

EQUICONTINUITY ON HARMONIC SPACES

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Dedicated to Professor KIYOSHI NOSHIRO on his 60th birthday

G. Mokobodzki proved [5] that on any harmonic space with countable basis satisfying the axioms 1, 2, T_+ , K_b [2] [1] any equally bounded set of harmonic functions is equicontinuous. P. Loeb and B. Walsh showed [4] that the same property holds on a harmonic space without countable basis, if Brelot's axiom 3 is fulfilled. The aim of this paper is to generalize these results to a harmonic space X satisfying only the axioms 1, 2_0 , K_1 , [2] [1] where 2_0 is a weakened form of axiom 2. As a corollary we get: if any point of X possesses two open neighbourhoods U , V such that the set of harmonic functions on U separates the points of $U \cap V$, then X has locally a countable basis.

Throughout this paper Bourbaki's notations and terminology will be used.

1. Family of measures

Throughout this paragraph we shall denote by X , Y two compact spaces and by $(\omega_x)_{x \in X}$ a family of (nonnegative) measures on Y such that for any equally bounded upper directed family $(f_i)_{i \in I}$ of Borel functions on Y the function on X

$$x \rightarrow \sup_{i \in I} \int f_i d\omega_x$$

is continuous. We denote for any bounded Borel function f on Y by f' the function on X

$$x \rightarrow \int f d\omega_x.$$

It is a continuous function. We denote further for any measure μ on X by μ' the measure on Y

$$f \rightarrow \int f' d\mu \quad (f \in \mathcal{X}(Y)).$$

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For any bounded Borel function f on Y and any measure μ on X we have

$$\int f' d\mu = \int f d\mu'.$$

LEMMA 1. *There exists a countable set $A \subset X$ such that for any nonnegative bounded Borel function f on Y f' vanishes on X if it vanishes on A .*

Let us denote by \mathfrak{M} the set of finite subsets of X and for any $M \in \mathfrak{M}$ by \mathcal{G}_M the set of Borel functions f on Y , $0 \leq f \leq 1$, such that f' vanishes on M . For any $f, g \in \mathcal{G}_M$ and any $x \in M$ we have

$$\int \sup(f, g) d\omega_x \leq \int f d\omega_x + \int g d\omega_x = 0.$$

Hence $\sup(f, g) \in \mathcal{G}_M$ and \mathcal{G}_M is upper directed. It follows that the least upper bound u_M of the family $(f')_{f \in \mathcal{G}_M}$ is continuous. Since it vanishes on M we deduce

$$\inf_{M \in \mathfrak{M}} u_M(x) = 0$$

for any $x \in X$. Hence there exists, by Dini's theorem, an increasing sequence (M_n) in \mathfrak{M} such that

$$\lim_{n \rightarrow \infty} u_{M_n}(x) = 0$$

for any $x \in X$. We set

$$A = \bigcup_{n=1}^{\infty} M_n.$$

Let f be a Borel function on Y , $0 \leq f \leq 1$, such that f' vanishes on A . f belongs to \mathcal{G}_M for any M . Hence for any $x \in X$ and any $M \in \mathfrak{M}$

$$f'(x) \leq u_M(x).$$

It follows that f' vanishes on X .

COROLLARY 1. *There exists an atomic measure μ on X such that for any measure ν on X ν' is absolutely continuous with respect to μ' .*

Let μ be an atomic measure on X such that

$$\mu(\{x\}) > 0$$

for any $x \in A$. Let ν be an arbitrary measure on X and let f be a nonnegative

bounded Borel function on Y such that

$$\int f d\mu' = 0.$$

Then f' vanishes on A and therefore on X and we get

$$\int f d\nu' = \int f' d\nu = 0.$$

THEOREM 1. *Let \mathcal{C} be the Banach space of real continuous functions on X with the norm of uniform convergence, let \mathcal{F} be the set of Borel functions f on Y such that $|f| \leq 1$ and let \mathcal{F}' be the set $\{f' | f \in \mathcal{F}\}$. \mathcal{F}' is compact with respect to the weak topology of \mathcal{C} and any sequence in \mathcal{F}' contains a convergent subsequence (also with respect to the weak topology of \mathcal{C}).*

Let μ be a measure on X such that for any measure ν on X ν' is absolutely continuous with respect to μ' . Let for any measure ν on X g_ν be a function of $\mathcal{L}^1(\mu')$ such that $\nu' = g_\nu \cdot \mu'$. The map φ of $\mathcal{L}^\infty(\mu')$ into \mathcal{C} defined by

$$\varphi(f)(x) = \int f g_{\omega_x} d\mu' \quad (f \in \mathcal{L}^\infty(\mu'))$$

is continuous with respect to the weak topology of \mathcal{C} and the topology $\sigma(\mathcal{L}^\infty(\mu'), \mathcal{L}^1(\mu'))$ of $\mathcal{L}^\infty(\mu')$ (i.e. the least fine topology on $\mathcal{L}^\infty(\mu')$ for which all linear forms $g \rightarrow \int g h d\mu'$ ($h \in \mathcal{L}^1(\mu')$) are continuous). Since $\mathcal{L}^\infty(\mu')$ is the dual of $\mathcal{L}^1(\mu')$ it follows that \mathcal{F} is quasi-compact with respect to the topology $\sigma(\mathcal{L}^\infty(\mu'), \mathcal{L}^1(\mu'))$ of $\mathcal{L}^\infty(\mu')$ (we used here an idea of Mokobodzki [4]). Hence $\mathcal{F}' = \varphi(\mathcal{F})$ is compact with respect to the weak topology of \mathcal{C} . The last assertion follows from Šmulian Dieudonné-Schwartz theorem ([3] page 314) (we followed in this point P. Loeb and B. Walsh [4]).

THEOREM 2. *Let X, Y, Z be compact spaces and $(\omega_x)_{x \in X}$ (resp. $(\rho_y)_{y \in Y}$) be a family of measures on Y (resp. on Z) such that for any equally bounded upper directed family $(f_i)_{i \in I}$ of Borel functions on Y (resp. on Z) the function on X (resp. on Y)*

$$x \rightarrow \sup_{i \in I} \int f_i d\omega_x \quad (\text{resp. } y \rightarrow \sup_{i \in I} \int f_i d\rho_y)$$

is continuous. Let \mathcal{G} be the set of Borel functions g on Z such that $|g| \leq 1$. The set of functions on X

$$\{x \rightarrow \int (\int g d\rho_y) d\omega_x(y) | g \in \mathcal{G}\}$$

is compact with respect to the topology of uniform convergence on X .

Let (g_n) be a sequence of functions of \mathcal{S} . By the preceding theorem there exists a subsequence (g_{n_k}) and a function $g \in \mathcal{S}$ such that

$$\lim_{k \rightarrow \infty} \int g_{n_k} d\rho_y = \int g d\rho_y$$

for any $y \in Y$. Let us denote for any natural number m by $f_{1,m}$ (resp. $f_{2,m}$) the function on Y

$$y \rightarrow \inf_{k \geq m} \int g_{n_k} d\rho_y \quad (\text{resp. } y \rightarrow \sup_{k \geq m} \int g_{n_k} d\rho_y).$$

The sequence $(f_{1,m})$ (resp. $f_{2,m}$) is nondecreasing (resp. nonincreasing) and converges to the function

$$y \rightarrow \int g d\rho_y.$$

Hence $(f'_{1,m})_m$ (resp. $f'_{2,m})_m$ is a nondecreasing (resp. nonincreasing) sequence of continuous functions on X converging to the continuous function

$$x \rightarrow \int \left(\int g d\rho_y \right) d\omega_x(y).$$

By Dini's theorem the convergence is uniform on X . Hence the sequence

$$(x \rightarrow \int \left(\int g_{n_k} d\rho_y \right) d\omega_x(y))_k$$

converges uniformly on X to the function

$$x \rightarrow \int \left(\int g d\rho_y \right) d\omega_x(y).$$

2. Harmonic spaces

Let X be a locally compact space and \mathcal{H} be a sheaf of real vector spaces of real continuous functions on X . More exactly this means that for any open set U of X $\mathcal{H}(U)$ is a set of real continuous functions on U , called **harmonic functions (on U)** such that: *a*) if $u, v \in \mathcal{H}(U)$ and if α, β are real numbers then $\alpha u + \beta v \in \mathcal{H}(U)$; *b*) if V is an open nonempty subset of the open set U , then the restriction to V of any harmonic function on U is a harmonic function on V ; *c*) if $(U_i)_{i \in I}$ is a family of open nonempty sets of X and if u is a real function on $\bigcup_{i \in I} U_i$ such that its restriction to any U_i is a harmonic function on U_i then u is a harmonic function on $\bigcup_{i \in I} U_i$. Let V be an open relatively compact

set of X . A family $(\omega_x)_{x \in V}$ of measures on the boundary ∂V of V is called a **family of harmonic measures on V** if:

a) for any bounded Borel function f on ∂V the function on V

$$x \rightarrow \int f d\omega_x$$

is harmonic;

b) if u is a harmonic function on U , $U \supset \bar{V}$, then for any $x \in V$

$$u(x) = \int u d\omega_x.$$

An open relatively compact set V of X for which there exists a family of harmonic measures on V will be called **pseudoregular**.

We suppose the following axioms.

AXIOM 2₀. *The set of pseudoregular sets is a basis of X .*

AXIOM K₁. *For any open set U of X the least upper bound of any equally bounded upper directed family of harmonic functions on U is harmonic on U .*

LEMMA 2. *Let V be an open relatively compact set of X and let $(\omega_x)_{x \in V}$ be a family of harmonic measures on V . If $(f_i)_{i \in I}$ is an equally bounded upper directed family of Borel functions on ∂V then the function on V*

$$x \rightarrow \sup_{i \in I} \int f_i d\omega_x$$

is continuous.

The lemma follows immediately from axiom K_1 .

THEOREM 3. *The set \mathcal{U} of harmonic functions u on X such that $|u| \leq 1$ is compact with respect to the uniform convergence on compact sets of X .*

Let U, V be pseudoregular sets, $\bar{U} \subset V$, and K be a compact subset of U . Let $(\omega_x)_{x \in U}$ (resp. $(\rho_y)_{y \in V}$) be a family of harmonic measures on U (resp. V). For any $y \in \partial U$, any $x \in K$ and any $u \in \mathcal{U}$ we have

$$u(y) = \int u d\rho_y, \quad u(x) = \int \left(\int u d\rho_y \right) d\omega_x(y).$$

Let \mathfrak{u} be an ultrafilter on \mathcal{U} . By lemma 2 and theorem 2 there exists a bounded Borel function f on ∂V such that u converges along \mathfrak{u} uniformly on

K to the function

$$x \rightarrow \int (\int f d\rho_y) d\omega_x(y).$$

This function being harmonic on U it follows that the function

$$x \rightarrow \lim_{u, \mathfrak{U}} u(x)$$

belongs to \mathcal{H} . Further it follows from axiom 2₀ that \mathfrak{U} converges to it uniformly on any compact set of X .

COROLLARY 2. *Any equally bounded set of harmonic functions on X is equicontinuous.*

This is a consequence of Ascoli's theorem.

COROLLARY 3. *If any point of X possesses two open neighbourhoods U, V such that the set of harmonic functions on U separates the points of $U \cap V$ then X has locally a countable basis.*

Let \mathcal{H} be the set of harmonic functions u on U such that $|u| \leq 1$. We may suppose that \mathcal{H} separates the points of $U \cap V$. Let K be a compact subset of $U \cap V$. For any $x, y \in K$ we set

$$d(x, y) = \sup_{u \in \mathcal{H}} |u(x) - u(y)|.$$

Since \mathcal{H} separates the points of K d is a distance on K . Since \mathcal{H} is equicontinuous the topology of K is finer than the topology associated to d . It follows that these topologies coincide and K possesses a countable basis.

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