REGULAR TSUJI FUNCTIONS WITH INFINITELY MANY JULIA POINTS

W. K. HAYMAN

To K. NOSHIRO on his 60th birthday

1. Introduction

Let D denote the unit disk $|z|<1$, and C the unit circle $|z|=1$. Corresponding to any function f meromorphic in D we denote by f^* the spherical derivative

$$f^*(z)=\frac{|f'(z)|}{1+|f(z)|^2}.$$

We write

$$L(r)=\int_0^{2\pi} f^*(re^{i\theta})rd\theta, \qquad 0<r<1,$$

and shall say that $f\in T_1(l)$ if

$$\overline{\lim_{r\to 1}}\, L(r)\leq l<+\infty.$$

The functions $f\in T_1(l)$ are called Tsuji functions by Collingwood and Piranian [1]. Following their notation we call a rectilinear segment S lying in D except for one end-point $e^{i\theta}$ on C a segment of Julia for f provided that in each open triangle in D having one vertex at $e^{i\theta}$ and meeting S, the function f assumes all values on the Riemann sphere except possibly two. A point $e^{i\theta}$ is called a Julia point for f provided that each rectilinear segment S lying except for one endpoint $e^{i\theta}$ in D is a segment of Julia for f.

Following Tsuji [3] Collingwood and Piranian [1] investigated the class $T_1(l)$ and provided a number of illuminating examples. They proved among other results [1, Theorems 1, 5]

THEOREM A. *There exists a meromorphic Tsuji function for which each point of C is a Julia point.*

THEOREM B. *The function*

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$$w = \exp\left\{\left(\frac{1+z}{1-z}\right)^2\right\}$$

is a regular Tsuji function with two segments of Julia at $z=1$. Their examples led Collingwood and Piranian to the following 3 conjectures concerning regular Tsuji functions.

I. If f is a regular Tsuji function then at most finitely many points of C are endpoints of segments of Julia for f.

II. If f is a regular Tsuji function then at most finitely many segments in D are segments of Julia for f.

III. If f is a regular normal Tsuji function then f has no segments of Julia.

In this paper we shall give a counter-example to I and II by proving

THEOREM 1. *There exist regular Tsuji functions with infinitely many Julia points.*

We shall prove elsewhere [2] that a normal meromorphic Tsuji function necessarily remains continuous in $|z|\leq 1$ in the metric of the closed sphere so that conjecture III holds even for meromorphic Tsuji functions. Also such a function can have no point other than $f(e^{i\theta})$ in its range set at $e^{i\theta}$. We shall prove however

THEOREM 2. *There exists a bounded Tsuji function, continuous in* $|z|\leq 1$ *and having zeros in each open triangle in* D *one of whose endpoints belongs to a certain infinite set on* C.

Thus the range at $e^{i\theta}$ need not be empty.

2. Preliminary results

We shall proceed by means of a series of lemmas We have first

LEMMA 1. *Let* Δ *be the domain defined by* $w=\rho e^{i\phi}$, *where*

$$2^{-n} < \rho < 1, \text{ if } \phi = \frac{\pi}{2^n}, \quad n = 1, 2, \ldots$$

$$0 < \rho < 1, \text{ if } 0 < \phi < \pi, \ \phi \neq \frac{\pi}{2^n}.$$

Then a function $w=f(z)$ *which maps* $D(1, 1)$ *conformally onto* Δ *is a bounded Tsuji function which remains continuous on* C *and vanishes at a countable set of points on* C *but no points of* D.

Clearly Δ is a simply connected domain whose boundary γ is rectifiable and of length

$$l = 2 + \pi + 2\sum_{1}^{\infty} 2^{-n} = 4 + \pi.$$

Thus (see e.g [2, Lemmas 8 and 10])

$$\lim_{r\to 1}\int_0^{2\pi} |f'(re^{i\theta})| r d\theta = 4 + \pi,$$

so that $f \in T_1(4+\pi)$. Also f remains continuous on C and maps C onto γ in such a way that each point of C corresponds in a (1, 1) manner to a prime end of γ. Since there are infinitely many prime ends of γ at the point $w = 0$, namely those for which

$$\frac{\pi}{2^{n+1}} < \phi < \frac{\pi}{2^n}, \quad n = 0, 1, 2, \ldots, \text{ and } \phi = 0,$$

there exists a corresponding sequence of points $z = e^{i\theta_n}$ on C which are mapped onto $w = 0$ by $f(z)$. Further since Δ does not contain $w = 0$, $f(z) \neq 0$ in D. This proves Lemma 1.

Theorems 1 and 2 will be a consequence of

Theorem 3. *Suppose that $f(z) \in T_1(l)$, $f(z) \not\equiv 0$, and that F is a finite or countable set on C such that $f(z)$ vanishes continuously at the points ζ of F. Then there exists a sequence z_ν of points in D such that*

(i) $\sum (1 - |z_\nu|) < +\infty$,

(ii) If $\Pi(z) = \prod_{\nu=1}^{\infty}\left(\frac{z_\nu - z}{1 - \bar{z}_\nu z}\right)\frac{\bar{z}_\nu}{|z_\nu|}$,

then $f(z)/\Pi(z)$ and $f(z)\Pi(z)$ both belong to $T_1(l')$ for some $l' < +\infty$.

(iii) *Each point $\zeta \in F$ is a Julia point for $f(z)/\Pi(z)$, with zero as the only possible exceptional value.*

(iv) *$f(z)\Pi(z)$ has infinitely many zeros in every triangle with vertex at $\zeta \in F$. Also $f(z)\Pi(z)$ remains continuous at every point $\zeta \in F$.*

We choose the sequence $z_\nu = \rho_\nu e^{i\phi_\nu}$ to satisfy the following conditions

a) $(1 - \rho_{\nu+1})/(1 - \rho_\nu) < \frac{1}{4}$, $\nu = 1, 2, \ldots, \rho_1 = \frac{1}{2}$.

b) Every triangle in D with vertex at a point ζ in F contains infinitely many of the points z_ν.

c) $|f(re^{i\theta})|<2^{-\nu}$, for $2\rho_\nu-1<r<1$, and $|\theta-\phi_\nu|<2^\nu(1-\rho_\nu)$.

d) $f(z_\nu)\neq 0$.

3. Proof of Theorem 3

We prove Theorem 3 in two stages.

LEMMA 2. *The conditions* a), b), c), d) *are compatible, i.e. a sequence* z_ν *exists satisfying them all.*

We assume that l_k, $k=1,2,\ldots$ is a countable system of rays, such that every l_k has one endpoint at a point $\zeta=e^{i\theta}\in F$, and further such that every Stolz angle with vertex at such a point ζ contains infinitely many of the rays l_k. Since F is finite or countable we can clearly choose such a system l_k. Next let n_p be a sequence of positive integers such that n_p assumes every positive integral value k infinitely often. For this we may choose for instance $n_p=1+p-[\nu p]^2$, where $[x]$ denotes the integral part of x. We then choose z_p to lie on the ray l_{n_p}. In this way condition b) is certainly satisfied. We can also satisfy a) and c). Suppose in fact that $\zeta=e^{i\theta}$ is the vertex of l_{n_p}. Then by hypothesis we have

$$|f(z)|<2^{-p}, \text{ if } |z-\zeta|<\varepsilon_p, \text{ say and } |z|<1.$$

We now choose ρ_p so near 1, that

$$2^{p+2}|\zeta-z_p|=\min\{(1-\rho_{p-1}),\ \varepsilon_p\}.$$

Then $(1-\rho_p)/(1-\rho_{p-1})\leq 2^{-p-2}$, so that a) holds. We also suppose that $f(z_p)\neq 0$, so that d) holds. Further if $z=re^{i\psi}$, and $2\rho_p-1<r<1$, $|\psi-\arg z_p|<2^p(1-\rho_p)$, then

$$|z-\zeta|<|z-z_p|+|z_p-\zeta|<|\psi-\arg z_p|+2(1-\rho_p)+|z_p-\zeta|$$
$$<(2^p+2)(1-\rho_p)+|z_p-\zeta|<(2^p+3)|\zeta-z_p|<\varepsilon_p.$$

Thus $|f(z)|<2^{-p}$ and c) is also satisfied. This proves Lemma 2.

We have finally.

LEMMA 3. *If the points* z_ν *satisfy* a), b), c) *and* d), *then the conclusions of Theorem* 3 *hold.*

In fact (i) is an immediate consequence of a). Again (iv) follows at once from b) and the fact that $|\Pi(z)|<1$ and so $f(z)\Pi(z)\to 0$ as $z\to\zeta\in F$ from $|z|<1$.

We next prove (iii). We note that

$$\left|\frac{1-\bar{z}_\nu z}{z-z_\nu}\right|^2-1=\frac{(1-|z_\nu|^2)(1-|z|^2)}{|z-z_\nu|^2}.$$

Thus

$$\log\left|\frac{1}{\Pi(z)}\right|^2<\frac{1}{2}\sum_{\nu=1}^{\infty}\frac{(1-|z_\nu|^2)(1-|z|^2)}{|z-z_\nu|^2}.$$

Suppose now that $|z|=r$, where $\frac{1}{2}<r<1$, and let q be the largest value of ν for which $|z_\nu|\leq 2r-1$. Then, for $0\leq t\leq q-1$, we have from a)

$$1-|z_{q-t}|\geq 4^t(1-|z_q|)>2.4^t(1-r).$$

Also

$$|z-z_{q-t}|\geq\frac{1}{2}(1-|z_{q-t}|) \text{ so that}$$

$$\frac{1-|z_{q-t}|}{|z-z_{q-t}|^2}\leq\frac{(1-|z_{q-t}|)}{\left[\frac{1}{2}(1-|z_{q-t}|)\right]^2}<\frac{4}{2[4^t(1-r)]}.$$

Thus

$$\frac{1}{2}\sum_{\nu\leq q}\frac{(1-|z_\nu|^2)(1-|z|^2)}{|z-z_\nu|^2}\leq 2\sum_{\nu\leq q}\frac{(1-|z_\nu|)(1-r)}{|z-z_\nu|^2}<4\sum_{t=0}^{\infty}4^{-t}<6.$$

Again if p is the least value of ν for which $|z_\nu|\geq\frac{1}{2}(1+r)$, we have for $t\geq 0$ in view of a)

$$(1-|z_{p+t}|)\leq 4^{-t}(1-|z_p|)\leq\frac{1}{2}4^{-t}(1-r)$$

and if $|z|=r$, $\nu\geq p$, then $|z-z_\nu|^2\geq\left\{\frac{1}{2}(1-r)\right\}^2$.

Thus

$$\frac{1}{2}\sum_{t=0}^{\infty}\frac{(1-|z_{p+t}|^2)(1-|z|^2)}{|z-z_{p+t}|^2}\leq\sum_{t=0}^{\infty}\frac{4^{-t}(1-r)(1-r)}{\left[\frac{1}{2}(1-r)\right]^2}\leq 4\sum_{t=0}^{\infty}4^{-t}<6.$$

Thus if $\Pi_1(z)$ denotes the product $\Pi(z)$ with the omission of the factor corresponding to the value z_ν, if any, for which

$$2r-1<|z_\nu|<\frac{1}{2}(1+r), \tag{1}$$

then we have on $|z|=r$

$$\frac{1}{|\Pi_1(z)|}<e^{12},$$

i.e.

$$A_1 < |\Pi_1(z)| < 1, \tag{2}$$

where $A_1 = e^{-12}$. We note that in view of a) there can be at most one ν for which z_ν lies in the range (1).

Suppose now that z_ν is a zero of $\Pi(z)$ and hence by d) a pole of $f(z)/\Pi(z)$ and consider $f(z)/\Pi(z)$ on the circle $|z-z_\nu| = 2^{-(1/2)\nu}(1-\rho_\nu)$. On this circle we have in view of c)

$$\left|\frac{f(z)}{\Pi(z)}\right| = \left|\frac{f(z)}{\Pi_1(z)}\right| \cdot \left|\frac{1-\bar{z}_\nu z}{z-z_\nu}\right| < A_1^{-1} 2^{-\nu} \cdot \frac{(1-|z_\nu|^2) + |z-z_\nu|\,|\bar{z}_\nu|}{2^{-(1/2)\nu}(1-\rho_\nu)}$$

$$< \frac{3A_1^{-1}2^{-\nu}(1-\rho_\nu)}{2^{-(1/2)\nu}(1-\rho_\nu)} = 3A_1^{-1}2^{-(1/2)\nu}.$$

Hence $\dfrac{f(z)}{\Pi(z)}$ assumes every value w, with $|w| > 3A_1^{-1}2^{-(1/2)\nu}$ equally often inside this circle, i.e. exactly once, and if w is fixed and $w \neq 0$, this condition is satisfied for all sufficiently large ν. It follows that, in any Stolz angle containing one of the lines l_k, $f(z)$ assumes infinitely often all values except possibly zero, and so these are all Julia lines. Since every Stolz angle at $\zeta \in F$ contains such lines l_k, it follows that every ray with endpoint at ζ is a Julia line, and so ζ is a Julia point.

4. Proof of (ii)

It remains to prove (ii) and this is by far the hardest part of the argument. We proceed in a number of stages.

LEMMA 4. *If* $\frac{1}{2} \leq r < 1$, *and* $\Pi_1(z)$ *is formed from* $\Pi(z)$ *by omitting the factor corresponding to that zero* z_ν, *if any, for which* (1) *holds, then if* $F(z) = f(z)/\Pi_1(z)$ *or* $F(z) = f(z)\Pi_1(z)$, *we have*

$$\int_0^{2\pi} F^*(re^{i\theta})\, r d\theta < l_1 < +\infty,$$

where l_1 *is independent of* r.

Consider first $F(z) = f(z)\Pi_1(z)$. We have

$$\frac{|F'(z)|}{1+|F|^2} \leq \frac{|f'\Pi_1|}{1+|f\Pi_1|^2} + \frac{|f\Pi_1'|}{1+|f\Pi_1|^2}. \tag{3}$$

In view of (2) we have $|f\Pi_1| > A_1|f|$, and so if $|f| > 1$, we have

$$\frac{1}{1+|f\Pi_1|^2} < \frac{1}{|A_1|^2|f|^2} < \frac{2}{A_1^2(1+|f|^2)}, \tag{4}$$

while if $|f|<1$

$$\frac{1}{1+|f\Pi_1|^2} < 1 < \frac{2}{1+|f|^2}.$$

Thus (4) holds in all cases and

$$\int_0^{2\pi} \frac{|f'(re^{i\theta})\Pi_1(re^{i\theta})|\,rd\theta}{1+|f(re^{i\theta})\Pi_1(re^{i\theta})|^2} \le \frac{2}{A_1^2}\int_0^{2\pi} \frac{|f'(re^{i\theta})|\,rd\theta}{1+|f(re^{i\theta})|^2} \le \frac{4\,l}{A_1^2}, \tag{5}$$

if r is sufficiently near 1.

We now consider the second term on the right hand side of (3). In view of (4) we may write

$$\frac{|f\Pi_1'|}{1+|f\Pi_1|^2} \le \frac{2}{A_1^2}\frac{|f|}{1+|f|^2}|\Pi_1'| \le \frac{2}{A_1^2}\frac{|f|}{1+|f|^2}\left|\frac{\Pi_1'}{\Pi_1}\right|.$$

Also

$$\left|\frac{\Pi_1'}{\Pi_1}\right| = \left|\sum_{\nu=1}^{\infty}\frac{1-|z_\nu|^2}{(1-\bar{z}_\nu z)(z-z_\nu)}\right| \le \sum_{\nu=1}^{\infty}\frac{1-|z_\nu|^2}{|z_\nu-z|^2}. \tag{6}$$

We therefore proceed to estimate

$$\int_{|z|=r}\frac{|f|}{1+|f|^2}\,\frac{1-|z_\nu|^2}{|z_\nu-z|^2}\,|dz|.$$

Suppose first that $|z_\nu|>\frac{1}{2}(1+r)$. Then if $z=re^{i\theta}$, $z_\nu=\rho_\nu e^{i\phi_\nu}$, we have

$$|z_\nu-z|^2=(\rho_\nu-r)^2+2\rho_\nu r[1-\cos(\phi-\phi_\nu)]\ge\frac{1}{4}(1-r)^2+\frac{(\phi-\phi_\nu)^2}{\pi^2},$$

for $\phi_\nu-\pi\le\phi\le\phi_\nu+\pi$. Thus

$$\int_{|z|=r}\frac{1}{|z_\nu-z|^2}|dz|\le\pi^2\int_{-\pi}^{\pi}\frac{d\phi}{\phi^2+(1-r)^2}\le\pi^2\int_{-\infty}^{\infty}\frac{d\phi}{\phi^2+(1-r)^2} = \frac{\pi^3}{1-r}.$$

Thus

$$\sum_{|z_\nu|>1/2(1+r)}\int_{|z|=r}\frac{|f|}{1+|f|^2}\,\frac{1-|z_\nu|^2}{|z_\nu-z|^2}|dz| < \frac{\pi^3}{2(1-r)}\sum_{|z_\nu|>1/2(1+r)}(1-|z_\nu|^2)<A_3, \tag{7}$$

in view of a).

Next suppose that $|z_\nu|\le 2r-1$. Then we have

$$|z_\nu-z|^2\ge\frac{1}{4}(1-\rho_\nu)^2+\frac{2\rho_\nu r(\phi-\phi_\nu)^2}{\pi^2}\ge\frac{(1-\rho_\nu)^2+(\phi-\phi_\nu)^2}{2\pi^2}, \tag{8}$$

since $\rho_\nu \geq \frac{1}{2}$, $r \geq \frac{1}{2}$. By c) we have, for $|\phi - \phi_\nu| < 2^{1/2\,\nu}(1-\rho_\nu)$,

$$\frac{|f|}{1+|f|^2}\frac{(1-|z_\nu|^2)}{|z_\nu - z^2|} < \frac{2\pi^2 2^{-\nu}(1-\rho_\nu^2)}{(1-\rho_\nu)^2} < \frac{2\pi^2 2^{1-\nu}}{(1-\rho_\nu)}.$$

Thus

$$\int_{|\phi-\phi_\nu|<2^{1/2\,\nu}(1-\rho_\nu)} \frac{|f|}{1+|f|^2}\frac{1-|z_\nu|^2}{|z_\nu - z|^2}|dz| < A_4 2^{-(1/2)\nu}. \tag{9}$$

Again if $|\phi - \phi_\nu| \geq 2^{(1/2)\nu}(1-\rho_\nu)$, then

$$\frac{1}{|z_\nu - z|^2} < \frac{2\pi^2}{(\phi-\phi_\nu)^2},$$

and so

$$\int_{|\phi-\phi_\nu|\geq 2^{(1/2)\nu}(1-\rho_\nu)} \frac{|dz|}{|z_\nu - z|^2} \leq 4\pi^2 \int_{2^{(1/2)\nu}(1-\rho_\nu)}^{\infty} \frac{dx}{x^2} = \frac{4\pi^2 2^{-(1/2)\nu}}{(1-\rho_\nu)}. \tag{10}$$

On combining (9) and (10) we deduce that if $|z_\nu| < 2r-1$,

$$\int_{|z|=r} \frac{|f|}{1+|f|^2}\frac{1-|z_\nu|^2}{|z_\nu - z|^2}|dz| < A_5 2^{-(1/2)\nu}. \tag{11}$$

Now using (6), (7) and (11) we see that

$$\int_{|z|=r} \frac{|f\Pi_1'|}{(1+|f\Pi_1|^2)}|dz| < A_6.$$

From this and (5) Lemma 4 follows for the case $F = f\Pi_1$, when we apply (3) and (4).

The case $F = f/\Pi_1$ is similar. We write

$$\frac{|F'|}{1+|F|^2} \leq \frac{|f'\Pi_1|}{|\Pi_1^2| + |f|^2} + \frac{|f\Pi_1'|}{|\Pi_1^2| + |f|^2} < A_7\left\{\frac{|f'|}{1+|f|^2} + \frac{|f|}{1+|f|^2}\left|\frac{\Pi_1'}{\Pi_1}\right|\right\}.$$

in view of (2). We now obtain our result as before, using (6), (7) and (11).

5. To complete the proof of Lemma 3 and so of Theorem 3 we now consider the possible effect of the single factor in $\Pi(z)$ corresponding to a zero z_ν, for which $2r-1 < |z_\nu| < \frac{1}{2}(1+r)$.

We consider first

$$F(z) = f(z)\Pi_1(z),\ G(z) = F(z)a(z),$$

where $a(z) = (z - z_\nu)/(1 - \bar{z}_\nu z)$ and $z_\nu = \rho_\nu e^{i\phi_\nu}$.

$$\frac{|G'(z)|}{1+|G|^2} \leq \frac{|F'(z)||a|}{1+|aF|^2} + \frac{\left|\frac{a'}{a}\right||aF|}{1+|aF|^2}.$$

If $|z-z_\nu|>\frac{1}{2}(1-|z_\nu|)$, then we see from (8) that

$$\frac{1}{2}<|a(z)|<1 \text{ and } \left|\frac{a'}{a}\right|<\frac{1-|z_\nu|^2}{|z-z_\nu|^2}<\frac{2\pi^2(1-\rho_\nu^2)}{(r-\rho_\nu)^2+|\phi-\phi_\nu|^2}.$$

Hence if E is the range of ϕ, for which $|re^{i\phi}-\rho_\nu e^{i\phi_\nu}|\geqq\frac{1}{2}(1-\rho_\nu)$, we have

$$\int_E \frac{|F'(z)||a||dz|}{1+|aF|^2} = \int_E \frac{\left|\frac{F'}{a}\right|d\phi}{\left|\frac{1}{a}\right|^2+|F|^2} < 2\int_{|z|=r}\frac{|F'(z)||dz|}{1+|F(z)|^2} < C,$$

say, while

$$\int_E \frac{|a'F|d\phi}{1+|aF|^2} \leq \int_E \left|\frac{a'}{a}\right|d\phi \leq 2\pi^2\int_E \frac{(1-\rho_\nu^2)d\phi}{(\phi-\phi_\nu)^2+(r-\rho_\nu)^2}.$$

If $|r-\rho_\nu|<\frac{1}{4}(1-\rho_\nu)$, we see that $|\phi-\phi_\nu|\geq\frac{1}{4}(1-\rho_\nu)$ in our range so that the righthand side is bounded by an absolute constant. If $|r-\rho_\nu|=\frac{1}{4}(1-\rho_\nu)$, then

$$\int_E \frac{(1-\rho_\nu^2)\,d\phi}{(\phi-\phi_\nu)^2+(r-\rho_\nu)^2} \leq \int_{-\infty}^{\infty}\frac{(1-\rho_\nu^2)\,dx}{x^2+(r-\rho_\nu)^2} = \frac{\pi(1-\rho_\nu^2)}{|r-\rho_\nu|} \leq 8\pi.$$

Thus in either case

$$\int_E \frac{|G'(z)|}{1+|G(z)|^2}|dz|<C_1, \tag{12}$$

where C_1 is independent of r.

Consider finally the range E' where $|z-\rho_\nu e^{i\phi_\nu}|<\frac{1}{2}(1-\rho_\nu)$. It follows from c) that in this range and even for ζ in a disk centre z and radius $\frac{1}{2}(1-\rho_\nu)$, we have $|f(\zeta)|<\frac{1}{2}$, and so also $|G(\zeta)|<\frac{1}{2}$, so that

$$|G'(z)|<\frac{2}{(1-\rho_\nu)}.$$

Thus if r is sufficiently near one, we have

$$\int_{E'}\frac{|G'(re^{i\theta})|}{1+|G(re^{i\theta})|^2}d\theta<\int_{E'}|G'(re^{i\theta})|\,d\theta<\frac{2}{1-\rho_\nu}2(1-\rho_\nu)=4. \tag{13}$$

On combining (12) and (13) we have Lemma 3 for $G(z)=f(z)\Pi(z)$.

It remains to consider the case where

$$G(z)=\frac{f(z)}{\Pi(z)}=\frac{F(z)}{a(z)},$$

and $F(z)=f(z)/\Pi_1(z)$. We consider now the two ranges E, where $|z-z_\nu|>\frac{1}{3}(1-|z_\nu|)$ and E', where $|z-z_\nu|<\frac{1}{3}(1-|z_\nu|)$. Since

$$\frac{|G'|}{1+|G|^2}\leq\frac{|F'|\left|\frac{1}{a}\right|}{1+\left|\frac{F}{a}\right|^2}+\frac{\left|\frac{a'}{a}\right|\left|\frac{F}{a}\right|}{1+\left|\frac{F}{a}\right|^2},$$

we prove just as before that (12) holds.

However in E' our argument is different. We note that $\frac{F(z)}{a(z)}$ has a pole of residue $r_0=F(z_\nu)(1-|z_\nu|^2)$ at $z=z_\nu$, and write

$$G(z)=\frac{F(z)}{a(z)}=\frac{r_0}{z-z_\nu}+G_1(z)=c(z)+G_1(z) \text{ say.}$$

Thus

$$G^*(re^{i\phi})=\frac{|G'(re^{i\phi})|}{1+|G|^2}\leq\frac{|G_1'(re^{i\phi})|}{1+|G|^2}+\frac{|c'(re^{i\phi})|}{1+|G|^2}$$

$$\leq|G_1'(re^{i\phi})|+\frac{|c'(re^{i\phi})|}{1+|G|^2}. \tag{14}$$

In view of c) and (2) $|F(z)|$, $|G(z)|$ and so $|G_1(z)|$ are small for $|z-z_\nu|=\frac{1}{2}(1-|z_\nu|)$ when ν is large and since $G_1(z)$ is regular in $|z-z_\nu|<\frac{1}{2}(1-|z_\nu|)$, we deduce that for large ν we have on E',

$$|G_1(z)|<1,\ |G_1'(z)|<(1-|z_\nu|)^{-1}.$$

Since the length of E' is at most $(1-|z_\nu|)$ for large ν we deduce that

$$\int_{E'}|G_1'(re^{i\phi})|d\phi<1 \tag{15}$$

for large ν.

To estimate the other term in (14) we let E'' be the part of E' where $|c(z)|>2$.

Then in E'' we have

$$|G(z)|\geq|c(z)|-\frac{1}{2}|c(z)|=\frac{1}{2}|c(z)|,$$

$$\frac{|c'(z)|}{1+|G|^2} \leq \frac{4|c'|}{|c|^2} = 4/|r_0|.$$

Since the length of E'' is at most $2|r_0|$ for large ν, we deduce that

$$\int_{E''} \frac{|c'(re^{i\phi})|}{1+|G|^2} d\phi \leq 8. \tag{16}$$

Finally if E''' is the part of E' outside E'', then

$$\int_{E'''} \frac{|c'(re^{i\phi})|}{1+|G|^2} d\phi \leq \int_{E'''} |c'(re^{i\phi})| d\phi = \int_{E'''} \frac{|r_0| d\phi}{|z-z_\nu|^2}. \tag{17}$$

We have in E''' $z = re^{i\phi}$, $z_\nu = \rho_\nu e^{i\phi_\nu}$, where

$$|z-z_\nu|^2 = (r-\rho_\nu)^2 + 4r\rho_\nu \sin^2 \frac{(\phi-\phi_\nu)}{2} > \frac{1}{4}|r_0|^2.$$

Suppose first that $|r-\rho_\nu| > \frac{1}{4}|r_0|$. Then since $r \geq \frac{1}{2}$, $\rho_\nu \geq \frac{1}{2}$ we have

$$\int_{E'''} \frac{|r_0| d\phi}{|z-z_\nu|^2} \leq \int_{-\infty}^{+\infty} \frac{\pi^2 |r_0| d\phi}{(r-\rho_\nu)^2 + (\phi-\phi_\nu)^2}$$
$$= \frac{\pi^3 |r_0|}{|r-\rho_\nu|} < 4\pi^3. \tag{18}$$

If on the other hand $|r-\rho_\nu| \leq \frac{1}{4}|r_0|$, then we must have in E''' $4r\rho_\nu \sin^2 \frac{(\phi-\phi_\nu)}{2} \geq \frac{1}{8}|r_0|^2$, so that

$$|\phi-\phi_\nu| \geq \frac{|r_0|}{4}.$$

Thus in this case

$$\int_{E'''} \frac{|r_0| d\phi}{|z-z_\nu|^2} \leq 2\int_{|r_0|/4}^{\infty} \frac{\pi^2 |r_0| dx}{x^2} = 2\pi^2 |r_0| \cdot \frac{4}{|r_0|} = 8\pi^2,$$

so that (18) still holds. On combining (14) to (18) we deduce

$$\int_{E'} G^*(re^{i\phi}) d\phi < A_7,$$

if r is sufficiently near one. On combining this with (12) we deduce Lemma 3.

6. Proof of Theorems 1 and 2. By choosing the function $f(z)$ of Lemma 1 and for F the corresponding countable set we see that Theorem 3 yields a non-zero Tsuji function $f(z)/\Pi(z)$ having every point of F as a Julia point.

Then the function $\Pi(z)/f(z)$ satisfies the conclusions of Theorem 1. Also $\Pi(z)f(z)$ satisfies the conclusions of Theorem 2.

In fact to see this we have only to show that $\Pi(z)f(z)$ remains continuous on C. This is obvious at all points of C which are not limits of zeros of $\Pi(z)$, since $\Pi(z)$ remains continuous at such points. The only other points of C are the points where $f(z)$ vanishes continuously and so $\Pi(z)f(z)$ vanishes and so remains continuous also at these points, since $|\Pi(z)|<1$.

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Bibliography

[1] E. F. Collingwood and G. Piranian, Tsuji functions with segments of Julia. Math. Zeit., **84** (1964), 246-253.
[2] W. K. Hayman, The boundary behaviour of Tsuji functions. Michigan Math. J. to appear.
[3] M. Tsuji, A theorem on the boundary behaviour of a meromorphic function in $|z|<1$. Comment. Math. Univ. St. Paul., 8 (1960), 53-55.

Imperial College,
London S.W. 7.
England.