J. Herzog and T. Hibi
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LEVEL RINGS ARISING FROM MEET-DISTRIBUTIVE MEET-SEMILATTICES

JÜRGEN HERZOG AND TAKAYUKI HIBI

Abstract. The homogenized ideal dual complex of an arbitrary meet-semilattice is introduced and described explicitly. Meet-distributive meet-semilattices whose homogenized ideal dual complex is level are characterized.

Introduction

In the present paper we continue our discussion in [6] and [5], and describe explicitly the generators of the homogenized ideal dual complex of an arbitrary meet-semilattice (Theorem 2.1). In case of meet-distributive meet-semilattices, a combinatorial formula (Proposition 1.2) to compute the h-vector of the homogenized ideal dual complex is given.

It is known [5] that the homogenized ideal dual complex $\Gamma_{\mathcal{L}}$ of a meet-semilattice $\mathcal{L}$ is Cohen-Macaulay if and only if $\mathcal{L}$ is meet-distributive. Thus it seems of interest to characterize the meet-distributive meet-semilattices $\mathcal{L}$ for which $\Gamma_{\mathcal{L}}$ is a level complex [8]. Our main theorem (Theorem 3.3) says that the homogenized ideal dual complex $\Gamma_{\mathcal{L}}$ is level if and only if a certain simplicial complex coming from $\mathcal{L}$ is pure. In particular, in case that $\mathcal{L}$ is a finite distributive lattice, $\Gamma_{\mathcal{L}}$ is level if and only if the simplicial complex consisting of all antichains of the poset of all join-irreducible elements of $\mathcal{L}$ is pure (Corollary 3.4).

§1. The h-vector of a finite meet-distributive meet-semilattice

First of all, we prepare notation and terminologies on finite lattices and finite posets (partially ordered sets). In a finite poset P we say that $\alpha \in P$ *covers* $\beta \in P$ (or β is a *lower neighbor* of α) if $\beta < \alpha$ and $\beta < \gamma < \alpha$ for no $\gamma \in P$. Let $N(\alpha)$ denote the set of lower neighbors of $\alpha \in P$. A *poset ideal*

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of P is a subset $\mathcal{I}$ of P such that $\alpha \in \mathcal{I}$ and $\beta \in P$ together with $\beta \leq \alpha$ imply $\beta \in \mathcal{I}$.

Let $\mathcal{L}$ be a finite meet-semilattice [7, p. 103] and $\hat{0}$ its unique minimal element. Since $\mathcal{L}$ is a meet-semilattice, it follows from [7, Proposition 3.3.1] that $\mathcal{L}$ is a lattice if and only if $\mathcal{L}$ possesses a unique maximal element $\hat{1}$. In other words, if $\mathcal{L}$ is a meet-semilattice and is not a lattice, then $\mathcal{L} \cup \{\hat{1}\}$ with a new element $\hat{1}$ such that $\alpha < \hat{1}$ for all $\alpha \in \mathcal{L}$ becomes a lattice. Thus, in a finite meet-semilattice $\mathcal{L}$, each element of $\mathcal{L}$ is the join of elements of $\mathcal{L}$. A *join-irreducible element* of $\mathcal{L}$ is an element $\alpha \in \mathcal{L}$ such that one cannot write $\alpha = \beta \vee \gamma$ with $\beta < \alpha$ and $\gamma < \alpha$. In other words, a join-irreducible element of $\mathcal{L}$ is an element $\alpha \in \mathcal{L}$ which covers exactly one element of $\mathcal{L}$.

Let $\mathcal{L}$ be a finite meet-semilattice and $P \subset \mathcal{L}$ the set of join-irreducible elements of $\mathcal{L}$. We will associate each element $\alpha \in \mathcal{L}$ with the subset

$$\ell(\alpha) = \{p \in P : p \leq \alpha\}. \tag{1}$$

Thus $\ell(\alpha)$ is a poset ideal of P, and $\alpha \in \ell(\alpha)$ if and only if α is join-irreducible. Moreover, for α and β belonging to $\mathcal{L}$, one has $\ell(\alpha) = \ell(\beta)$ if and only if $\alpha = \beta$.

Lemma 1.1. *One has $\ell(\alpha \wedge \beta) = \ell(\alpha) \cap \ell(\beta)$ for all $\alpha, \beta \in \mathcal{L}$.*

Proof. Let $\gamma = \alpha \wedge \beta$. Then $\ell(\gamma) \subset \ell(\alpha) \cap \ell(\beta)$. Since $\mathcal{L} \cup \{\hat{1}\}$ with a new element $\hat{1}$ is a lattice, if $\ell(\gamma) \neq \ell(\alpha) \cap \ell(\beta)$ and if $p \in (\ell(\alpha) \cap \ell(\beta)) \setminus \ell(\gamma)$, then $\delta = \gamma \vee p \in \mathcal{L}$ with $\gamma < \delta \leq \alpha$ and $\delta \leq \beta$. This contradicts $\gamma = \alpha \wedge \beta$. ∎

Let K be a field and $K[\mathbf{x}, \mathbf{y}] = K[\{x_p, y_p\}_{p \in P}]$ denote the polynomial ring in $2|P|$ variables over K with each $\deg x_p = \deg y_p = 1$. We associate each element $\alpha \in \mathcal{L}$ with the squarefree monomial

$$u_\alpha = \Big(\prod_{p \in \ell(\alpha)} x_p \Big) \Big(\prod_{p \in P \setminus \ell(\alpha)} y_p \Big) \in K[\mathbf{x}, \mathbf{y}]$$

and set

$$H_{\mathcal{L}} = (u_\alpha)_{\alpha \in \mathcal{L}} \subset K[\mathbf{x}, \mathbf{y}].$$

Since the ideal $H_{\mathcal{L}}$ is squarefree, there is a simplicial complex $\Sigma_{\mathcal{L}}$ on the vertex set $\{x_p, y_p\}_{p \in P}$ whose Stanley-Reisner ideal $I_{\Sigma_{\mathcal{L}}}$ coincides with $H_{\mathcal{L}}$. We call $\Sigma_{\mathcal{L}}$ the *homogenized ideal complex of $\mathcal{L}$*.

Let $\Gamma_{\mathcal{L}}$ denote the *Alexander dual* ([2], [4]) of $\Sigma_{\mathcal{L}}$ and call $\Gamma_{\mathcal{L}}$ *the homogenized ideal dual complex of* $\mathcal{L}$. We write $\mathcal{F}(\Gamma_{\mathcal{L}})$ for the set of facets (maximal faces) of $\Gamma_{\mathcal{L}}$. One has

$$\mathcal{F}(\Gamma_{\mathcal{L}}) = \{F_\alpha : \alpha \in \mathcal{L}\} \tag{2}$$

where

$$F_\alpha = \{x_q : q \in P \setminus \ell(\alpha)\} \cup \{y_q : q \in \ell(\alpha)\}.$$

Hence,

$$I_{\Gamma_{\mathcal{L}}} = \bigcap_{\alpha \in \mathcal{L}} \big(\{x_p : p \in \ell(\alpha)\} \cup \{y_q : q \in P \setminus \ell(\alpha)\}\big).$$

In particular $\Gamma_{\mathcal{L}}$ is a pure simplicial complex of dimension $|P| - 1$.

A finite meet-semilattice $\mathcal{L}$ is called *meet-distributive* [7, p. 156] if each interval $[\alpha, \beta] = \{\gamma \in \mathcal{L} : \alpha \leq \gamma \leq \beta\}$ of $\mathcal{L}$ such that α is the meet of the lower neighbors of β in $[\alpha, \beta]$ is boolean. For example, every poset ideal of a finite distributive lattice is a meet-distributive meet-semilattice.

Let $\mathcal{L}$ be an arbitrary finite meet-distributive meet-semilattice and, as before, $P \subset \mathcal{L}$ the set of join-irreducible elements of $\mathcal{L}$. The *distributive closure* of $\mathcal{L}$ is the finite distributive lattice $\mathcal{J}(P)$ consisting of all poset ideals of P ordered by inclusion.

Recall that Birkhoff's fundamental structure theorem on finite distributive lattices [7, Theorem 3.4.1] guarantees that every finite distributive lattice is of the form $\mathcal{J}(P)$ for a unique finite poset P. In fact, if P is the set of join-irreducible element of a finite distributive lattice $\mathcal{L}$, then $\mathcal{L} = \mathcal{J}(P)$.

It is not difficult to see that the map $\ell\colon \mathcal{L} \to \mathcal{J}(P)$ defined by (1) is an embedding of meet-semilattices if and only if $\mathcal{L}$ is meet-distributive. Consult [3] for further information about meet-distributive lattices.

Proposition 1.2. *Let $\mathcal{L}$ be a finite meet-distributive meet-semilattice and $\Gamma_{\mathcal{L}}$ its homogenized ideal dual complex. Let $h(\Gamma_{\mathcal{L}}) = (h_0, h_1, \dots)$ be its h-vector. Then, for all i, one has*

$$h_i = |\{\alpha \in \mathcal{L} : |N(\alpha)| = i\}|.$$

Proof. Let $\alpha \in \mathcal{L}$ with $|N(\alpha)| = i$ and $\ell(\alpha) = \{q_1, \dots, q_\delta\}$. Let $N(\alpha) = \{r_1, \dots, r_i\}$ with each $\ell(r_j) = \ell(\alpha) \setminus \{q_{\delta-j+1}\}$. Let $r = r_1 \wedge \cdots \wedge r_i$. Thus $\ell(r) = \bigcap_{j=1}^{i} \ell(r_j) = \{q_1, \dots, q_{\delta-i}\}$ and the interval $[r, \alpha]$ in $\mathcal{L}$ is the boolean lattice of rank i. Since a subset $A \subset \ell(\alpha)$ is contained in none of the sets $\ell(r_1), \dots, \ell(r_i)$ if and only if A contains $\{q_\delta, q_{\delta-1}, \dots, q_{\delta-i+1}\}$, it follows that the number of subsets $A \subset \ell(\alpha)$ with $|A| = k$ such that $A \subset \ell(q)$ for no $q \in \mathcal{L}$ with $q < \alpha$ is $\binom{\delta-i}{k-i}$. In other words, the number of those faces $F \subset F_\alpha$ of Γ with $|F| = j+1$ such that $F \subset F_q$ for no $q \in \mathcal{L}$ with $q < \alpha$ is $\sum_{k=i}^{\delta} \binom{|P|-\delta}{j-k+1}\binom{\delta-i}{k-i}$, which is equal to $\binom{|P|-i}{j-i+1} = \binom{|P|-i}{|P|-j-1}$. Thus the number of faces F of $\Gamma_\mathcal{L}$ with $|F| = j+1$ is

$$f_j(\Gamma_\mathcal{L}) = \sum_{i=0}^{j+1} \binom{|P|-i}{|P|-j-1} |\{\alpha \in \mathcal{L} : |N(\alpha)| = i\}|.$$

On the other hand, in general, one has

$$f_j(\Gamma_\mathcal{L}) = \sum_{i=0}^{j+1} \binom{|P|-i}{|P|-j-1} h_i(\Gamma_\mathcal{L}).$$

Hence $h_i(\Gamma_\mathcal{L}) = |\{\alpha \in \mathcal{L} : |N(\alpha)| = i\}|$, as desired. □

COROLLARY 1.3. *Let $\mathcal{L}$ be a finite meet-distributive meet-semilattice, P the set of join-irreducible elements of $\mathcal{L}$ and $\Gamma_\mathcal{L}$ the homogenized ideal dual complex of $\mathcal{L}$. Let $n = |P|$ and $(h_0, h_1, \dots, h_n)$ the h-vector of $\Gamma_\mathcal{L}$. Then $h_1 = n$, and the a-invariant of $\Gamma_\mathcal{L}$ (which is the nonpositive integer $\max\{i : h_i \neq 0\} - n$) is equal to $\max\{|N(\alpha)| : \alpha \in \mathcal{L}\} - n$.*

EXAMPLE 1.4. Let $\mathcal{B}_{[n]}$ denote the boolean lattice of rank n and $\mathcal{L}$ a poset ideal of $\mathcal{B}_{[n]}$ which contains all join-irreducible elements (i.e., $\{1\}, \dots, \{n\}$) of $\mathcal{B}_{[n]}$. Then the meet-distributive meet-semilattice $\mathcal{L}$ is a simplicial complex on $[n]$ and the h-vector of $\Gamma_\mathcal{L}$ coincides with the f-vector of $\mathcal{L}$.

(a) By using (2) the Stanley-Reisner ideal $I_{\Gamma_\mathcal{L}}$ of $\Gamma_\mathcal{L}$ is generated by those squarefree monomials $\prod_{q \in \ell(\beta)} y_q$ such that $\beta \in \mathcal{B}_{[n]}$ is a minimal nonface of $\mathcal{L}$ and by the quadratic monomials $x_{\{i\}} y_{\{i\}}$ for all $i \in [n]$.

(b) Let $T = K[y_{\{1\}}, \dots, y_{\{n\}}]$ and $J \subset T$ the ideal generated by those squarefree monomials $\prod_{q \in \ell(\beta)} y_q$ such that $\beta \in \mathcal{B}_{[n]}$ is a minimal nonface of $\mathcal{L}$ and by $y_{\{i\}}^2$ for all $i \in [n]$. The quotient ring T/J is 0-dimensional and its h-vector is $(f_{-1}, f_0, \dots)$, the f-vector of $\mathcal{L}$ with $f_{-1} = 1$. It turns out that $I_{\Gamma_\mathcal{L}}$ is the polarization [1, Lemma 4.3.2] of the ideal J. Since T/J is Cohen-Macaulay, it follows immediately that $\Gamma_\mathcal{L}$ is Cohen-Macaulay. This fact is a special case of [5, Corollary 1.6].

(c) Since T/J is a level ring [8, p. 91] if and only if the simplicial complex $\mathcal{L}$ is pure, it follows that the homogenized ideal dual complex $\Gamma_\mathcal{L}$ of $\mathcal{L}$ is a level complex if and only if the simplicial complex $\mathcal{L}$ is pure.

(d) Let Δ be a simplicial complex on the vertex set $V = \{y_{\{1\}}, \ldots, y_{\{n\}}\}$, and let $W = \{x_{\{1\}}, \ldots, x_{\{n\}}\}$. We write $\Delta^\sharp$ for the simplicial complex on the vertex set $V \cup W$ whose facets are those of Δ together with all edges $\{x_{\{i\}}, y_{\{i\}}\}$ for $i = 1, \ldots, n$. By the observation (a) for a simplicial complex $\mathcal{L}$ ($\subset \mathcal{B}_{[n]}$) on $[n]$ one has a simplicial complex Δ on V such that the facet ideal of $\Delta^\sharp$, i.e., the ideal generated by all monomials corresponding to the facets, coincides with the Stanley-Reisner ideal $I_{\Gamma_{\mathcal{L}}}$ of $\Gamma_{\mathcal{L}}$. Conversely, given a simplicial complex Δ on V, there is a simplicial complex $\mathcal{L}$ ($\subset \mathcal{B}_{[n]}$) on $[n]$ such that the facet ideal of $\Delta^\sharp$ coincides with $I_{\Gamma_{\mathcal{L}}}$. Since $\Gamma_{\mathcal{L}}$ is always Cohen-Macaulay, the facet ideal of $\Delta^\sharp$ is Cohen-Macaulay. This argument is a direct and easy proof of [5, Corollary 4.4].

(e) By using (b) and (c), it follows that every f-vector of a pure simplicial complex is the h-vector of a level complex.

It would, of course, be of interest to generalize the fact (c) of Example 1.4 to arbitrary meet-distributive meet-semilattices $\mathcal{L}$.

§2. Alexander duality of meet-distributive meet-semilattices

A nice description of the homogenized ideal dual complex of a finite distributive lattice is obtained in [6, Lemma 3.1]. On the other hand, the homogenized ideal dual complex of a meet-distributive meet-semilattice of a special kind, namely, a poset ideal of a finite distributive lattice is described in [5, Theorem 4.2]. An explicit description of the homogenized ideal dual complex of an arbitrary finite meet-semilattice will be obtained in Theorem 2.1 below.

If, in general, P is a finite poset and $B \subset P$, then we write $\langle B \rangle$ for the poset ideal of P *generated by* B, i.e., $p \in P$ belongs to $\langle B \rangle$ if and only if $p \leq q$ for some $q \in B$.

THEOREM 2.1. *Let $\mathcal{L}$ be an arbitrary finite meet-semilattice and P the set of join-irreducible elements of $\mathcal{L}$. Then the Stanley-Reisner ideal $I_{\Gamma_{\mathcal{L}}}$ of the homogenized ideal dual complex $\Gamma_{\mathcal{L}}$ of $\mathcal{L}$ is generated by the following squarefree monomials*:

(i) $x_p y_q$, *where $p, q \in P$ with $p < q$*;

(ii) $\prod_{q \in B} y_q$, *where B is an antichain of P with $\langle B \rangle \not\subset \ell(\alpha)$ for all $\alpha \in \mathcal{L}$*;

(iii) $x_p \prod_{q \in B} y_q$, *where B is an antichain of P with $\ell(\beta) \neq \langle B \rangle$ for all $\beta \in \mathcal{L}$, but with $\langle B \rangle \subset \ell(\alpha)$ for some $\alpha \in \mathcal{L}$ and where $p \in \ell\big(\bigwedge_{\langle B \rangle \subset \ell(\alpha)} \alpha\big) \setminus \langle B \rangle$.*

Proof. Let $A \subset P$ and $B \subset P$ with $A \cap B = \emptyset$. We write $\mathbf{x}_A\mathbf{y}_B$ for the squarefree monomial $\prod_{p\in A} x_p \prod_{q\in B} y_q$ of $K[\mathbf{x}, \mathbf{y}]$. By the definition of the Stanley-Reisner ideal $I_{\Gamma_\mathcal{L}}$ of $\Gamma_\mathcal{L}$, it follows that $\mathbf{x}_A\mathbf{y}_B$ belongs to $I_{\Gamma_\mathcal{L}}$ if and only if there is no facet $\mathcal{F}_\alpha$ of $\Gamma_\mathcal{L}$ with $\{x_p : p \in A\} \cup \{y_q : q \in B\} \subset \mathcal{F}_\alpha$. Thus by using (2) one has $\mathbf{x}_A\mathbf{y}_B \in I_{\mathcal{L}_\Gamma}$ if and only if there is no $\alpha \in \mathcal{L}$ such that $A \subset P \setminus \ell(\alpha)$ and $B \subset \ell(\alpha)$. In other words, one has $\mathbf{x}_A\mathbf{y}_B \in I_{\mathcal{L}_\Gamma}$ if and only if the following condition $(*)$ is satisfied:

$$(*) \qquad \text{each } \alpha \in \mathcal{L} \text{ with } B \subset \ell(\alpha) \text{ satisfies } A \cap \ell(\alpha) \neq \emptyset.$$

We say that a pair (A, B), where $A \subset P$ and $B \subset P$ with $A \cap B = \emptyset$, is an *independent pair* of $\mathcal{L}$ if the condition $(*)$ is satisfied. Thus $I_{\mathcal{L}_\Gamma}$ is generated by all monomials x_py_p with $p \in P$ together with those monomials $\mathbf{x}_A\mathbf{y}_B$ such that (A, B) is an independent pair of $\mathcal{L}$.

Let $M(B)$ denote the set of maximal elements of B. Thus one has $\langle B\rangle = \langle M(B)\rangle$. Hence (A, B) is independent if and only if $(A, M(B))$ is independent. Since $\mathbf{x}_A\mathbf{y}_{M(B)}$ divides $\mathbf{x}_A\mathbf{y}_B$ and since $M(B)$ is an antichain of P, it follows that $I_{\mathcal{L}_\Gamma}$ is generated by all monomials x_py_p with $p \in P$ together with those monomials $\mathbf{x}_A\mathbf{y}_B$ such that (A, B) is an independent pair of $\mathcal{L}$ and B is an antichain of P.

Let p and q belong to P. Since $\ell(q) = \langle\{q\}\rangle \in \mathcal{L}$ for all $q \in \mathcal{L}$, the pair $(\{p\}, \{q\})$ with $p \neq q$ is an independent pair of $\mathcal{L}$ if and only if $p < q$. Let $\ell(\beta) = \langle B\rangle$ for some $\beta \in \mathcal{L}$. Then a pair (A, B) is independent if and only if $A \cap \langle B\rangle \neq \emptyset$. On the other hand, $A \cap \langle B\rangle \neq \emptyset$ if and only if there are $p \in A$ and $q \in B$ with $p < q$.

Consequently, $I_{\mathcal{L}_\Gamma}$ is generated by all monomials x_py_q, where $p, q \in P$ with $p < q$ together with those monomials $\mathbf{x}_A\mathbf{y}_B$, where (A, B) is an independent pair of $\mathcal{L}$ such that B is an antichain of P with $\ell(\beta) = \langle B\rangle$ for no $\beta \in \mathcal{L}$ and with $A \cap \langle B\rangle = \emptyset$.

Now, let B be an antichain of P with $\ell(\beta) = \langle B\rangle$ for no $\beta \in \mathcal{L}$ and $A \subset P$ with $A \cap \langle B\rangle = \emptyset$.

(a) First, if $\langle B\rangle \subset \ell(\alpha)$ for no $\alpha \in \mathcal{L}$, then (A, B) is independent for all $A \subset P$ with $A \cap B = \emptyset$. Thus in particular $(\emptyset, B)$ is an independent pair of $\mathcal{L}$.

(b) Second, if $\langle B\rangle \subset \ell(\alpha)$ for some $\alpha \in \mathcal{L}$, then (A, B) is independent if and only if $A \cap \big(\bigcap_{\langle B\rangle\subset\ell(\alpha)} \ell(\alpha)\big) \neq \emptyset$. Since $\bigcap_{\langle B\rangle\subset\alpha} \ell(\alpha) = \ell\big(\bigwedge_{\langle B\rangle\subset\alpha} \alpha\big)$, it follows that (A, B) is independent if and only if there is $p \in A$ with $p \in \ell\big(\bigwedge_{\langle B\rangle\subset\alpha} \alpha\big) \setminus \langle B\rangle$. ∎

§3. Level rings arising from meet-distributive meet-semilattices

It is known [5, Corollary 1.6] that the homogenized ideal dual complex $\Gamma_{\mathcal{L}}$ of a finite meet-semilattice $\mathcal{L}$ is Cohen-Macaulay if and only if $\mathcal{L}$ is meet-distributive. The problem when the homogenized ideal dual complex of a meet-distributive meet-semilattice is a level ring is now studied.

Recall that a Cohen-Macaulay graded ring $R = R_0 \oplus R_1 \oplus \cdots$ over a field $K = R_0$ is called *level* [8, p. 91] if the canonical module of R is generated in one degree. Every Gorenstein ring is level.

Let $\mathcal{L}$ be a finite meet-distributive meet-semilattice and P the set of join-irreducible elements of $\mathcal{L}$. Let, as before, $K[\mathbf{x}, \mathbf{y}] = K[\{x_p, y_p\}_{p \in P}]$ denote the polynomial ring in $2|P|$ variables over a field K with each $\deg x_p = \deg y_p = 1$.

For each $\alpha \in \mathcal{L}$ we write $\alpha' \in \mathcal{L}$ for the meet of all $\beta \in N(\alpha)$, where $N(\alpha)$ is the set of lower neighbors of α. Since $\mathcal{L}$ is a meet-distributive meet-semilattice, it follows that the interval

$$\mathcal{B}_\alpha = [\alpha', \alpha] = \{\gamma \in \mathcal{L} : \alpha' \leq \gamma \leq \alpha\}$$

of $\mathcal{L}$ is a boolean lattice. Let $S(\alpha) \subset P$ denote the antichain

$$S(\alpha) = \ell(\alpha) \setminus \ell(\alpha').$$

Each element belonging to $S(\alpha)$ is a maximal element of $\ell(\alpha)$, and $t \in \ell(\alpha)$ belongs to $S(\alpha)$ if and only if $\ell(\beta) = \ell(\alpha) \setminus \{t\}$ for some $\beta \in N(\alpha)$.

Lemma 3.1. *If α and β belong to $\mathcal{L}$ with $\alpha \neq \beta$, then $S(\alpha) \neq S(\beta)$.*

Proof. Let $\gamma = \alpha \wedge \beta$. If $S(\alpha) = S(\beta)$, then $S(\alpha) \subset S(\gamma)$. In fact, for each $t \in S(\alpha) = S(\beta)$, there are $\alpha_0 \in N(\alpha)$ and $\beta_0 \in N(\beta)$ with $\ell(\alpha_0) = \ell(\alpha) \setminus \{t\}$ and $\ell(\beta_0) = \ell(\beta) \setminus \{t\}$. By using Lemma 1.1, one has $\ell(\alpha_0 \wedge \beta_0) = \ell(\gamma) \setminus \{t\}$. Hence $t \in S(\gamma)$. Now, since $\gamma < \alpha$, one has $\delta \in N(\alpha)$ with $\gamma \leq \delta < \alpha$. Since $\ell(\delta) = \ell(\alpha) \setminus \{t\}$ for some $t \in S(\alpha)$, it follows that $t \notin \ell(\gamma)$. This contradict $S(\alpha) \subset S(\gamma)$. Hence $S(\alpha) \neq S(\beta)$, as desired. ∎

Recall from the proof of Theorem 2.1 that a pair (A, B), where $A \subset P$ and $B \subset P$ with $A \cap B = \emptyset$, is said to be an independent pair of $\mathcal{L}$ if each $\alpha \in \mathcal{L}$ with $B \subset \ell(\alpha)$ satisfies $A \cap \ell(\alpha) \neq \emptyset$.

Lemma 3.2. *Let $\alpha \in \mathcal{L}$ and $T \subset S(\alpha)$. Then the pair $(\emptyset, T)$ cannot be independent. Moreover, for $p \in S(\alpha) \setminus T$, the pair $(\{p\}, T)$ cannot be independent.*

Proof. Since $T \subset \ell(\alpha)$, the pair $(\emptyset, T)$ cannot be an independent pair of $\mathcal{L}$. On the other hand, since $p \in S(\alpha)$, one has $\beta \in N(\alpha)$ with $\ell(\beta) = \ell(\alpha) \setminus \{p\}$. Since $T \subset \ell(\beta)$ and since $\{p\} \cap \ell(\beta) = \emptyset$, it follows that $(\{p\}, T)$ cannot be an independent pair of $\mathcal{L}$, as desired. ∎

Let $I_{\Gamma_{\mathcal{L}}}$ denote the Stanley-Reisner ideal of $\mathcal{L}$ and $K[\Gamma_{\mathcal{L}}] = K[\mathbf{x}, \mathbf{y}]/I_{\Gamma_{\mathcal{L}}}$ the Stanley-Reisner ring of $\Gamma_{\mathcal{L}}$. Since the dimension of $\Gamma_{\mathcal{L}}$ is $|P|-1$ and the Krull dimension of $K[\Gamma_{\mathcal{L}}]$ coincides with $|P|$, it follows easily that $\{x_p - y_p : p \in P\}$ is a linear system of parameters of $K[\Gamma_{\mathcal{L}}]$. Since $K[\Gamma_{\mathcal{L}}]$ is Cohen-Macaulay, by using Proposition 1.2, the Hilbert series of the quotient ring

$$K[\Gamma_{\mathcal{L}}]/(x_p - y_p : p \in P)$$

is $h_0 + h_1\lambda + h_2\lambda^2 + \cdots$, where $(h_0, h_1, h_2, \ldots)$ is the h-vector of $\Gamma_{\mathcal{L}}$.

Let $J_{\Gamma_{\mathcal{L}}}$ be the monomial ideal of $K[\mathbf{x}] = K[\{x_p\}_{p \in P}]$ generated by those monomials

(i) $x_p x_q$, where $p, q \in P$ with $p < q$;

(ii) $\prod_{q \in B} x_q$, where B is an antichain of P with $\langle B \rangle \subset \ell(\alpha)$ for no $\alpha \in \mathcal{L}$;

(iii) $x_p \prod_{q \in B} x_q$, where B is an antichain of P with $\ell(\beta) = \langle B \rangle$ for no $\beta \in \mathcal{L}$, but with $\langle B \rangle \subset \ell(\alpha)$ for some $\alpha \in \mathcal{L}$ and where $p \in \ell\big(\bigwedge_{\langle B \rangle \subset \ell(\alpha)} \alpha\big) \setminus \langle B \rangle$.

By virtue of Theorem 2.1 it follows that

$$K[\mathbf{x}]/J_{\Gamma_{\mathcal{L}}} = K[\Gamma_{\mathcal{L}}]/(x_p - y_p : p \in P).$$

We associate each $\alpha \in \mathcal{L}$ with the monomial

$$u_\alpha = \prod_{p \in S(\alpha)} x_p$$

of degree $|N(\alpha)|$.

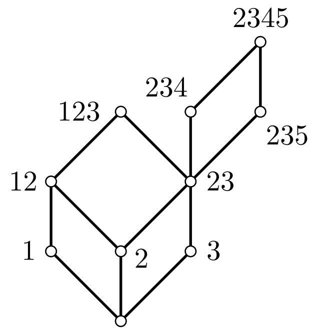
THEOREM 3.3. *Let $\mathcal{L}$ be a finite meet-distributive meet-semilattice, P the set of join-irreducible elements of $\mathcal{L}$, and $K[\mathbf{x}] = K[\{x_p\}_{p \in P}]$ the polynomial ring in $|P|$ variables over a field K. Then the set of monomials $\{u_\alpha \,;\, \alpha \in \mathcal{L}\}$ is a K-basis of the quotient ring $K[\mathbf{x}]/J_{\Gamma_{\mathcal{L}}}$. Thus in particular $\{S(\alpha) : \alpha \in \mathcal{L}\}$ is a simplicial complex on the vertex set $\{x_p : p \in P\}$ whose f-vector coincides with the h-vector of $\mathcal{L}$.*

Proof. Lemma 3.2 says that, for each $\alpha \in \mathcal{L}$, the monomial u_α does not belongs to $J_{\Gamma_\mathcal{L}}$. Moreover, Lemma 3.1 guarantees that, for $\alpha \neq \beta$ belonging to $\mathcal{L}$, one has $u_\alpha \neq u_\beta$. Hence, for each $i = 0, 1, 2, \ldots$, the number of monomials u_α with $\alpha \in \mathcal{L}$ of degree i is equal to h_i. Since the Hilbert series of $K[\mathbf{x}]/J_{\Gamma_\mathcal{L}}$ is $h_0 + h_1\lambda + h_2\lambda^2 + \cdots$, it follows that $\{u_\alpha \ ; \ \alpha \in \mathcal{L}\}$ is a K-basis of $K[\mathbf{x}]/J_{\Gamma_\mathcal{L}}$, as required. □

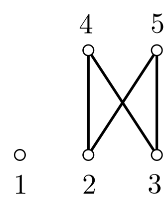
We now come to the combinatorial characterization for the homogenized ideal dual complex $\Gamma_\mathcal{L}$ of a finite meet-distributive meet-semilattice $\mathcal{L}$ to be level.

COROLLARY 3.4. *The homogenized ideal dual complex $\Gamma_\mathcal{L}$ of a finite meet-distributive meet-semilattice $\mathcal{L}$ is a level complex if and only if the simplicial complex $\{S(\alpha) : \alpha \in \mathcal{L}\}$ is pure. Thus in particular the homogenized ideal dual complex $\Gamma_\mathcal{L}$ of a finite distributive lattice $\mathcal{L} = \mathcal{J}(P)$ is level if and only if the simplicial complex consisting of all antichains of P is pure.*

Consider the following example of a meet-distributive meet-semilattice $\mathcal{L}$



with the following poset of join-irreducible elements



By using Theorem 2.1 the Stanley-Reisner ideal of the homogenized ideal dual complex of $\mathcal{L}$ is generated by the following monomials:

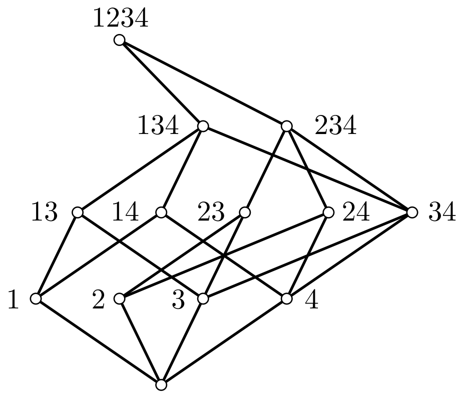
(i) $x_1y_1, \ldots, x_5y_5$, x_2y_4, x_2y_5, x_3y_4 and x_3y_5;

(ii) y_1y_4 and y_1y_5;

(iii) $x_2y_1y_3$.

The h-vector of $\mathcal{L}$ is $(1, 5, 4)$. By using Corollary 3.4 the homogenized ideal dual complex $\Gamma_{\mathcal{L}}$ is level.

In the following meet-distributive meet-semilattice $\mathcal{L}$ the facets of the simplicial complex $\{S(\alpha) : \alpha \in \mathcal{L}\}$ are $\{1, 2\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$. Since this simplicial complex is not pure it follows from Corollary 3.4 that $\Gamma_{\mathcal{L}}$ is not level. The h-vector of $\mathcal{L}$ is $(1, 4, 6, 2)$.



By Theorem 3.3 the h-vector of the homogenized ideal dual complex of a meet-distributive meet-semilattice is just the f-vector of a simplicial complex, and the h-vector of a level simplicial complex coming from a meet-distributive meet-semilattice is just the f-vector of a pure simplicial complex. These facts lead us to the following

Question 3.5. (a) Characterize the h-vectors of the homogenized ideal dual complex of finite distributive lattices.

(b) Characterize the h-vectors of the homogenized ideal dual complex of meet-distributive lattices.

(c) Find a nice class of level simplicial complexes whose h-vector is not the f-vector of a pure simplicial complex.

For example $(1, 3, 3)$ is the h-vector of the the homogenized ideal dual complex of the meet-distributive lattice $\mathcal{B}_{[3]} \setminus \{1, 3\}$, but is not the h-vector of the homogenized ideal dual complex of a distributive lattice.

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Jürgen Herzog
Fachbereich Mathematik und Informatik
Universität Duisburg-Essen
Campus Essen
45117 Essen
Germany
`juergen.herzog@uni-essen.de`

Takayuki Hibi
Department of Pure and Applied Mathematics
Graduate School of Information Science and Technology
Osaka University
Toyonaka
Osaka 560-0043
Japan
`hibi@math.sci.osaka-u.ac.jp`