

SPLITTING OF ALGEBRAS BY FUNCTION FIELDS OF ONE VARIABLE

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To the memory of TADASI NAKAYAMA

§ 1. Introduction

Let K be a field and $\mathfrak{B}(K)$ the Brauer group of K . It consists of the similarity classes of finite central simple algebras over K .¹⁾ For any field extension F/K there is a natural mapping $\mathfrak{B}(K) \rightarrow \mathfrak{B}(F)$ which is obtained by assigning to each central simple algebra A/K the tensor product $A \otimes_K F$ which is a central simple algebra over F . The kernel of this map is the relative Brauer group $\mathfrak{B}(F/K)$, consisting of those $A \in \mathfrak{B}(K)$ which are split by F .

If F/K is finite algebraic, the investigation of $\mathfrak{B}(F/K)$ is part of the general theory of central simple algebras. In particular, if the ground field K is a number field or a local number field,²⁾ the relative Brauer group $\mathfrak{B}(F/K)$ can then explicitly be determined, using class field theory.

In this paper, we propose to investigate $\mathfrak{B}(F/K)$ in the case where F/K is a function field of one variable.³⁾ Our results will give a complete description of $\mathfrak{B}(F/K)$ if K is a local number field.

For any transcendental field extension F/K , let \mathfrak{p} be a place of F/K such that the image field $F\mathfrak{p}$ is algebraic over K .⁴⁾ Any central simple algebra A/K which is split by F is also split by $F\mathfrak{p}$, as we have shown in an earlier

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¹⁾ For the general theory of central simple algebras see e.g. Deuring [6], chap. IV and V, or Artin-Nesbitt-Thrall [2], chap. V-VIII.

²⁾ A field K is called a number field if it is a finite-dimensional extension of the field \mathbb{Q} of rational numbers. A field K is called a local number field if it is the completion of a number field k with respect to a non-trivial valuation of K . This is the case if and only if K is either the field of real numbers or the field of complex numbers (archimedean case), or if K is a finite-dimensional extension of the rational p -adic field \mathbb{Q}_p , for some prime number p .

³⁾ A field extension F/K is called a function field of one variable, if F/K is finitely generated, K is algebraically closed in F , and the degree of transcendency of F/K is 1.

⁴⁾ For the general theory of places see e.g. Zariski-Samuel [15], vol. II, chap. VI.

paper.⁵⁾ That is, we have $\mathfrak{B}(F/K) \subset \mathfrak{B}(F\mathfrak{p}/K)$. Let us put

$$\mathfrak{B}(F/K) = \bigcap_{\mathfrak{p}} \mathfrak{B}(F\mathfrak{p}/K),$$

\mathfrak{p} ranging over the places of F/K such that $F\mathfrak{p}/K$ is algebraic. We then have

$$\mathfrak{B}(F/K) \subset \mathfrak{B}(F/K).$$

If F/K is a separable function field of one variable, we shall show in § 3 that the factor group $\frac{\mathfrak{B}(F/K)}{\mathfrak{B}(F/K)}$ can be described by a certain cohomological invariant $X(F/K)$ which is connected with the one-dimensional Galois cohomology of the idèle class group.

The interpretation of this result is as follows: As we have said above, the investigation of $\mathfrak{B}(L/K)$ for an algebraic field extension L of K is part of the classical theory of central simple algebras. Hence we may regard $\mathfrak{B}(F/K)$, which concerns only Brauer groups of algebraic extensions $F\mathfrak{p}/K$, as essentially known, in particular if the ground field is a number field or a local number field. Hence the invariant $X(F/K)$, which is explicitly defined in § 3, will describe the deviation of the group $\mathfrak{B}(F/K)$ from the (known) group $\mathfrak{B}(F/K)$.

In the special case where K is a local number field we shall see that $X(F/K) = 1$. In the archimedean case, this will follow from the results of Witt [12] while in the non-archimedean case we shall refer to the corresponding results of Tate [11]. This then shows that $\mathfrak{B}(F/K) = \mathfrak{B}(F/K)$. On the other hand, the known structure of $\mathfrak{B}(K)$ for local number fields permits to determine $\mathfrak{B}(F/K)$ explicitly. We then will obtain the following result which constitutes the main result of this paper:

THEOREM 1. *Let F/K be a function field of one variable over a local number field K . Let $d(F/K)$ be the smallest positive integer which is a degree of a divisor of F/K .*

Then:

The group $\mathfrak{B}(F/K)$ is cyclic of order $d(F/K)$; it consists of all $A \in \mathfrak{B}(K)$ whose Schur index divides $d(F/K)$.

As to global number fields K as ground fields, we shall show by examples

⁵⁾ [9], page 428, prop. 8.

that the equality $\mathfrak{B}(F/K) = \mathfrak{B}(F/K)$ is not true in general. This case has still to be investigated.

§ 2. The cohomological language

Let \bar{K}/K be a finite Galois extension of K , with Galois group $G = G(\bar{K}/K)$. As shown in the theory of crossed product algebras, we have

$$(1) \quad \mathfrak{B}(\bar{K}/K) = H^2(G, \bar{K}^\times)^{6)}$$

where \bar{K}^\times denotes the multiplicative group of the field \bar{K} . Here, $H^2(G, \bar{K}^\times)$ denotes the second cohomology group of G in the multiplicative group of \bar{K} .⁷⁾

If L/K is any field extension and $\bar{L} = L\bar{K}$ a field compositum of L with \bar{K} , then the subgroup

$$G_L = G(\bar{K}/L \cap \bar{K})$$

of G can be regarded as the Galois group of \bar{L}/L . The restriction from G to G_L together with the inclusion map $\bar{K}^\times \subset \bar{L}^\times$ gives a cohomology map

$$(2) \quad H^2(G, \bar{K}^\times) \rightarrow H^2(G_L, \bar{L}^\times).$$

On the other hand, the map $\mathfrak{B}(K) \rightarrow \mathfrak{B}(L)$ described in § 1 induces a map

$$(3) \quad \mathfrak{B}(\bar{K}/K) \rightarrow \mathfrak{B}(\bar{L}/L).$$

In the theory of crossed products it is shown that the two maps (2) and (3) coincide after the identification (1) and the corresponding identification for \bar{L}/L .⁸⁾ The kernel of (3) consists of those algebras over K , which are split by \bar{K} and L . That is, this kernel is $\mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(L/K)$.

Hence :

$$(4) \quad \mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(L/K) = \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G_L, \bar{L}^\times).$$

If $L = F$ is a separable function field of one variable, then F has a separable place \mathfrak{p} ; hence we may choose \bar{K} so as to contain $F\mathfrak{p}$. As said in § 1, $\mathfrak{B}(F/K) \subset \mathfrak{B}(F\mathfrak{p}/K) \subset \mathfrak{B}(\bar{K}/K)$. On the other hand, F is linearly disjoint to \bar{K} over K and hence $G_{\mathfrak{p}} = G$ can be regarded as the Galois group of $\bar{F} = F \cdot \bar{K}$ over F .

⁶⁾ See e.g. the books mentioned in ¹⁾. For another approach see Serre [10], chap. X, § 5-6.

⁷⁾ For the general cohomology theory we refer to [10] chap. VII, or Cartan-Eilenberg [3], or Artin [1].

⁸⁾ Deuring [6], page 61, Satz 1.

Hence :

PROPOSITION 1. *Let F/K be a separable function field of one variable. Then there exists a finite Galois extension \bar{K}/K such that $\mathfrak{B}(F/K) \subset \mathfrak{B}(\bar{K}/K)$. If this is so, then*

$$\mathfrak{B}(F/K) = \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{F}^\times).$$

Next we shall give a cohomological interpretation of $\mathfrak{B}(F/K)$.

Let \mathfrak{p} be a place of F/K and $F_{\mathfrak{p}}$ its image field. Let $F_{\mathfrak{p}} \cdot \bar{K}$ be a field compositum of $F_{\mathfrak{p}}$ and \bar{K} over K and denote by $G_{\mathfrak{p}}$ the group of \bar{K} over $F_{\mathfrak{p}} \cap \bar{K}$. From (4) we obtain :

$$(5) \quad \mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(F_{\mathfrak{p}}/\bar{K}) = \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G_{\mathfrak{p}}, (F_{\mathfrak{p}} \cdot \bar{K})^\times).$$

Let now \mathfrak{p} range over all places of F/K and

$$H^2(G, \bar{K}^\times) \rightarrow \prod_{\mathfrak{p}} H^2(G_{\mathfrak{p}}, (F_{\mathfrak{p}} \cdot \bar{K})^\times)$$

be the map which in each component of the direct product induces the map mentioned in (5). Its kernel is the intersection of the kernels in (5). Hence

$$(6) \quad \mathfrak{B}(\bar{K}/K) \cap \mathfrak{B}(F/K) = \text{kernel } H^2(G, \bar{K}^\times) \rightarrow \prod_{\mathfrak{p}} H^2(G_{\mathfrak{p}}, (F_{\mathfrak{p}} \cdot \bar{K})^\times)$$

If we choose \bar{K} such that it contains $F_{\mathfrak{p}}$ for some \mathfrak{p} , which is possible if F/K is separable, then $\mathfrak{B}(F/K)$ is contained in $\mathfrak{B}(\bar{K}/K)$ and hence we may replace the intersection on the left hand side of (6) by $\mathfrak{B}(F/K)$.

On the right hand side of (6), the image group is a direct product of cohomology groups with respect to various subgroups $G_{\mathfrak{p}}$ of G . However, this group can be interpreted as a cohomology group of G in a certain group $\bar{W} = \bar{W}(\bar{F}/\bar{K})$, as follows.

For a given prime \mathfrak{p} , the field compositum $F_{\mathfrak{p}} \cdot \bar{K}$ is in general not uniquely determined. There may be several inequivalent field composita of $F_{\mathfrak{p}}$ with \bar{K} over K . Let $\bar{\mathfrak{p}}$ range over the primes of \bar{F}/\bar{K} which lie above \mathfrak{p} (we then write $\bar{\mathfrak{p}}|\mathfrak{p}$). It is well known that the inequivalent field composita of $F_{\mathfrak{p}}$ with \bar{K} correspond 1-1 to the $\bar{\mathfrak{p}}|\mathfrak{p}$. For any $\bar{\mathfrak{p}}|\mathfrak{p}$, the image field $\bar{F}_{\bar{\mathfrak{p}}}$ contains $F_{\bar{\mathfrak{p}}}$, which is K -isomorphic to $F_{\mathfrak{p}}$ under the map

$$a_{\mathfrak{p}} \rightarrow a_{\bar{\mathfrak{p}}} \quad (a \in F).$$

We have

$$\overline{F\mathfrak{p}} = F\mathfrak{p} \cdot \overline{K},$$

and this is the field compositum belonging to $\overline{\mathfrak{p}}$.⁹⁾

We now form the direct product $\prod_{\mathfrak{p}|\mathfrak{p}} \overline{F\mathfrak{p}}$. Since the $\overline{F\mathfrak{p}}$ are all the inequivalent field composita of $F\mathfrak{p}$ and \overline{K} , we have a natural isomorphism

$$(7) \quad F\mathfrak{p} \otimes_K \overline{K} = \prod_{\mathfrak{p}|\mathfrak{p}} \overline{F\mathfrak{p}}$$

which is obtained by mapping \overline{K} diagonally into $\prod_{\mathfrak{p}|\mathfrak{p}} \overline{F\mathfrak{p}}$ (\overline{K} is contained in each $\overline{F\mathfrak{p}}$) and by mapping

$$a\mathfrak{p} \rightarrow \prod_{\mathfrak{p}|\mathfrak{p}} a\overline{\mathfrak{p}} \quad (a \in F).$$

The Galois group G acts naturally on $F\mathfrak{p} \otimes \overline{K}$ (on the right factor) and hence on the direct product on the right hand side of (7), thereby permuting the factors $\overline{F\mathfrak{p}}$ transitively. If $\mathfrak{p}|\mathfrak{p}$ is fixed and $G_{\mathfrak{p}}$ denotes the subgroup of G leaving the elements of $\overline{F\mathfrak{p}}$ fixed, then we may write

$$(8) \quad F\mathfrak{p} \otimes_K \overline{K} = \prod_{\sigma \in G \bmod G_{\mathfrak{p}}} (\overline{F\mathfrak{p}})^{\sigma}.$$

Let $\overline{W}_{\mathfrak{p}}$ be the group of units of the algebra $F\mathfrak{p} \otimes \overline{K}$. We obtain

$$(9) \quad \overline{W}_{\mathfrak{p}} = \prod_{\sigma \bmod G_{\mathfrak{p}}} (\overline{F\mathfrak{p}})^{\times \sigma}.$$

Shapiros lemma from cohomology theory¹⁰⁾ now shows that

$$(10) \quad H^i(G, \overline{W}_{\mathfrak{p}}) = H^i(G_{\mathfrak{p}}, (\overline{F\mathfrak{p}})^{\times}) \quad (i \geq 0).$$

This isomorphism is obtained by the restriction of G to the subgroup $G_{\mathfrak{p}}$, followed by the projection $\overline{W}_{\mathfrak{p}} \rightarrow (\overline{F\mathfrak{p}})^{\times}$.

Observe that on the right hand side in (10) we have one fixed compositum $\overline{F\mathfrak{p}} = F\mathfrak{p} \cdot \overline{K}$ of $F\mathfrak{p}$ and \overline{K} . This may take the place of what we have denoted by $F\mathfrak{p} \cdot \overline{K}$ in (5). The diagonal imbedding $\overline{K}^{\times} \rightarrow \overline{W}_{\mathfrak{p}}$ followed by the projection $\overline{W}_{\mathfrak{p}} \rightarrow (\overline{F\mathfrak{p}})^{\times}$ is precisely the natural injection $\overline{K}^{\times} \rightarrow (\overline{F\mathfrak{p}})^{\times} = (F\mathfrak{p} \cdot \overline{K})^{\times}$. Hence we obtain from (5) and (10) (for $i = 2$) that

$$(11) \quad \mathfrak{B}(\overline{K}/K) \cap \mathfrak{B}(F\mathfrak{p}/K) = \text{kernel } H^2(G, \overline{K}^{\times}) \rightarrow H^2(G, \overline{W}_{\mathfrak{p}}).$$

⁹⁾ Chevalley [5], page 92, theorem 3.

¹⁰⁾ [10], page 125, exercice.

Now let us put all places \mathfrak{p} of F/K together :

$$\overline{W} = \prod_{\mathfrak{p}} \overline{W}_{\mathfrak{p}} = \prod_{\overline{\mathfrak{p}}} (\overline{F}_{\overline{\mathfrak{p}}})^{\times}$$

G acts on \overline{W} componentwise on each $\overline{W}_{\mathfrak{p}}$. We have

$$(12) \quad H^i(G, \overline{W}) = \prod_{\mathfrak{p}} H^i(G, \overline{W}_{\mathfrak{p}}) \quad (i \geq 0)$$

and we obtain :

PROPOSITION 2. *Let F/K be a separable function field of one variable. Then there is a finite Galois extension \overline{K}/K such that $\mathfrak{B}(F/K) \subset \mathfrak{B}(\overline{K}/K)$. If this is so, we have*

$$\mathfrak{B}(F/K) = \text{kernel } H^2(G, \overline{K}^{\times}) \rightarrow H^2(G, \overline{W}),$$

where

$$\overline{W} = \prod_{\overline{\mathfrak{p}}} (\overline{F}_{\overline{\mathfrak{p}}})^{\times}$$

($\overline{\mathfrak{p}}$ ranging over the places of $\overline{F}/\overline{K}$), and G acts on \overline{W} naturally as described above.

§ 3. The kernel theorem

Let F/K be a function field of one variable and \overline{K}/K a finite Galois extension with group G . According to propositions 1 and 2, we shall study in this § 3 the maps

$$H^2(G, \overline{K}^{\times}) \rightarrow H^2(G, \overline{F}^{\times})$$

and

$$H^2(G, \overline{K}^{\times}) \rightarrow H^2(G, \overline{W})$$

described in § 2 and we shall compare their kernels.

We introduce the following notations :

D the divisor group of F/K

H the group of principal divisors in D

$CD = D/H$ the divisor class group

J the idèle group of F/K

$CJ = J/F^{\times}$ the group of idèle classes

U the group of idèle units in J

$CU = UF^{\times}/F^{\times} = U/K^{\times}$ the idèle unit classes.

As to the definitions, D is defined to be the free abelian multiplicative

group generated by the prime divisors (places) \mathfrak{p} of F/K . Hence every divisor $a \in D$ is a product

$$a = \prod_{\mathfrak{p}} \mathfrak{p}^{a(\mathfrak{p})}$$

with uniquely determined integers $a(\mathfrak{p})$ such that $a(\mathfrak{p}) = 0$ for all but a finite number of \mathfrak{p} .

Let $w_{\mathfrak{p}}$ be the additive normalized valuation of F belonging to \mathfrak{p} . The principal divisor for $a \in F^{\times}$ is

$$(a) = \prod_{\mathfrak{p}} \mathfrak{p}^{w_{\mathfrak{p}}(a)}.$$

H is defined to be the image of the map $a \rightarrow (a)$ from F^{\times} into D . The kernel of this map is K^{\times} , so that the sequence

$$1 \rightarrow K^{\times} \rightarrow F^{\times} \rightarrow H \rightarrow 1$$

is exact.

J is defined to consist of all functions $\mathfrak{p} \rightarrow \alpha(\mathfrak{p})$, defined on the primes \mathfrak{p} of F/K , with values $\alpha(\mathfrak{p})$ in the multiplicative group of $F_{\mathfrak{p}}$, the \mathfrak{p} -adic completion of F with respect to \mathfrak{p} .

These functions α have to satisfy the finiteness condition that $w_{\mathfrak{p}}(\alpha(\mathfrak{p})) \neq 0$ for all but a finite number of \mathfrak{p} .¹¹⁾ There is a mapping $J \rightarrow D$ obtained by assigning to each $\alpha \in J$ its divisor $(\alpha) = \prod_{\mathfrak{p}} \mathfrak{p}^{w_{\mathfrak{p}}(\alpha(\mathfrak{p}))}$.

This mapping is epimorphic; its kernel is called U , so that the sequence

$$1 \rightarrow U \rightarrow J \rightarrow D \rightarrow 1$$

is exact.

There is a mapping $F^{\times} \rightarrow J$ obtained by assigning to each $a \in F^{\times}$ the idèle α_a given by $\alpha_a(\mathfrak{p}) = a$, for all \mathfrak{p} (diagonal imbedding). This mapping is monomorphic and we identify F^{\times} with its image in J . This identification is coherent with the mappings $F^{\times} \rightarrow D$ and $J \rightarrow D$, i.e. we have $(a) = (\alpha_a)$. In other words, the diagram

$$\begin{array}{ccc} F^{\times} & \longrightarrow & J \\ \downarrow & & \downarrow \\ H & \longrightarrow & D \end{array}$$

is commutative.

¹¹⁾ By continuity, the valuation $w_{\mathfrak{p}}$ of F extends uniquely to a valuation of the completion $F_{\mathfrak{p}}$, and this extension is again denoted by $w_{\mathfrak{p}}$.

From the above definitions and discussions it follows that the diagram

$$\begin{array}{ccccc}
 & 1 & 1 & 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 1 \rightarrow & K^\times & \rightarrow U & \rightarrow CU \rightarrow 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 1 \rightarrow & F^\times & \rightarrow J & \rightarrow CJ \rightarrow 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 1 \rightarrow & H & \rightarrow D & \rightarrow CD \rightarrow 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 & 1 & 1 & 1 &
 \end{array}$$

in which the arrows denote the natural maps in question, is commutative with exact rows and columns.

For the function field \bar{F}/\bar{K} we have a similar diagram whose corresponding groups will be denoted by $\bar{D}, \bar{H}, \bar{CD}, \bar{J}$, etc :

$$\begin{array}{ccccc}
 & 1 & 1 & 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 1 \rightarrow & \bar{K}^\times & \rightarrow \bar{U} & \rightarrow \bar{CU} \rightarrow 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 1 \rightarrow & \bar{F}^\times & \rightarrow \bar{J} & \rightarrow \bar{CJ} \rightarrow 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 1 \rightarrow & \bar{H} & \rightarrow \bar{D} & \rightarrow \bar{CD} \rightarrow 1 & \\
 & \downarrow & \downarrow & \downarrow & \\
 & 1 & 1 & 1 &
 \end{array}$$

As said in § 2, the Galois group $G = G(\bar{K}/K)$ can be regarded as the Galois group of \bar{F}/F , since F and \bar{K} are linearly disjoint over K . Hence G acts on \bar{F}^\times . Also, G acts on all the other groups of our diagram, as follows.

G acts on the primes \bar{p} of \bar{F} : If $w_{\bar{p}}$ is the additive normalized valuation of \bar{F} belonging to \bar{p} then \bar{p}^σ is defined by

$$w_{\bar{p}^\sigma}(a^\sigma) = w_{\bar{p}}(a) \quad (a \in F, \sigma \in G).$$

The map $\sigma : \bar{F} \rightarrow \bar{F}$ is continuous if \bar{F} as the domain of this map is topologized by $w_{\bar{p}}$, and it is topologized by $w_{\bar{p}^\sigma}$ if considered as the range of σ . Hence σ extends, by continuity, uniquely to a map $\sigma : \bar{F}_{\bar{p}} \rightarrow \bar{F}_{\bar{p}^\sigma}$ of the corresponding completions. According to these maps, G acts on \bar{J} , namely :

$$\alpha^\sigma(\bar{p}^\sigma) = \alpha(\bar{p})^\sigma \quad (\alpha \in \bar{J}, \sigma \in G).$$

By definition, it is clear that the maps

$$\bar{F}^\times \rightarrow \bar{J}$$

(diagonal imbedding) and

$$\bar{J} \rightarrow \bar{D}$$

(divisor map) are G -permissible. Hence all the other maps of our diagram, being based on the two maps mentioned above, are G -permissible, G acting on the groups of the diagram in the natural way. In other words: our diagram is G -permissible.

In particular, for each group \bar{M} of our diagram we can form the cohomology groups $H^i(G, \bar{M})$, and for each exact row or column $1 \rightarrow \bar{M}_1 \rightarrow \bar{M}_2 \rightarrow \bar{M}_3 \rightarrow 1$ of our diagram we obtain a cohomological connecting map $H^i(G, \bar{M}_3) \rightarrow H^{i+1}(G, \bar{M}_1)$.

From the lower horizontal sequence of the diagram we thus obtain a cohomology map

$$H^i(\bar{CD}) \rightarrow H^{i+1}(\bar{H}).$$

From the left vertical sequence we obtain also

$$H^{i+1}(\bar{H}) \rightarrow H^{i+2}(\bar{K}^\times)$$

which combined with the map above yields a map

$$h^i : H^i(\bar{CD}) \rightarrow H^{i+2}(\bar{K}^\times).$$

Similarly, using first the right vertical sequence and then the upper horizontal sequence of the diagram we obtain another map

$$g^i : H^i(\bar{CD}) \rightarrow H^{i+2}(\bar{K}^\times).$$

It is known from general cohomology theory¹²⁾ that both maps h^i and g^i differ only by a sign; in particular, both maps have the same kernel and the same image.

Let us investigate these maps in the case $i = 0$.

Investigation of h^0 :

By definition, h^0 is obtained by considering the left lower corner of the diagram, namely:

¹²⁾ Cartan-Eilenberg [3], page 56, prop. 2.1.

$$\begin{array}{c}
 1 \\
 \downarrow \\
 \overline{K}^\times \\
 \downarrow \\
 \overline{F}^\times \\
 \downarrow \\
 1 \rightarrow \overline{H} \rightarrow \overline{D} \rightarrow \overline{CD} \rightarrow 1 \\
 \downarrow \\
 1
 \end{array}$$

This portion of our diagram gives the two maps

$$H^0(\overline{CD}) \rightarrow H^1(\overline{H})$$

and

$$H^1(\overline{H}) \rightarrow H^2(\overline{K}^\times)$$

the composite of which is h^0 .

We begin by observing that

$$(13) \quad H^1(\overline{F}^\times) = 1$$

and

$$(14) \quad H^1(\overline{D}) = 1.$$

The first of these formulae is well known as the celebrated 'Hilbert theorem 90'. The second follows from the fact that \overline{D} is the free abelian group generated by the primes \bar{p} of $\overline{F}/\overline{K}$ which are only permuted under G .¹³⁾

From (13) it follows, using the exactness of the column of our diagram portion, that

$$H^1(\overline{H}) \rightarrow H^2(\overline{K}^\times) \text{ is monomorphic.}$$

Similarly, from (14) it follows that

$$H^0(\overline{CD}) \rightarrow H^1(\overline{H}) \text{ is epimorphic.}$$

Putting both statements together we obtain

$$(15) \quad \begin{aligned} \text{image } (h_0) &= \text{image } H^1(\overline{H}) \rightarrow H^2(\overline{K}^\times) \\ &= H^1(\overline{H}). \end{aligned}$$

On the other hand,

$$\text{image } H^1(\overline{H}) \rightarrow H^2(\overline{K}^\times) = \text{kernel } H^2(\overline{K}^\times) \rightarrow H^2(\overline{F}^\times)$$

¹³⁾ See e.g. [9], page 437,

so that we finally obtain

$$(16) \quad \text{image } (h^0) = \text{kernel } H^2(\overline{K}^\times) \rightarrow H^2(\overline{F}^\times).$$

Investigation of g^0 :

Now we have to consider the right upper corner of our diagram:

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & \downarrow & & \\ & & & & \overline{CU} & & \\ 1 & \rightarrow & \overline{K}^\times & \rightarrow & \overline{U} & \rightarrow & 1 \\ & & & & \downarrow & & \\ & & & & \overline{CJ} & & \\ & & & & \downarrow & & \\ & & & & \overline{CD} & & \\ & & & & \downarrow & & \\ & & & & 1 & & \end{array}$$

g^0 is the composite of the two maps

$$H^0(\overline{CD}) \rightarrow H^1(\overline{CU})$$

and

$$H^1(\overline{CU}) \rightarrow H^2(\overline{K}^\times).$$

First we have, in analogy to (13), the formula

$$(17) \quad H^1(\overline{U}) = 1.$$

Proof. Let $\overline{U}_{\overline{p}}$ be the group of \overline{p} -adic units in the \overline{p} -adic completion $\overline{F}_{\overline{p}}$ of \overline{F} . By definition, \overline{U} is the direct product

$$\overline{U} = \prod_{\overline{p}} \overline{U}_{\overline{p}}.$$

Each place \overline{p} induces an epimorphic map

$$\overline{U}_{\overline{p}} \rightarrow (\overline{F}_{\overline{p}})^\times.$$

These maps define an epimorphic map

$$\overline{U} \rightarrow \overline{W}$$

where \overline{W} is the direct product of the $(\overline{F}_{\overline{p}})^\times$ as in §2. By comparing the definitions of the actions of G on \overline{U} (as part of J) and on \overline{W} (see §2) we see that this map is G -permissible.

Let \overline{V} be the kernel, so that

$$1 \rightarrow \overline{V} \rightarrow \overline{U} \rightarrow \overline{W} \rightarrow 1$$

is exact. We shall show in a moment that

$$(18) \quad H^i(G, \bar{V}) = 1 \quad (i \geq 1).$$

This shows that $\bar{U} \rightarrow \bar{W}$ induces an isomorphism

$$(19) \quad H^i(G, \bar{U}) = H^i(G, \bar{W}) \quad (i \geq 1).$$

Using (12) and (10) we obtain

$$H^i(G, \bar{U}) = \prod_{\mathfrak{p}} H^i(G_{\bar{\mathfrak{p}}}, (\bar{F}_{\bar{\mathfrak{p}}})^{\times}) \quad (i \geq 1)$$

where \mathfrak{p} ranges over the places of F/K and $\bar{\mathfrak{p}}$ denotes always a *fixed* extension of \mathfrak{p} to \bar{F}/\bar{K} . For $i = 1$, the right hand side of (19) is 1, using Hilbert's theorem 90 for each field $\bar{F}_{\bar{\mathfrak{p}}}$. Hence (17).

Proof of (18). Let $\bar{V}_{\bar{\mathfrak{p}}}$ be the kernel of the map $\bar{U}_{\bar{\mathfrak{p}}} \rightarrow (\bar{F}_{\bar{\mathfrak{p}}})^{\times}$, consisting of the elements $a \in \bar{F}_{\bar{\mathfrak{p}}}$ with $a\bar{\mathfrak{p}} = 1$. Then $\bar{V} = \prod_{\bar{\mathfrak{p}}} \bar{V}_{\bar{\mathfrak{p}}}$. Put $\bar{V}_{\mathfrak{p}} = \prod_{\bar{\mathfrak{p}}|\mathfrak{p}} \bar{V}_{\bar{\mathfrak{p}}}$. Then $\bar{V} = \prod_{\mathfrak{p}} \bar{V}_{\mathfrak{p}}$ is a G -permissible direct product. Hence

$$H^i(G, \bar{V}) = \prod_{\mathfrak{p}} H^i(G, \bar{V}_{\mathfrak{p}}).$$

From Shapiro's lemma¹⁰⁾ we infer that

$$H^i(G, \bar{V}_{\mathfrak{p}}) = H^i(G_{\bar{\mathfrak{p}}}, \bar{V}_{\bar{\mathfrak{p}}}),$$

$\bar{\mathfrak{p}}$ being a fixed extension of \mathfrak{p} . Hence we have to show that $H^i(G_{\bar{\mathfrak{p}}}, \bar{V}_{\bar{\mathfrak{p}}}) = 1$ for $i \geq 1$. Changing notation, this amounts to show the following

LEMMA. *Let F be a complete field with respect to a non-archimedean, discrete valuation $w_{\mathfrak{p}}$ with corresponding prime \mathfrak{p} . Let V be the multiplicative subgroup of elements $a \in F$ with $a\mathfrak{p} = 1$ (i.e. $w_{\mathfrak{p}}(a - 1) > 0$). If G is a finite group of continuous automorphisms of F whose induced action on the image field $F\mathfrak{p}$ is faithful, then*

$$H^i(G, V) = 1 \quad (i \geq 1).$$

This lemma is well known from local class field theory. For the proof see e.g. Witt [14], page 154, no. 2 or Serre [10], page 193, lemma 2.

Let us return to our original notation. We now have proved (17) which is, for the map g^0 , the analogue to (13). The analogue to (14) would be $H^1(G, \bar{C}\bar{J}) = 1$. This is not true in general (although we shall see later that it is true in the case where K is a local number field). We therefore introduce the group

$$(20) \quad X = X(\bar{F}/\bar{K}) = \text{kernel } H^1(G, \bar{CJ}) \rightarrow H^1(G, \bar{CD}).$$

From the exactness of the column of our diagram portion we infer that

$$(21) \quad \begin{aligned} X &= \text{image } H^1(G, \bar{CU}) \rightarrow H^1(G, \bar{CJ}) \\ &= H^1(G, \bar{CU})/Y \end{aligned}$$

where

$$Y = \text{image } H^0(G, \bar{CD}) \rightarrow H^1(G, \bar{CU}).$$

From (17) it follows that

$$H^1(G, \bar{CU}) \rightarrow H^2(G, \bar{K}^\times) \text{ is monomorphic.}$$

Its image is the kernel of $H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{U})$. Hence

$$H^1(G, \bar{CU}) \simeq \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{U}).$$

In this isomorphism, the image Y of $H^0(G, \bar{CD}) \rightarrow H^1(G, \bar{CU})$ corresponds to the image of g^0 (by definition of g^0). Hence we obtain from (21):

(22) *The image of g^0 is contained in the kernel of $H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{U})$ and the corresponding factor group is isomorphic to X .*

Finally, we claim:

$$(23) \quad \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{U}) = \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{W}).$$

Proof. As shown in (19), the map $\bar{U} \rightarrow \bar{W}$ induces an isomorphism of cohomology groups. Hence the map $\bar{K}^\times \rightarrow \bar{U} \rightarrow \bar{W}$ induces a map $H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{W})$ which has the same kernel as $H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{U})$. Q.e.d.

Observe that the map $\bar{K}^\times \rightarrow \bar{U} \rightarrow \bar{W}$ is the diagonal imbedding of \bar{K}^\times in \bar{W} which we have considered in §2.

Now remember that the maps h^0 and g^0 have the same image, as mentioned above. Comparing (16), (22) and (23) we obtain therefore the following ‘kernel theorem’:

THEOREM 2. *Let F/K be a function field of one variable and \bar{K}/K a finite Galois extension with Galois group G . Then the kernel of $H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{F}^\times)$ is contained in the kernel of $H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{W})$ and the corresponding factor group is isomorphic to X , where X is defined to be the kernel of $H^1(G, \bar{CJ}) \rightarrow H^1(G, \bar{CD})$.*

In particular, if $H^1(G, \bar{CJ}) = 1$ then $X = 1$ and therefore $\text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{F}^\times) = \text{kernel } H^2(G, \bar{K}^\times) \rightarrow H^2(G, \bar{W})$.

Using propositions 1 and 2 of §2 we obtain as an immediate consequence :

THEOREM 3. *Let F/K be a separable function field of one variable. Then the factor group $\mathfrak{B}(F/K)$ modulo $\mathfrak{B}(F/K)$ can be cohomologically described as the group X of theorem 2, where \bar{K}/K has to be chosen such that $\mathfrak{B}(F/K) \subset \mathfrak{B}(\bar{K}/K)$.*

(As mentioned in §2, the latter inclusion is true if \bar{K} contains the image field $F\mathfrak{p}$ of a separable place \mathfrak{p} of F/K .)

In particular, if the Galois cohomology of the idèle classes \overline{CJ} vanishes in dimension 1, then $\mathfrak{B}(F/K) = \mathfrak{B}(F/K)$.

§ 4. Proof of theorem 1

Now let F/K be a function field of one variable over a local number field K . If the valuation of K is non-archimedean, then there is a theorem of Tate which says that the Galois cohomology of the idèle classes vanishes in dimension 1.¹⁴⁾ Hence $\mathfrak{B}(F/K) = \mathfrak{B}(F/K)$.

Now let the valuation of K be archimedean. Then K is either the field of complex numbers, or the field of real numbers. In the first case K is algebraically closed and hence $\mathfrak{B}(F/K) = \mathfrak{B}(F/K) = 1$. In the second case, assume first that F/K has a real place \mathfrak{p} . Then $F\mathfrak{p} = K$, $\mathfrak{B}(F/K) \subset \mathfrak{B}(K/K) = 1$, hence $\mathfrak{B}(F/K) = \mathfrak{B}(F/K) = 1$. Secondly, if all places \mathfrak{p} of F/K are complex, then $F\mathfrak{p} = \bar{K}$ is the field of complex numbers for all \mathfrak{p} . Hence $\mathfrak{B}(F/K) = \mathfrak{B}(\bar{K}/K) = \mathfrak{B}(K)$ is of order two, the only non-trivial element of $\mathfrak{B}(K)$ corresponding to the quaternion algebra over K . On the other hand, Witt has shown that if F/K has no real places, then -1 is a sum of two squares in F , hence -1 is a norm of $F\bar{K}/F$, i.e. the quaternion algebra splits over F .¹⁵⁾ This implies that $\mathfrak{B}(F/K)$ contains the quaternion algebra and is therefore equal to $\mathfrak{B}(K)$.

Hence, in any case, $\mathfrak{B}(F/K) = \mathfrak{B}(F/K)$.¹⁶⁾

¹⁴⁾ Tate [11], page 156-02, line 2-5.

¹⁵⁾ Witt [12], page 7 Satz 2.

¹⁶⁾ Using Witts results, it can be shown that Tates relation $H^1(G, \overline{CJ})=1$ holds also if K is real and \bar{K} complex. For, if one interprets Witts statement I' ([12], page 5) cohomologically, it says that the map $H^2(G, \bar{F}^\times) \rightarrow H^2(G, \bar{J})$ is injective. On the other hand, from our diagram in § 3 we obtain an exact sequence $H^1(G, \bar{J}) \rightarrow H^1(G, \overline{CJ}) \rightarrow H^2(G, \bar{F}^\times) \rightarrow H^2(G, \bar{J})$ and we have $H^1(G, \bar{J})=1$ from Hilberts theorem 90 for the completions $\bar{F}_{\mathfrak{p}}$. Hence $H^1(G, \overline{CJ})=1$.

In order to complete the proof of theorem 1 we have to describe the group $\mathfrak{B}(F/K)$.

Consider first the non-archimedean case. As is well known from local class field theory, the Brauer group $\mathfrak{B}(K)$ is isomorphic to the additive group \mathbf{Q}/\mathbf{Z} of rational numbers modulo integers.¹⁷⁾ The isomorphism

$$\mathfrak{B}(K) \approx \mathbf{Q}/\mathbf{Z}$$

is obtained by assigning to each central simple algebra A/K its Hasse invariant $\text{inv}_K(A)$. If L/K is a finite algebraic extension field, then $\text{inv}_L(A \otimes_K L) = (L:K) \cdot \text{inv}_K(A)$. In particular, L splits A if and only if $(L:K)$ is a multiple of the order of A in $\mathfrak{B}(K)$. In other words $\mathfrak{B}(L/K)$ consists of all those $A \in \mathfrak{B}(K)$ for which $A^{(L:K)} = 1$. The group structure of \mathbf{Q}/\mathbf{Z} implies moreover that $\mathfrak{B}(L/K)$ is cyclic of order $(L:K)$.

In particular, $\mathfrak{B}(F\mathfrak{p}/K)$ is cyclic of order $(F\mathfrak{p}:K) = \text{deg}(\mathfrak{p})$, and $\mathfrak{B}(F\mathfrak{p}/K)$ consists of all elements $A \in \mathfrak{B}(K)$ with $A^{\text{deg}(\mathfrak{p})} = 1$. Taking the intersection for all \mathfrak{p} , we see that if

$$0 < d(F/K) = \text{gcd}_p \text{deg}(\mathfrak{p})$$

then $\mathfrak{B}(F/K)$ is cyclic of order $d(F/K)$ and consists of all elements $A \in \mathfrak{B}(K)$ with $A^{d(F/K)} = 1$.

If $a = \prod_p \mathfrak{p}^{a(\mathfrak{p})}$ is a divisor of F/K then $\text{deg}(a) = \sum_p a(\mathfrak{p}) \cdot \text{deg}(\mathfrak{p})$ is a linear combination of the degrees $\text{deg}(\mathfrak{p})$, hence a multiple of $d(F/K)$, and conversely. Hence $d(F/K)$ can be characterized as the least positive degrees of divisors of F/K .

This proves theorem 3 in the non-archimedean case, if one uses the fact (proved by studying the Hasse invariant as above) that the Schur index of any $A \in \mathfrak{B}(K)$ is equal to its order in $\mathfrak{B}(K)$.

In the archimedean real case, we have

$$\mathfrak{B}(K) \approx \frac{1}{2} \mathbf{Z}/\mathbf{Z}.$$

If one defines the Hasse invariant of the quaternion algebra to be $\frac{1}{2}$ modulo \mathbf{Z} , then the above considerations carry over verbally in order to prove theorem 1.

¹⁷⁾ See e.g. Deuring [6], page 112, Satz 3.

In the archimedean complex case, there is nothing to prove.

§ 5. Some additional remarks

(a) *Examples of function fields F/K over a number field K for which $\mathfrak{B}(F/K) \neq \tilde{\mathfrak{B}}(F/K)$:*

Let K be a number field and F/K be a function field of one variable which is of genus 0 but not rational. It has been shown by Witt [13] that $F = F(A)$ is a generic splitting field of a certain quaternion algebra A over K which is uniquely determined by F . Then $\mathfrak{B}(F/K)$ is of order 2, the only non-trivial element of $\mathfrak{B}(F/K)$ being A ; this follows also from our general theory of generic splitting fields¹⁸⁾. Let \mathfrak{q} range over the primes of K including the primes at infinity. Let M be the set of primes \mathfrak{q} at which A is ramified, i.e. for which the \mathfrak{q} -adic Hasse invariant $\text{inv}_{\mathfrak{q}}(A) \equiv \frac{1}{2} \pmod{\mathbf{Z}}$. According to the Hasse sum formula $\sum_{\mathfrak{q}} \text{inv}_{\mathfrak{q}}(A) \equiv 0 \pmod{\mathbf{Z}}$ ¹⁹⁾ the number m of primes $\mathfrak{q} \in M$ is even. To every non-empty subset $N \subset M$ which consists of an even number of primes \mathfrak{q} there exists one and only one quaternion algebra $A(N)$ with the primes in N as its ramification primes.¹⁹⁾ In particular, $A = A(M)$. These quaternion algebras generate a subgroup of $\mathfrak{B}(K)$ of order 2^{m-1} . We claim that this subgroup coincides with $\tilde{\mathfrak{B}}(F/K)$. Let \mathfrak{p} be a prime of F/K . Then $F\mathfrak{p}$ splits A , hence the \mathfrak{q} -adic completion $(F\mathfrak{p})_{\mathfrak{q}}$ splits $A_{\mathfrak{q}} = A \otimes_{\mathbf{K}} K_{\mathfrak{q}}$ for every \mathfrak{q} . Hence $((F\mathfrak{p})_{\mathfrak{q}} : K_{\mathfrak{q}}) \equiv 0 \pmod{2}$ for $\mathfrak{q} \in M$. In particular, this holds for $\mathfrak{q} \in N$. Hence $(F\mathfrak{p})_{\mathfrak{q}}$ splits $A(N)_{\mathfrak{q}}$. Since this is true for all \mathfrak{q} , it follows²⁰⁾ that $F\mathfrak{p}$ splits $A(N)$. Hence $A(N) \in \tilde{\mathfrak{B}}(F/K)$ for all N . Conversely, let $B \in \tilde{\mathfrak{B}}(F/K)$, $B \neq 1$. Let L/K be a finite algebraic splitting field of A . Since $F = F(A)$ is a generic splitting field for A , there is a place \mathfrak{p} of F/K such that $F\mathfrak{p} \subset L$.²¹⁾ Since $F\mathfrak{p}$ splits B it follows that L splits B . Hence every finite algebraic splitting field L of A is also a splitting field for B . According to the existence theorem of Grunwald²²⁾ there exists a finite algebraic extension field L/K such that $(L_{\mathfrak{q}} : K_{\mathfrak{q}}) = 2$ for $\mathfrak{q} \in M$ and $L_{\mathfrak{q}_0} = K_{\mathfrak{q}_0}$, if $\mathfrak{q}_0 \notin M$ is arbitrarily chosen. This field L splits A by construction and hence B .

¹⁸⁾ [9], page 414, theorem 5.

¹⁹⁾ Deuring [6], page 119, Satz 9.

²⁰⁾ Deuring [6], page 117, Satz 1.

²¹⁾ [9] page 413, theorem 2.

²²⁾ Hasse [7], page 40, Ganz schwacher Existenzsatz.

It follows $\text{inv}_q(B) \equiv 0 \pmod{\frac{1}{2}}$ for $q \in M$ and $\text{inv}_{q_0}(B) = 1$. Since $q_0 \notin M$ is arbitrary, we see that B is unramified outside of M . For $q \in M$, the invariant $\text{inv}_q(B)$ is either 0 or $\frac{1}{2} \pmod{\mathbf{Z}}$. Hence $B = A(N)$ is a quaternion algebra belonging to some subset $N \subset M$.

We have now shown that $\mathfrak{B}(F/K)$ is of order 2 while $\tilde{\mathfrak{B}}(F/K)$ is of order 2^{m-1} . If we choose A such that the number m of ramification points of A is $m > 2$, which is possible¹⁹⁾, then for the field $F = F(A)$ we have $\mathfrak{B}(F/K) \neq \tilde{\mathfrak{B}}(F/K)$.

(b) *Examples of function fields F/K over a number field K such that $d(F/K) \neq 1$ and $\mathfrak{B}(F/K) = 1$.*

Let K be a number field and F/K a function field of one variable and genus 1 with the property that $d(F/K) > 1$ but $d(F_q/K_q) = 1$ for all primes q of K , where $F_q = FK_q$ is the constant extension of F/K with respect to the completion K_q of q .²³⁾ If $A \in \mathfrak{B}(F/K)$, then for every q the completion A_q is split by F_q/K_q . Since $d(F_q/K_q) = 1$ it follows from theorem 1 that A_q splits too. Hence $\text{inv}_q(A) \equiv 0 \pmod{\mathbf{Z}}$ for all q , i.e. $A = 1$.¹⁹⁾

REFERENCES

- [1] Artin, Kohomologietheorie endlicher Gruppen; Vorlesungsarbeit Hamburg, (1957).
- [2] Artin-Nesbitt-Thrall, Rings with minimum condition, Ann Arbor (1944).
- [3] Cartan-Eilenberg, Homological Algebra, Princeton, (1956).
- [4] Cassels, Arithmetic on Curves of genus one. VI. The Tate-Šafarevič group can be arbitrarily large, Journal für die reine und angewandte Mathematik, vol. **214/215** (1964), 65-70.
- [5] Chevalley, Introduction to the theory of functions of one variable, New York, (1951).
- [6] Deuring, Algebren, Berlin, (1935).
- [7] Hasse, Zum Existenzsatz von Grunwald in der Klassenkörpertheorie, Journal für die reine und angewandte Mathematik, vol. **188** (1950), 40-64.
- [8] Reichardt, Einige im Kleinen überall lösbare, im Grossen unlösbare diophantische Gleichungen; Journal für die reine und angewandte Mathematik vol. **184** (1942), 12-18.
- [9] Roquette, On the Galois cohomology of the projective linear group and its applications to the construction of generic splitting fields of algebras; Mathematische Annalen, vol. **150** (1963), 411-439.

²³⁾ The existence of such function fields has been proved by Reichardt [8]. See also Cassels [4], page 65, theorem.

- [10] Serre, *Corps locaux*, Paris, (1962).
- [11] Tate, *WC-groups over p -adic fields*; Séminaire Bourbaki no. 156, Paris, (1957).
- [12] Witt, *Zerlegung reeller algebraischer Funktionen in Quadrate. Schiefkörper über reellem Funktionenkörper*. *Journal für die reine und angewandte Mathematik*, vol. **171** (1934), 4-11.
- [13] Witt, *Über ein Gegenbeispiel zum Normensatz*, *Mathematische Zeitschrift*, vol. **39** (1935), 492-467.
- [14] Witt, *Schiefkörper über diskret bewerteten Körpern*. *Journal für die reine und angewandte Mathematik*, vol. **176** (1937), 153-156.
- [15] Zariski-Samuel, *Commutative Algebra*, vol. **II**, Princeton, (1960).

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