lambda$. They are defined over F.

From the product $\star_v$ and F_v we deduce the ingredients of an E_6-structure over F, as follows. Identify E_v with $k \oplus k$ (over F), as before.

For $a \in A_v$, $b \in B_v$ define

$$\langle a, b\rangle_v = H_v((a,0),(0,b)),$$
$$(a,b) \star_v (a,b) = (bb, aa),$$
$$F_v((a,b)) = (f_v(a), g_v(b)).$$

COROLLARY 4.9. (i) $S_v = (A_v, B_v, \langle\ ,\ \rangle_v, f_v, g_v)$ *is an* E_6*-structure over* F;

(ii) V *is equivalent with* $V(S_v)$.

Proof. (i) is a reformulation of Prop. 4.7, for the present case.

Put $V_v = V(S_v)$. We identify its underlying space with V. Indicate its ingredients by a suffix v. Then there is $\lambda \in F$ with $\lambda^2 = -4h(v)$ such that

$$[\ ,\ ]_v = \lambda[\ ,\ ], \quad h_v = \lambda^2 h,$$

by the definition (6) of $[\ ,\ ]$ and Cor. 4.8. By (9), $\{\ \}_v = \lambda\{\ \}$, proving (ii).

Let V be an E_7-structure over k. We maintain the previous notations. Let G be the automorphism group of V, with Lie algebra $\mathfrak{g}$. By Theorem 2.2, G is a simply connected group of type E_7. Fix $v \in V$ with $h(v) \neq 0$.

COROLLARY 4.10. *The isotropy group* G_v *of* v *in* G *is a connected, quasi-simple, simply connected group of type* E_6. *Its Lie algebra is the annihilator of* v *in* $\mathfrak{g}$.

Proof. For $v = e$ this follows from Cor. 2.7 (ii). For the general case apply Cor. 2.8.

An E_7-structure over F of the form $V(S)$ is said to be *reduced* (over F). This notion is stable under equivalence.

THEOREM 4.11. *The following conditions are equivalent*:

(a) *There is* $v \in V(F)$ *with* $h(v) \in -(F^*)^2$,

(b) V *is reduced over* F,

(c) *The hypersurface* $h = 0$ *in* V *contains a non-singular* F*-rational point,*

(d) $h(v)$ *takes all values in* F *for* $v \in V(F)$.

Proof. The implication (a) $\Rightarrow$ (b) follows from Cor. 4.9. To prove (b) $\Rightarrow$ (c) we may assume that we are in the situation considered in 2.1, the ingredients being defined over F. Take $v = (a, 0, 0, 1)$ with $a \in A(F)$, $f(a) \neq 0$. From (7) and (11) we see that $h(v) = 0$ and $\{vvv\} = (0, 0, 0, 2) \neq 0$. By (9) this means that v is a non-singular point of $h = 0$.

Next assume that $v \in V(F)$ is as in (c). Then $h(v) = 0$, $\{vvv\} \neq 0$. By Lemma 4.4

$$t_{\{vvv\}}(w) = [\{vvv\}, w]\{vvv\}.$$

Put $n = \{vvv\}$. By the definition of t_n and of the triple product the preceding formula implies that $[n, n, w, v'] = 0$ for all $v' \in V$ and $w \in H = \{w \mid [n, w] = 0\}$. For $w \in H$

$$h(w + \alpha n) = h(w) + 4\alpha[w, w, w, n].$$

If there is $w \in H(F)$ with $[w, w, w, n] \neq 0$ the preceding formula shows that there is $v \in V$ such that $h(v)$ has any preassigned value, whence (d).

The case remains that $[w, w, w, n] = 0$ for all $w \in H(F)$. Then $[w, w, w', n] = 0$ for $w, w' \in H(F)$. Using (9) we see that $t_w(n) = Q(w)n$, where Q is a quadratic form on H which is defined over F. As a consequence of Lemma 4.2 (i)

$$(Q(w)^2 + 4h(w))(Q(w)^2 + 12h(w)) = 0.$$

Hence $h(w)$ is a non-zero multiple of $Q(w)^2$. If Q were zero on $H(F)$ then Q would be zero and h would vanish on H. But thus is impossible as H would be stable under the group G°, contradicting Lemma 2.5 (ii). The implication (c) $\Rightarrow$ (d) follows. Since (d) $\Rightarrow$ (a) is obvious the Theorem is proved.

§5. Rationality questions

As before, k is algebraically closed of characteristic $\neq 2, 3$ and F is a subfield of k. Let F_s be the separable closure of F in k. For the facts on Galois cohomology to be used we refer to [Ser].

V is an E_7-structure over F with automorphism group G. By Th. 2.2 it is a simply connected group of type E_7.

PROPOSITION 5.1. (i) *G is defined over F;*
(ii) *If V is the standard E_7-structure V_0 then G is split over F.*

Proof. Let $\mathcal{F}$ be the space of symmetric trilinear maps of $V \times V \times V$ to V. The group $GL(V)$ acts on it and by Cor. 3.4 (i), G is the isotropy group in $GL(V)$ of $\{\ \} \in \mathcal{F}$. To prove that it is defined over F apply [Sp3, 12.1.2 (i)] to the action of $GL(V)$ on $\mathcal{F}$ (which is defined over F). The kernel of the tangent map of [loc. cit.] at the identity element is the space of derivations of $\{\ \}$ and by Prop. 3.7 the condition of [loc. cit.] is satisfied. This proves (i).

To prove (ii) we have to show that if $V = V_0$ the group G contains a maximal torus over F which is F-split. In the proof of Th. 2.2 we introduced a maximal torus T_1 of G. It is of the form $\mathbb{G}_m.T$, where T is a maximal torus of the group H introduced in 1.5. Now the underlying E_6-structure is the standard one of 1.4. In that case one easily constructs an F-split maximal torus T of H (cf. [Sp2, 14.21]). For such a T the torus T_1 is also F-split.

LEMMA 5.2. *V is F_s-isomorphic to V_0.*

Proof. Assume that $F = F_s$. Choose $v \in V(F)$ with $h(v) \neq 0$. Then $h(v) \in -(F^*)^2$. Let S_v be as in Cor. 4.9. Then V is equivalent with $V(S_v)$ and even isomorphic since $F = F_s$ (cf. the end of 1.2). For the same reason S_v is F-isomorphic to S_0. Now use that over a separably closed field all Albert algebras are isomorphic (this is proved as in the algebraically closed case, see [SV, p. 153]).

Let G_0 be the F-split simply connected group of type E_7 and let G_1 be an F-form of G_0. After [T] we say that G_1 is a *strong form* of G_0 if it is a twist of G_0 by a cocycle in a cohomology class in $H^1(F, G_0)$ (the adjective "inner" of [loc. cit.] is superfluous since in the present case all forms are inner).

PROPOSITION 5.3. (i) *There is a bijection of $H^1(F, G_0)$ onto the set of isomorphism classes of E_7-structures over F;*

(ii) *There is a bijection of the set of isomorphism classes of strong forms of G_0 over F onto the set of equivalence classes of of E_7-structures over F.*

Proof. (i) follows from the preceding results, by standard arguments, cf. [Ser, Ch. III, §1].

Let $\overline{G_0}$ be the quotient of G_0 by its center. We have an exact sequence of groups

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G_0 \longrightarrow \overline{G_0} \longrightarrow 1,$$

inducing an exact sequence of Galois cohomology sets. The isomorphism classes of strong forms of G_0 are in bijection with the image of $H^1(F, G_0)$ in $H^1(F, \overline{G_0})$. (ii) follows by applying [loc. cit. Prop. 42, p. 54] in the present case. We skip the details.

Let G be a strong F-form of G_0. The proof of Prop. 5.3 also shows the following.

Corollary 5.4. *G is the automorphism group of an E_7-structure over F.*

Assume that G is the automorphism group of an E_7-structure V over F, with quartic form h.

Proposition 5.5. *Let $v \in V(F)$, $h(v) \neq 0$.*

(i) *The isotropy group G_v of v in G is a connected, quasi-simple, simply connected F-group of type E_6;*

(ii) *G_v is of inner type over F if and only if $h(v) \in -(F^*)^2$.*

Proof. (i) was established in Cor. 4.10, except for the fact that G_v is defined over F. This follows from [Sp3, 12.1.2 (i)], using the last point of Cor. 4.10.

If $h(v) \in -(F^*)^2$ it follows from Cor. 4.9 and the proof of Prop. 1.6 that G_v is F-isomorphic to the invariance group of the cubic form of an Albert algebra over F. Such a group is a strong inner form of the split group of type E_6 (see e.g. [T, p. 666, equivalence of (I) and (III)]).

It remains to show that G_v is an outer form if $h(v) \notin -(F^*)^2$. Assume this and put $E = F(\sqrt{(-4h(v))})$, a quadratic extension of F (with the notations of 4.1, $E = E_v(F)$). Denote by A and B the 27-dimensional spaces like A_v and B_v in Cor. 4.9. They are defined over E and their sum is defined over F. They are G_v-stable. The non-trivial automorphism of E/F permutes them. It follows that the 27-dimensional irreducible representations of the F-group G_v of type E_6 are not defined over F and by [loc. cit., equivalence of (II) and (III)] G_v cannot be of inner type.

LEMMA 5.6. (i) *G is isotropic over F if V is reduced*;
(ii) *If G is isotropic over F there is $v \in V(F) - \{0\}$ with $h(v) = 0$.*

Proof. Let V be reduced over F. It can be described as in 2.1, all ingredients being defined over F. The linear maps

$$(a, b, \xi, \eta) \longmapsto (a, b, x\xi, x^{-1}\eta) \quad (x \in k^*)$$

form a one-dimensional F-split subtorus of G. Hence G is isotropic over F.

Let G be isotropic over F and let S be a one-dimensional F-split subtorus of G. If $v \in V(F) - \{0\}$ is a weight vector for a non-zero weight of S then $h(v) = 0$, by an easy argument.

§6. Comments and problems

V is an E_7-structure over F with quartic form h. Its automorphism group is G, as before.

6.1. By Prop. 5.4, G always contains simply connected F-subgroups of type E_6. In particular, outer forms of E_6 will appear, as automorphism groups of Hermitian E_6-structures W_v of Prop. 4.7.

A further study of Hermitian E_6-structures will be helpful in understanding E_7-structures. We shall not go into this study now.

At this point mention should be made of a construction of E_7-structures out of Hermitian E_6-structures, suggested by Cor. 4.8.

Let $\Sigma = (E/F, W, H, G)$ be a Hermitian E_6-structure over the quadratic extension field $E = F(\sqrt{\lambda})$ and F (the notations are as in 1.8).

Put $V = W \oplus R$ and define on V a quartic form h and an alternating bilinear form $[\ ,\ ]$ by

$$\lambda h((v, \zeta)) = H(ww, ww) + \mathrm{tr}(\zeta F(w)) - \frac{1}{4}(H(w, w) - \mathrm{n}(\zeta))^2,$$
$$\lambda[(w, \zeta), (w', \zeta')] = -\,\mathrm{tr}(H(w, w') + \zeta\sigma(\zeta'))\sqrt{\lambda}).$$

tr and n denote again trace and norm maps.

PROPOSITION 6.2. *V, h and $[\ ,\ ]$ are the ingredients of an E_7-structure over F.*

Proof. Working over k one translates the definitions of $[\ ,\ ]$ and h into (6) and (7). We omit the details.

Corollary 6.3. *The automorphism group of the Hermitian E_6-structure Σ is an outer F-form of the simply connected group of type E_6.*

Proof. With the notations of the Proposition, the automorphism group in question is the isotropy group in G of the point $(0, 1) \in V = W \oplus R$. Then apply Prop. 5.5.

6.4. We say that V is *isotropic* over F if there is $v \in V(F) - \{0\}$ with $h(v) = 0$. This is the case if V is reduced, by Th. 4.11. But there are other cases.

Consider the index (or Tits diagram) of our F-group G. It is the Dynkin diagram D of type E_7, in which certain vertices, called isotropic, are marked (see e.g. [T, 1.5.5]). We use the numbering of [B, p. 265] for the vertices of D.

It follows from [T, 5.2] that if G is not split or anisotropic over F, the possible sets of isotropic vertices of D are $\{1\}$, $\{7\}$, $\{1, 6, 7\}$. The second and third possibility are realized by automorphism groups of reduced E_7-structures, coming from an Albert division algebra over F or a non-split reduced Albert algebra over F (use the properties of a strongly inner forms of groups of type E_6 discussed in [loc. cit., p. 666]).

But groups G realizing the first possibility also exist for certain F. In that case $h(v) = 0$ has non-zero solutions in $V(F)$ by Lemma 5.6 (ii). It follows from Prop. 4.11 that $\{vvv\} = 0$ for all such v. The existence problem over a given F of such G is briefly discussed in [Sel, p. 94], it is tied up with the existence of certain anisotropic Hermitian forms over quaternion division algebras. But the situation is not very clear. A further study in the context of E_7-structures is desirable.

6.5. V is *anisotropic* if it is not isotropic. In this case G is anisotropic over F.

In [T, 3.1], such a G is constructed in the case that E is a field of rational functions $E_0(t)$, where E_0 is a field over which there exists a central division algebra of degree and exponent 4.

The construction of [loc. cit.] uses Bruhat-Tits theory, for groups over $E_0((t))$. It would be interesting to find a direct construction of a corresponding E_7-structure.

In this context the question should be mentioned (cf. [loc. cit., p. 667]) of the existence of an anisotropic E_7-structure over F if there is a central division algebra over F of degree and exponent 4, for which the reduced norm map is not surjective.

6.6. Finally, some questions about the Rost invariant $R_G \in H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ of G (Rost invariants are discussed in Merkurjev's contribution in [GMS]).

Is there an elementary description in terms of an E_7-structure of the 2-torsion part of R_G in the spirit of the description the 3-torsion invariant of an Albert division algebra (see e.g. [SV, Ch. 8]).

The case of Albert algebras also suggests the question whether reducedness of V can be read off from R_G.

References

[B] N. Bourbaki, Groupes et algèbres de Lie, Chap. IV, V, VI, Hermann, 1968.

[Br] R. Brown, *Groups of type E_7*, J. reine angew. Math., **236** (1969), 79–110.

[Ca] É. Cartan, Oeuvres, vol. I, CNRS, Paris, 1984.

[Car] R. W. Carter, Finite groups of Lie type, conjugacy classes and representations, Wiley, 1985.

[F] H. Freudenthal, *Beziehungen der E_7 und E_8 zur Oktavenebene* I, Proc. Kon. Ned. Ak. v. Wet., **57** (1954), 218–230; VIII, ibid., **62** (1959), 447–465.

[G1] R. S. Garibaldi, *Structurable algebras and groups of type E_6 and E_7*, J. Alg., **236** (2001), 651–691.

[G2] R. S. Garibaldi, *Groups of type E_7 over arbitrary fields*, Comm. Alg., **29** (2001), 2689–2710.

[GMS] S. Garibaldi, A. Merkurjev and J.-P. Serre, Cohomological invariants in Galois cohomology, Univ. Lect. Series vol. 28, Amer. Math. Soc., 2003.

[M] K. Meyberg, *Eine Theorie der Freudenthalschen Tripelsysteme,* I, II, Proc. Kon. Ned. Ak. v. Wet., **71** (1968), 162–190.

[R] R. W. Richardson, *Affine coset spaces of reductive algebraic groups*, Bull. London Math. Soc., **9** (1977), 38–41.

[Sel] M. Selbach, Klassifikationstheorie halbeinfacher algebraischer Gruppen, Bonner Mathematische Schriften, Nr. 83, 1976.

[Ser] J.-P. Serre, Cohomologie Galoisienne (5me éd.), Lect. Notes in Math. no. 5, Springer-Verlag, 1994.

[SK] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J., **65** (1977), 1–158.

[Sp1] T. A. Springer, *Characterization of a class of cubic forms*, Proc. Kon. Ned. Ak. v. Wet., **65** (1962), 259–265.

[Sp2] T. A. Springer, Jordan algebras and algebraic groups, Springer-Verlag, 1973.

[Sp3] T. A. Springer, Linear algebraic groups, Birkhäuser, 1998.

[SV] T. A. Springer and F. D. Veldkamp, Octonions, Jordan algebras and exceptional groups, Springer-Verlag, 2000.

[T] J. Tits, *Strongly inner anisotropic forms of simple algebraic groups*, J. Alg., **131** (1990), 648–677.

Mathematisch Instituut
Budapestlaan 6
3584 CD Utrecht
the Netherlands
springer@math.uu.nl