LIFTING PROJECTIVES

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In memory of TADASI NAKAYAMA

1. Introduction and statement of result

Let R be a ring with radical $\mathfrak{N}$ (all rings have a unit element, all modules are unital). Often, one wishes to lift modules modulo $\mathfrak{N}$, that is, to a given, say, left $R/\mathfrak{N}$-module U find a left R-module E with the property that $E/\mathfrak{N}E \simeq U$. This is of course not always possible. Here I prove, roughly, that if a finitely generated projective U can be lifted at all, it can be lifted to a projective. Or rather, if U can be lifted to an E satisfying a certain mild condition, then E is projective (Lemma).

It is convenient to introduce the notion of "cover". In any category, an epimorphism $f : A \to B$ is called a cover if any morphism $g : X \to A$ such that fg is an epimorphism, must needs be an epimorphism. Sloppily, we also say that A is a cover of B. In the category of R-modules, Nakayama's Lemma asserts that f is a cover if A is finitely generated and $\ker f \subset \mathfrak{N}A$. Repeated application of this simple remark will prove the result, which I dedicate to the memory of T. Nakayama.

LEMMA. *Let R be a left noetherian ring, $\mathfrak{A}$ a two-sided ideal contained in its radical. Let U be a finitely generated projective $R/\mathfrak{A}$-module. Suppose the left R-module E is an R-cover of U and that* $\mathrm{Tor}_1^R(R/\mathfrak{A}, E) = 0$. *Then E, uniquely determined up to isomorphism, is finitely generated projective. Moreover, $E/\mathfrak{A}E \simeq U$.*

This fact is useful in the theory of homological dimension. For commutative rings, it is easily derived from the "critère de platitude" [4, Ch. III, Th. 1, p. 98], bearing in mind that finitely presented flat modules are projective. Even here, however, the approach using covers is more direct. A variant of the lemma was proved in [8, Lemma 1.13, p. 6] with a different application in view. Since theses are seldom produced in order to be read, it seems worth

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while making the result more widely available.

2. Proof

First we show that taking a cover of a projective U amounts to lifting U.

LEMMA 0. *Let $\mathfrak{A}$ be a two-sided ideal in the ring R and U a finitely generated left $R/\mathfrak{A}$-module. If E is an R-cover of U, then E is finitely generated. If, in addition, U is $R/\mathfrak{A}$-projective, then $E/\mathfrak{A}E \simeq U$.*

Proof. For any R-module X, write $\overline{X} = X/\mathfrak{A}X$ and t_X for the residue class map $X \to \overline{X}$, and for any R-map $f : X \to Y$ write $\overline{f} : \overline{X} \to \overline{Y}$ for the corresponding $R/\mathfrak{A} = \overline{R}$-map.

With this notation fixed, let $\overline{f}$ be an $\overline{R}$-epimorphism from a finitely generated free $\overline{R}$-module $\overline{L}$ onto U. Raise to a free R-module L on the same number of generators. If $s : E \to X$ is our R-cover, let $f : L \to E$ be such that $sf = \overline{f}t_L$. The latter map being surjective, the cover property implies that f is too, which proves E is finitely generated.

To show that the surjection $\overline{s} : \overline{E} \to \overline{U} = U$ is injective, we need our assumption that U is $\overline{R}$-projective and hence may be identified with a direct summand of $\overline{E}$. Consider the submodule $F = t_E^{-1}(U)$ of E and observe that $\overline{s}t_E(F) = s(F) = U$. Since s is a cover, $F = E$ and $\overline{E} \simeq U$.

Proof of Lemma. From the above, we know that E is finitely generated and that $E/\mathfrak{A}E = \overline{E} \simeq U$. Let f be an epimorphism of a finitely generated free module L (projective would do as well) onto E and put $\ker f = g : D \to L$. Since $\mathrm{Tor}_1^R(\overline{R}, E) = 0$ the bottom row in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & D & \xrightarrow{g} & L & \xrightarrow{f} & E & \longrightarrow & 0 \\ & & \downarrow t_D & & \downarrow t_L & & \downarrow t_E & & \\ 0 & \longrightarrow & \overline{D} & \xrightarrow{\overline{g}} & \overline{L} & \xrightarrow{\overline{f}} & \overline{E} & \longrightarrow & 0 \end{array}$$

is also exact. Since $\overline{E}$ is $\overline{R}$-projective, this row splits and we have a map $\overline{h}$: $\overline{L} \to \overline{D}$ such that $\overline{h}\overline{g} = 1_{\overline{D}}$. Use the projectivity of L to find a map $h : L \to D$ such that $t_D h = \overline{h} t_L$.

We wish to prove that hg is an automorphism of D, so that the top row splits too, making E a direct summand of L and hence projective. Our commutative diagram shows that $t_D hg = \overline{h} t_L g = \overline{h}\overline{g} t_D = t_D$. The ring R being noetherian, D is finitely generated, so t_D is a cover and hg surjective. Then hg is an epimorphism of the noetherian module D onto itself, therefore it is an automorphism [3, Lemma 3, p. 23]. Thus E is a projective cover of $\overline{E}$, and as such uniquely determined up to isomorphism [2, Lemma 2.3, p. 472]. This finishes the proof.

3. Applications

The following device answers a question of Kaplansky, who uses it in homological dimension theory [7].

COROLLARY 1. *Let R be a left noetherian ring, E a finitely generated left R-module. Let x be an element both in the centre and in the radical of R. Assume that x is a non-zero divisor on E and that E/xE is projective over the residue class ring R/xR. Then E is projective.*

Proof. The residue class map $E \to E/xE$ is a cover, and the injectivity of $x : E \to E$ is easily seen to imply $\mathrm{Tor}_1^R(R/xR, E) = 0$, so that the Lemma applies.

Let us define, as I believe one should in the non-commutative case, a semilocal ring as a ring which modulo its radical becomes an Artin ring. The Lemma then yields a generalization of a fact which is standard fare for commutative noetherian local rings [4, Ch. II, Cor. 2, p. 107] and is also known for semi-primary rings [1, Prop. 7, p. 71] and semi-perfect rings [5, Th. 11, p. 333].

COROLLARY 2. *Let R be a left noetherian semilocal ring with radical $\mathfrak{N}$. Then for a finitely generated left module E the following conditions are equivalent:*

1. *E is projective.*
2. *E is flat.*
3. $\mathrm{Tor}_1^R(R/\mathfrak{N}, E) = 0$.

Proof. 1. implies 2. implies 3. is true for any ring and any module. 3. implies 1. follows from the Lemma since every module, in particular $E/\mathfrak{N}E$, is projective over the semisimple Artin ring $R/\mathfrak{N}$.

This enables one to prove various results on global dimension, replacing the residue class field of the local ring by $R/\mathfrak{N}$. It suffices to adapt the arguments in [6, Ch. 0, 17.2], cf. also [1]. As an example, I mention

PROPOSITION. *Let R be a noetherian semilocal ring with radical $\mathfrak{N}$. For* gl dim R *to be* $\leq n$, *it is necessary that* $\mathrm{Tor}_i^R(R/\mathfrak{N}, R/\mathfrak{N}) = 0$ *for* $i > n$ *and sufficient that* $\mathrm{Tor}_{n+1}^R(R/\mathfrak{N}, R/\mathfrak{N}) = 0$.

4. Denoetherizing

One may try to relax the assumption in the Lemma that R be noetherian by imposing conditions on E, $\mathfrak{A}$ and/or R. Various combinations seem reasonable. I only treat one which generalizes the previous result.

LEMMA′. *In the Lemma, we can drop the noetherianness of R if we decree that*

1. *R is the direct limit of a directed system of left noetherian rings R_i (certainly true for all commutative rings);*
2. *E is finitely presented.*

Proof. Let I be our directed set and assume every R_i is a subring of $R = \varinjlim R_i$; if not, we could replace each R_i by its canonical image in R which remains noetherian. We proceed as before, remarking that condition 2. guarantees that D is finitely generated. Again we find a map $h : L \to D$ with the property that hg is an epimorphism of D onto itself and we wish to prove that hg is an automorphism.

Let $s : D \to D$ be a surjection and suppose $s(x) = 0$ for some $x \in D$. Choose a set of generators of D over R, say d_k, $k = 1, \ldots, n$. Pick n elements $c_k \in D$ such that $s(c_k) = d_k$. Now x, the c_k and the images $s(d_k)$ can all be expressed as linear combinations of the generators d_k with coefficients from R. Since only finitely many of these appear, there is an i in the directed set I such that R_i contains them all. Let D_i be the module generated by the d_k over R_i as a subset of D. Our construction achieves that s maps D_i onto D_i. Therefore the restriction $s_i : D_i \to D_i$ is a surjection of a noetherian R_i-module, hence injective. But $x \in D_i$, so $s_i(x) = s(x) = 0$. This means $x = 0$ and we are through.

A discussion of the applications in section 3. using the modified Lemma′ is left to the gentle reader.

Remark added in proof. In the tome recently out, Grothendieck obtains that a surjection $S : D \to D$ is injective if D is finitely presented [9, Ch. IV, Prop. 8.9.3, p. 35]. Curiously enough, our naive approach proves more. I suspect that the technique developed in this note has a bearing on certain questions discussed in that treatise, e.g. [9, 11.3.10.2 and 11.3.12, pp. 138-140]. Compare [8, Lemma 1.13, p. 6].

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