PL-SUBMANIFOLDS AND HOMOLOGY CLASSES OF A *PL*-MANIFOLD[*)]

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Dedicated to Professor K. Noshiro for his 60th birthday

This paper is devoted to the problem of the realisation of homology classes of a *PL*-manifold by *PL*-submanifolds.

The present study is founded on the consideration of Thom complexes $M(PL_k)$, $M(SPL_k)$ for *PL*-microbundles which is defined by R. Williamson [5]. We shall apply Thom's method [4] to *PL*-manifolds.

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1. Generalities

Following Milnor [3] and Williamson [5] we shall work in the category of locally finite simplicial complexes and piecewise linear maps (briefly, *PL*-maps).

A mapping $F: K \rightarrow L$ between locally finite simplicial complexes is *PL-map*, if there exists a rectilinear subdivision K' of K so that f maps each simplex of K' linearly into a simplex of L.

Let X be a locally finite simplicial complex and Y be a closed subspace of it. Then we shall say that Y is a *PL-subspace* of X, if Y can be triangulated so that the inclusion $i: Y \rightarrow X$ is a *PL*-map. It follows that some subdivision of Y is a subcomplex of some subdivision of X (cf. Williamson [5], § 1). Given two such triangulations the identity is a *PL*-homeomorphism from one to the other.

Let V^n be a closed *PL*-manifold[1)] of dimension n. Then we shall say that W^p is a *PL-submanifold* of dimension p, if W^p is a closed *PL*-manifold of

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1) By a *PL*-manifold we shall mean a combinatorial manifold.

dimension p and a PL-subspace of V^n.

In the following we suppose that V^n is a closed PL-manifold of dimension n. Let W^p be a PL-submanifold of dimension p. The inclusion map $i: W^p \to V^n$ induces the homomorphism $i_*: H_p(W^p, Z_2) \to H_p(V^n, Z_2)$. Let $z \in H_p(V^n, Z_2)$ be the image by i_* of the fundamental class w of the PL-manifold W^p. Then we say that the homology class z is *realized* by the PL-submanifold W^p. Let V^n be oriented, and W^p be an oriented PL-submanifold of dimension p. The inclusion map $i: W^p \to V^n$ induces the homomorphism $i_*: H_p(W^p, Z) \to H_p(V^n, Z)$. Let $z \in H_p(V^n, Z)$ be the image by i_* of the fundamental class w of the oriented PL-manifold W^p. Then we say that the homology class z is *realized* by the oriented PL-submanifold W^p.

Here the following questions are considered : Let a homology class z mod 2 of the PL-manifold V^n be given. Is it realisable by a PL-submanifold?; Let an integral homology class z of the oriented PL-manifold V^n be given. Is it realisable by an oriented PL-submanifold?

2. Thom complexes $M(PL_k)$, $M(SPL_k)$

We shall recall the definition of Thom complexes for PL-microbundles (cf. Williamson [5], §4). Let ξ be a PL-microbundle:

$$\xi : B(\xi) \xrightarrow{i_\xi} E(\xi) \xrightarrow{j_\xi} B(\xi).$$

Let E be an open neighborhood of $i_\xi(B(\xi))$ in $E(\xi)$ such that $E(\xi) - E$ is a PL-subspace of $E(\xi)$. If $E(\xi) - E$ is a strong deformation retract of $E(\xi) - i_\xi(B(\xi))$, we shall say that E is an *admissible neighborhood in Williamson's sense.* Then we call the quotient space formed by collapsing $E(\xi) - E$ to a point $*$ a *Thom complex* of ξ (although it may not be locally finite at $*$) and denote it by $T(\xi)$ or $T_E(\xi)$. We point out that $T_E(\xi) - i_\xi(B(\xi))$ is contractible.

Let U be any neighborhood of $i_\xi(B(\xi))$ in $E(\xi)$. Then there exists an admissible neighborhood E in Williamson's sense such that E is open and $\overline{E} \subset U$. Moreover, the homotopy type of $T_E(\xi)$ does not depend on the particular choice of an admissible neighborhood E (cf. Williamson [5], §4).

We know that for each n there exists a universal PL-microbundle for fibre dimension n

$$\gamma(PL_n) : B(PL_n) \xrightarrow{i_n} E(PL_n) \xrightarrow{j_n} B(PL_n)$$

and a universal orientable *PL*-microbundle for fibre dimension n

$$\Upsilon(SPL_n) : B(SPL_n) \xrightarrow{i_n} E(SPL_n) \xrightarrow{j_n} B(SPL_n)$$

(cf. Milnor [3], § 5, Williamson [5], § 2). For $T(\Upsilon(PL_n))$, $T(\Upsilon(SPL_n))$ we write $M(PL_k)$, $M(SPL_k)$ respectively.

Let ξ be a *PL*-microbundle of dimension n. A *PL*-microbundle ξ is considered as a topological microbundle. Therefore, by Kister [1], there exists an admissible neighborhood $E_1(\xi)$ of $i_\xi(B(\xi))$ in Kister's sense such that $\{E_1(\xi), j_\xi|E_1(\xi), B(\xi)\}$ is a fibre bundle with fibre R^n and structure group $H_0(n)$. We have the Thom isomorphism

$$\varphi_\xi^* : H^0(B(\xi), Z_2) \longrightarrow H^n(E_1(\xi), E_1(\xi) - i_\xi(B(\xi)) ; Z_2),$$

(cf. Milnor [2]). As is remarked above, there exists an admissible neighborhood E of $i_\xi(B(\xi))$ in Williamson's sense such that E is open and $\bar{E} \subset E_1(\xi)$. Now we consider n-th cohomology group of Thom complex $T_E(\xi)$:

$$\begin{aligned} H^n(T_E(\xi), Z_2) &= H^n(E(\xi)/E(\xi) - E;\ Z_2) \\ &\cong H^n(E(\xi), E(\xi) - E;\ Z_2) \\ &\cong H^n(E(\xi), E(\xi) - i_\xi(B(\xi)) ;\ Z_2) \\ &\cong H^n(E_1(\xi), E_1(\xi) - i_\xi(B(\xi)) ;\ Z_2), \end{aligned}$$

where the last isomorphism is the excision. We shall denote this isomorphism by ι_E. Composing two isomorphisms φ_ξ^* and ι_E, we have

$$\iota_E \circ \varphi_\xi^* : H^0(B(\xi), Z_2) \longrightarrow H^n(T_E(\xi), Z_2).$$

Let ω denote the unit of the cohomology ring $H^*(B(\xi), Z_2)$. The cohomology class $U_\xi \in H^n(T_E(\xi), Z_2)$ defined by

$$U_\xi = \iota_E \circ \varphi_\xi^*(\omega)$$

will be called the *fundamental class* of Thom complex $T_E(\xi)$. In the case where ξ is orientable, we have the Thom isomorphism

$$\varphi_\xi^* : H^0(B(\xi), Z) \longrightarrow H^n(E_1(\xi), E_1(\xi) - i_\xi(B(\xi)) ;\ Z)$$

and the *fundamental class* $U_\xi \in H^n(T_E(\xi), Z)$, in quite an analogous way (cf. Milnor [2]).

We shall denote by U_n the fundamental classes of Thom complexes $M(PL_n)$ and $M(SPL_n)$, and φ_n^* the Thom isomorphisms of universal *PL*-microbundles $\Upsilon(PL_n)$ and $\Upsilon(SPL_n)$.

3. Fundamental theorem

DEFINITION. We say that a cohomology class $u \in H^k(A, Z_2)$ of a space A is *PL_k-realisable,* if there exists a mapping $f : A \to M(PL_k)$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class U_k of the Thom complex $M(PL_k)$. We say that a cohomology class $u \in H^k(A, Z)$ of a space A is *SPL_k-realisable,* if there exists a mapping $f : A \to M(SPL_k)$ such that u is the image, for the homomorphism f^* induced by f, of the fundamental class U_k of the Thom complex $M(SPL_k)$.

Then we have the following

THEOREM. *Let V^n be a closed PL-manifold of dimension n.*

a) In order that a homology class $z \in H_{n-k}(V^n, Z_2)$ $k>0$, can be realized by a PL-submanifold W^{n-k} which has a normal PL-microbundle in V^n, it is necessary and sufficient that the cohomology class $u \in H^k(V^n, Z_2)$, corresponding to z by the Poincaré duality, is PL_k-realisable.

b) Let V^n be oriented. In order that a homology class $z \in H_{n-k}(V^n, Z)$, $k>0$, can be realized by an oriented PL-submanifold W^{n-k} which has an orientable normal PL-microbundle in V^n, it is necessary and sufficient that the cohomology class $u \in H^k(V^n, Z)$, corresponding to z by the Poincaré duality, is SPL_k-realisable.

Proof. We shall prove the case a) of the theorem. The case b) can be proved quite in parallel with the case a).

i) Necessity. Suppose that there exists a PL-submanifold W^{n-k} in V^n which have a normal PL-microbundle of dimension k

$$\nu : B(\nu) \xrightarrow{i_\nu} E(\nu) \xrightarrow{j_\nu} B(\nu) = W^{n-k}.$$

The normal PL-microbundle ν is induced from the universal PL-microbundle

$$\gamma(PL_k) : B(PL_k) \xrightarrow{i_k} E(PL_k) \xrightarrow{j_k} B(PL_k)$$

by a mapping $f : W^{n-k} \to B(PL_k)$. Therefore, there exists a mapping $\bar{f} : E(\nu) \to E(PL_k)$ such that the following diagram

$$\begin{array}{ccc} E(\nu) & \xrightarrow{\bar{f}} & E(PL_k) \\ {\scriptstyle j_\nu}\downarrow & & \downarrow{\scriptstyle j_k} \\ W^{n-k} = B(\nu) & \xrightarrow{f} & B(PL_k) \end{array}$$

is commutative. The universal *PL*-microbundle $\gamma(PL_k)$ admits an admissible fibre bundle

$$\gamma_1(PL_k) = \{E_1(PL_k),\ j_k | E_1(PL_k),\ B(PL_k),\ R^k,\ H_0(k)\}$$

in Kister's sense (cf. Kister [1]). Moreover, by the uniqueness of the admissible fibre bundle (cf. Kister [1]), the induced bundle $f^*\gamma_1(PL_k)$ is an admissible fibre bundle

$$\{E_1(\nu),\ j_\nu | E_1(\nu),\ B(\nu),\ R^k,\ H_0(k)\}$$

of the normal *PL*-microbundle ν. Since $\bar{f}$ maps $E(\nu) - i_\nu(B(\nu))$ into $E(PL_k) - i_k(B(PL_k))$, the following diagram

$$\begin{array}{ccc}
H^k(E(\nu),\ E(\nu) - i_\nu(B(\nu))\ ;\ Z_2) & \xleftarrow{\bar{f}^*} & H^k(E(PL_k),\ E(PL_k) - i_k(B(PL_k));\ Z_2) \\
\alpha\uparrow & & \uparrow\alpha \\
H^k(E_1(\nu),\ E_1(\nu) - i_\nu(B(\nu))\ ;\ Z_2) & \xleftarrow{(\bar{f}|E_1(\nu))^*} & H^k(E_1(\gamma_k),\ E_1(\gamma_k) - i_k(B(PL_k))\ ;\ Z_2) \\
\varphi_\nu^*\uparrow & & \uparrow\varphi_k^* \\
H^0(B(\nu),\ Z_2) & \xleftarrow{f^*} & H^0(B(PL_k),\ Z_2)
\end{array}$$

is commutative, where α are the excision isomorphisms (cf. Milnor [2]), and $E_1(\gamma_k)$ denotes $E_1(PL_k)$.

Let E_k be an admissible neighborhood of $i_k(B(PL_k))$ in $E(PL_k)$. Let us denote by $g : E(\nu) \to M(PL_k)$ the composite map, $p \circ \bar{f}$ of $\bar{f} : E(\nu) \to E(PL_k)$ and the natural projection $p : E(PL_k) \to E(PL_k)/E(PL_k) - E_k = M(PL_k)$. Now we can define mapping $\bar{g} : V^n \to M(PL_k)$ such that $\bar{g}|E(\nu) = g$; it is sufficient to map $V^n - E(\nu)$ to the point $*$. Then we have the following commutative diagram:

$$\begin{array}{ccc}
H^k(V^n,\ Z_2) & \nwarrow^{\bar{g}^*} & \\
j^*\uparrow & & \\
H^k(V^n,\ V^n - W^{n-k};\ Z_2) & & H^k(M(PL_k),\ Z_2) \\
\beta\uparrow & & \uparrow\iota_k \\
H^k(E(\nu),\ E(\nu) - i_\nu(B(\nu));\ Z_2) & \xleftarrow{\bar{f}^*} & H^k(E(PL_k),\ E(PL_k) - i_k(B(PL_k))\ ;\ Z_2),
\end{array}$$

where j^* is the relativisation and β is the excision isomorphism.

Then we have

$$\begin{aligned}
\bar{g}^*(U_k) &= \bar{g}^* \circ \iota_k \circ \alpha \circ \varphi_k^*(\omega) \\
&= j^* \circ \beta \circ \alpha \circ \varphi_\nu^*(\omega) \\
&= \psi(i_W)(\omega),
\end{aligned}$$

where $\psi(i_W)$ is the Gysin homomorphism of the inclusion map $i_W : W^{n-k} \to V^n$. Therefore,

$$\begin{aligned} \bar{g}^*(U_k) &= D_V \circ (i_W)_* \circ D_W(\omega) \\ &= D_V \circ (i_W)_*(w) \\ &= D_V(z) = u, \end{aligned}$$

where D_V and D_W are the Poincaré dualities of V^n and W^{n-k}, respectively.

ii) *Sufficiency.* Suppose that there exists a mapping f of V^n into $M(PL_k)$ such that $f^*(U_k) = u$. The Thom complex $M(PL_k)$, deprived the point $*$, is considered as a locally finite simplicial complex, and the PL-subspace $B(PL_k)$ has the normal PL-microbundle $\Upsilon(PL_k)$ in $M(PL_k) - *$. By the theorem 3.3.1. in Williamson [5], we have a mapping f_1, homotopic to f, t-regular for $(\nu, \Upsilon(PL_k))$, where ν is a normal PL-microbundle of $f_1^{-1}(B(PL_k))$ in V^n. However, by the lemma 4.2. in Williamson [5], $f_1^{-1}(B(PL_k))$ is a PL-submanifold W^{n-k} in V^n. Moreover, by the definition of t-regularity, the induced PL-microbundle $f_1^*\Upsilon(PL_k)$ is isomorphic to ν. We know $f_1^*(U_k) = f^*(U_k) = u$. Then, as in the case i), we can see that the PL-submanifold W^{n-k} realizes the homology class z, corresponding to u by the Poincaré duality.

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