ON MEROMORPHISMS OF ALGEBRAIC SYSTEMS

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Dedicated to the memory of Professor TADASI NAKAYAMA

1. Introduction

In the present paper by an algebraic system (algebra) A we shall mean a system with a set F of operations $f_\lambda : (x_1, \ldots, x_n) \in A \times \cdots \times A \to f_\lambda(x_1, \ldots, x_n) \in A$. A *polynomial* $p(x_1, \ldots, x_r)$ is a function of variables $x_1, \ldots, x_r$ which is either one of the x_i, or (recursively) a result of some operation $f_\lambda(p_1, \ldots, p_n)$ performed on other polynomials p_i. An algebra A may satisfy a set R of identities $p(x_1, \ldots, x_r) = q(x_1, \ldots, x_s)$, and then A shall be called an *(F, R)-algebra.*

By a *meromorphism* between two algebras admitting the same operations, we mean a many-many correspondence of elements which preserves all algebraic combinations. If φ is a meromorphism of A onto B, under which the correspondence of elements shall be written $a \to b(\varphi)$ or $a\varphi b$, then $a_i\varphi b_i$ $(i=1, \ldots, n)$ imply $f_\lambda(a_1, \ldots, a_n)\varphi f_\lambda(b_1, \ldots, b_n)$. We shall write $b\bar{\varphi}a$ to mean $a\varphi b$, and then $\bar{\varphi}$ becomes a meromorphism of B onto A. Let φ and ψ be meromorphisms from A onto B and from B onto C respectively, and define $a\varphi\psi c$ to mean $a\varphi b$ and $b\psi c$ for some $b \in B$. Then $\varphi\psi$ becomes a meromorphism from A onto C.

Now on a meromorphism of any algebra the following theorem similar to the Homomorphism Theorem holds.

MEROMORPHISM THEOREM. *Let φ be a meromorphism of A onto B. If we define the relation φ^* in A by*

$a\varphi^* a'$ means that for some finite number of elements $a_0, a_1, \ldots, a_n \in A$ and $b_1, \ldots, b_n \in B$,

$$a_0 = a,\ a' = a_n,\ a_{i-1}\varphi b_i,\ a_i\varphi b_i \qquad (i=1, \ldots, n),$$

then φ^* is a congruence relation on A, and similarly $\bar{\varphi}^*$ is that on B. Further their homomorphic images are isomorphic: $\varphi^*(A) \cong \bar{\varphi}^*(B)$.

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If, given $b \in B$, $\{x\ ;\ x\varphi b\}$ is necessarily a congruence class under φ^* in the above theorem and, given $a \in A$, $\{y\ ;\ a\varphi y\}$ is necessarily that under $\bar{\varphi}^*$, then φ is called a *class-meromorphism*. As is already known, a meromorphism φ is a class-meromorphism if and only if $a\varphi b$, $a'\varphi b$ and $a'\varphi b'$ imply $a\varphi b'$. When φ and ψ are two meromorphisms of A onto B, we define $\varphi \leqq \psi$ to mean that $a\varphi b$ implies $a\psi b$. Then the above condition that φ be a class-meromorphism is written $\varphi\bar{\varphi}\varphi \leqq \varphi$.

In Shoda's theory for abstract algebraic systems the following condition on an algebra A is often assumed:

(α) Every meromorphism between two homomorphic images of A is a class-meromorphism.

In the present paper we shall deal with meromorphisms of an algebra A onto itself. We shall first show in § 2 that the above condition (α) is equivalent to the condition

(β) Every meromorphism of A onto itself is a class-meromorphism.

A meromorphism φ of A onto itself may be regarded as a relation between elements of A. If φ is reflexive, i.e. $a\varphi a$ holds for all $a \in A$, we shall call φ a *quasi-congruence*. We shall show that a quasi-congruence on A is a class-meromorphism if and only if it is a congruence relation. We shall inquire in § 2 mainly into the symmetricity and transitivity of quasi-congruences in abstract algebras, and discuss the connections among the transitivity, symmetricity and permutability of quasi-congruences.

In § 3 and § 4 we shall deal with quasi-congruences on some real algebraic systems. Especially we shall discuss in § 3 the conditions that quasi-congruences on a semigroup be symmetric and in § 4 that quasi-congruences on a lattice be transitive. The lattice of quasi-congruences on a lattice is not necessarily distributive. We shall lastly give some sufficient conditions for that lattice to be distributive.

2. Meromorphisms of an abstract algebra onto itself

Let φ and ψ be homomorphisms of A and θ a meromorphism between $\varphi(A)$ and $\psi(A)$. If we define $a\Theta b$ to mean $\varphi(a)\theta\psi(b)$, then it is easy to see that Θ is a meromorphism of A onto itself. Suppose that $\varphi(a)\theta\psi(b)$, $\varphi(a')\theta\psi(b)$ and $\varphi(a')\theta\varphi(b')$. Then $a\Theta b$, $a'\Theta b$ and $a'\Theta b'$; hence if Θ is a class-meromorphism we get $a\Theta b'$ and $\varphi(a)\theta\psi(b')$, which shows that θ is a class-meromorphism between $\varphi(A)$ and $\psi(A)$. Thus we have

THEOREM 2.1. *Every meromorphism between two homomorphic images of an algebra A is a class-meromorphism if and only if every meromorphism of A onto itself is a class-meromorphism.*

Meromorphisms of A onto itself form a partially ordered semigroup $M(A)$ under the multiplication and the ordering defined in § 1:

$a\varphi\psi b$ means that $a\varphi c$ and $c\psi b$ for some $c \in A$;
$\varphi \leqq \psi$ means that $a\varphi b$ implies $a\psi b$.

Further, it is rather evident that $\varphi \leqq \varphi_1$ and $\psi \leqq \psi_1$ imply $\varphi\psi \leqq \varphi_1\psi_1$.

A meromorphism θ of A onto itself is regarded as a relation in A, and it becomes a *congruence relation* if it is reflexive, symmetric (symbolically $\bar{\theta} \leqq \theta$) and transitive ($\theta^2 \leqq \theta$). A *quasi-congruence* on A is a meromorphism of A onto itself which is reflexive. The set $Q(A)$ of quasi-congruences on A becomes a subsemigroup of $M(A)$ mentioned above and a complete lattice under the ordering defined in $M(A)$. In $Q(A)$ $a \to b(\Lambda_\alpha \theta_\alpha)$ means that $a\theta_\alpha b$ for all θ_α.

Now let P be a set of ordered pairs (a, b) of elements of A, and define the relation θ in the following way:

$u\theta v$ means that a polynomial $p(x_1, \ldots, x_m, y_1, \ldots, y_n)$ exists such that

$$u = p(a_1, \ldots, a_m, c_1, \ldots, c_n) \text{ and } v = p(b_1, \ldots, b_m, c_1, \ldots, c_n) \quad \text{for some } (a_i, b_i) \in P.$$

Then it is easily seen that θ becomes a quasi-congruence, which is the least of elements φ of $Q(A)$ satisfying $a\varphi b$ for every pair $(a, b) \in P$. This θ is called the quasi-congruence *generated* by P and denoted by $\theta(P)$. It follows that $\theta(P) = V_{(a,b)\in P}\,\theta(a, b)$, where $\theta(a, b)$ is the quasi-congruence generated by one pair (a, b).

We intend to discuss the symmetricity and transitivity of quasi-congruences. We first show

THEOREM 2.2. *Let $\{\theta_\alpha\}$ be a set of quasi-congruences on an algebra A. Then $\overline{\Lambda_\alpha \theta_\alpha} = \Lambda_\alpha \bar{\theta}_\alpha$ and $\overline{V_\alpha \theta_\alpha} = V_\alpha \bar{\theta}_\alpha$; accordingly symmetric quasi-congruences form a closed sublattice of $Q(A)$.*

Proof. It is clear by the meaning that $\overline{\Lambda_\alpha \theta_\alpha} = \Lambda_\alpha \overline{\theta}_\alpha$. Let P be a set of ordered pairs (a, b) of elements of A and put $\overline{P} = \{(b, a) ; (a, b) \in P\}$. If $u \to v(\theta(P))$, then a polynomial p exists such that $u = p(a_1, \ldots, a_m, c_1, \ldots, c_n)$, $v = p(b_1, \ldots, b_m, c_1, \ldots, c_n)$ and $(a_i, b_i) \in P$. Then $(b_i, a_i) \in \overline{P}$ and hence we infer $v \to u(\theta(\overline{P}))$, which shows $\overline{\theta(P)} = \theta(\overline{P})$. Now put $\theta_\alpha = \theta(P_\alpha)$. Then $V_\alpha \theta_\alpha = \theta(V_\alpha P_\alpha)$, where $V_\alpha P_\alpha$ is the set-sum of P_α. So we can deduce

$$\overline{V_\alpha \theta_\alpha} = \overline{\theta(V_\alpha P_\alpha)} = \theta(\overline{V_\alpha P_\alpha}) = \theta(V_\alpha \overline{P}_\alpha) = V_\alpha \theta(\overline{P}_\alpha) = V_\alpha \overline{\theta}_\alpha,$$

completing the proof.

If quasi-congruences θ_α are transitive, then $\Lambda_\alpha \theta_\alpha$ is also transitive but $V_\alpha \theta_\alpha$ is not necessarily transitive; hence the set $\Theta(A)$ of congruences on A is meet-closed in $Q(A)$ but not always a sublattice of $Q(A)$.

Now let S be a subalgebra of an algebra A and every quasi-congruence on S be transitive. Suppose $x, y, z \in S$, $x\theta y$ and $y\theta z$ under a quasi-congruence θ on A. Since θ can be regarded as a quasi-congruence θ_0 on S, provided the range of elements is restricted in S, and θ_0 is transitive, we infer $x\theta_0 z$ and $x\theta z$. So we have

THEOREM 2.3. *Quasi-congruences on an algebra A are transitive if every triple $\{x, y, z\}$ is contained in a subalgebra $S = S(x, y, z)$ on which quasi-congruences are transitive.*

And similarly,

THEOREM 2.4. *Quasi-congruences on an algebra A are symmetric if every pair $\{x, y\}$ is contained in a subalgebra $S = S(x, y)$ on which quasi-congruences are symmetric.*

Two quasi-congruences φ and ψ are called *permutable* if and only if $\varphi\psi = \psi\varphi$. We see some connections among the transitivity, symmetricity and permutability of quasi-congruences.

THEOREM 2.5. *If the join $\varphi \cup \psi$ of two quasi-congruences φ and ψ is transitive, then $\varphi\psi = \psi\varphi = \varphi \cup \psi$.*

Proof. When φ and ψ are quasi-congruences on A, $a\varphi b$ implies $a\varphi b\psi b$; hence we have $\varphi \leqq \varphi\psi$, $\psi \leqq \varphi\psi$ and $\varphi \cup \psi \leqq \varphi\psi$. So we can deduce from $(\varphi \cup \psi)^2 \leqq \varphi \cup \psi$, $\varphi\psi \leqq (\varphi \cup \psi)^2 \leqq \varphi \cup \psi \leqq \varphi\psi$.

THEOREM 2.6. *If quasi-congruences φ, ψ and $\varphi\psi$ are symmetric, then φ and ψ are permutable.*

Proof. It is easily seen that $\overline{\varphi\psi} = \bar{\psi}\bar{\varphi}$. Hence the symmetricity implies $\varphi\psi = \overline{\varphi\psi} = \bar{\psi}\bar{\varphi} = \psi\varphi$.

Next we deal with congruence relations regarded as quasi-congruences. Given a quasi-congruence θ, it follows from the Meromorphism Theorem mentioned in § 1 that $\theta^* = V_n(\theta\bar{\theta})^n$ is a congruence, which is called *generated* by θ, and if θ is originally a congruence, $\theta^* = \theta$.

THEOREM 2.7. *A quasi-congruence is a class-meromorphism if and only if it is a congruence.*

Proof. If θ is a congruence on A, then $\theta = V_n(\theta\bar{\theta})^n \geqq \theta\bar{\theta}\theta\bar{\theta} \geqq \theta\bar{\theta}\theta$, whence θ is a class-meromorphism. Conversely if $\theta\bar{\theta}\theta \leqq \theta$, then $\bar{\theta} \leqq \theta\bar{\theta}\theta \leqq \theta$ and $\theta^2 \leqq \theta\bar{\theta}\theta \leqq \theta$; hence θ is a congruence.

The set $\Theta(A)$ of congruences on A is not always a sublattice or a subsemigroup of $Q(A)$. We shall give below some conditions for $\Theta(A)$ to be so.

The product $\varphi\psi$ of two congruences φ and ψ becomes a congruence if and only if they are permutable; hence

THEOREM 2.8. *Congruences on an algebra A form a subsemigroup of $Q(A)$ if and only if they are permutable.*

Let φ and ψ be congruences on A and $\varphi \vee \psi$ the congruence generated by $\varphi\psi$. Then $\varphi \cup \psi \leqq \varphi\psi \leqq \varphi \vee \psi$. Hence we can infer from Theorem 2.5,

THEOREM 2.9. *If quasi-congruences on an algebra A are transitive, then congruences on A form a sublattice of $Q(A)$. If congruences on A form a sublattice of $Q(A)$, then they are permutable.*

As shown above the transitivity or symmetricity of quasi-congruences implies the permutability of congruences. Hence if quasi-congruences are class-meromorphisms, then congruences are permutable. But the converse is not true. On the other hand Malcev [2] has proved the following theorem.

THEOREM 2.10 (Malcev). *If congruences on every (F, R)-algebra are permutable, then there exists a polynomial $p(x, y, z)$ such that $p(x, y, y) = x$ and $p(x, x, y) = y$.*

If such a polynomial $p(x, y, z)$ exists, then $a\varphi b$, $a'\varphi b$ and $a'\varphi b'$ imply $a = p(a, a', a')\varphi p(b, b, b') = b'$. Hence

THEOREM 2.11. *If congruences on every* (F, R)*-algebra are permutable, then meromorphisms of every* (F, R)*-algebra onto itself are class-meromorphisms.*

3. Quasi-congruences on a semigroup

We intend to obtain the condition for a semigroup G that every quasi-congruence on G be a congruence. We have succeeded to solve this problem for a commutative semigroup.

THEOREM 3.1. *For a commutative semigroup G the following conditions are equivalent:*

(1) *every quasi-congruence on G is symmetric,*

(2) *G is a group in which every element has a finite order.*

Proof. (1)→(2). Let a be any element of G. If we define $x\theta y$ to mean either $x=y$ or $x=ya^n$ with $n=1, 2, \ldots$, then it is easy to see that θ is a quasi-congruence on G. Since $a^2\theta a$ and θ is symmetric, we get $a\theta a^2$ and $a = a^{n+1}$ $(n=1, 2, \ldots)$. Put $e=a^n$. If $n=1$, then $e^2=a^2=a=e$, and if $n\geqq 2$, then $e^2=a^{n+1}a^{n-1}=aa^{n-1}=a^n=e$. Since $ex\theta x$, we have $x\theta ex$, that is either $x=ex$ or $x=exa^n$, and then we can show $ex=x$ by $e^2=e$; namely e is an identity. Similarly, given $b\in G$, we can find $e'=b^m$ such that $e'x=x$ for all $x\in G$, and then we have $e'=ee'=e'e=e$ and either $b=e$ or $b^{m-1}b=e$; so b has an inverse and a finite order.

Now the implication (2)→(1) can be shown without the commutativity of G. Namely

THEOREM 3.2. *If G is a group in which every element has a finite order, then every quasi-congruence θ on G, regarded as a semigroup, is a congruence.*

Proof. $a\theta b$ and $b\theta c$ imply $ab^{-1}b\theta bb^{-1}c$, that is $a\theta c$. Hence every quasi-congruence on a group is transitive. Suppose that $a\theta b$ and the order of $c=ab^{-1}$ is n. If $n=1$, then $a=b$ and $b\theta a$. If $n\geqq 2$, then $c=ab^{-1}\theta 1$ implies $c^{-1}=c^{n-1}\theta 1$ and $ba^{-1}\theta 1$; whence we get $b\theta a$. Thus θ is a congruence.

As is already known, a congruence θ on a group G regarded as a semigroup becomes that on G regarded as a group; namely preserves the operation $f(x) = x^{-1}$. On the other hand every meromorphism between groups, preserving

$f(x)=x^{-1}$, is a class-meromorphism. Hence Theorem 3.1. shows that a quasi-congruence on a group G regarded as a semigroup is not necessarily that on G regarded as a group and further the permutability of quasi-congruences on a semigroup does not imply the symmetricity of those.

4. Quasi-congruences on a lattice

In the present section we intend to discuss the properties of quasi-congruences on a lattice with the operations $\cup$ and $\cap$. A semilattice on which quasi-congruences are symmetric is trivial. For every element of a semilattice L, regarded as a commutative semigroup under the multiplication $\cup$, is idempotent, and so L can contain no element other than one element 1 if it forms a group. This follows also from the fact that the relation $\leqq$ becomes a quasi-congruence in a semilattice or a lattice; hence

THEOREM 4.1. *Some quasi-congruence on a lattice (semilattice) L is not symmetric, provided L contains two or more elements.*

Then we consider the transitivity of quasi-congruences on a lattice L.

LEMMA 4.1. *Let θ be a quasi-congruence on a lattice L. If the implication $a\theta b\theta c \rightarrow a\theta c$ holds for the cases $a\leqq b\leqq c$ and $a\geqq b\geqq c$, then $\theta^2=\theta$.*

Proof. $a\theta b\theta c$ implies $a\cup a\theta a\cup b$, $a\cup b\cup b\theta a\cup b\cup c$ and $a\theta a\cup b\cup c$, since $a\leqq a\cup b\leqq a\cup b\cup c$. Similarly $a\cup b\cup c\theta b\cup c\theta c$ implies $a\cup b\cup c\theta c$. Then we have $a\cap(a\cup b\cup c)\theta(a\cup b\cup c)\cap c$, that is $a\theta c$.

Now we call an element m of a lattice *modular* if $x\leqq y$ implies $x\cup(m\cap y)=(x\cup m)\cap y$.

THEOREM 4.2. *Let m be a modular element in a lattice L. If all intervals containing m are complemented, then quasi-congruences on L are transitive.*

Proof. We shall show for $a\leqq b\leqq c$ that $a\theta b\theta c$ implies $a\theta c$. Let x be a relative complement of $b\cup m$ in the interval $[a\cap m, c\cup m]$ and y that of $(b\cup x)\cap m$ in $[a\cap m, m]$. Then we get

$$a=a\cup(a\cap m)=a\cup(x\cap(b\cup m))\theta b\cup(x\cap(c\cup m))=b\cup x,$$
$$y=(a\cap m)\cup y\theta((b\cup x)\cap m)\cup y=m$$

and

$$a = a \cup (a \cap m) = a \cup (y \cap ((b \cup x) \cap m)) = a \cup (y \cap (b \cup x))\theta$$
$$(b \cup x) \cup (m \cap (c \cup x)) = (b \cup x \cup m) \cap (c \cup x) = (c \cup m) \cap (c \cup x);$$

accordingly $c \cap a\theta c \cap (c \cup m) \cap (c \cup x)$, that is $a\theta c$.

Dually we can show that $a \geqq b \geqq c$ and $a\theta b\theta c$ imply $a\theta c$. Hence it follows from Lemma 4.1 that θ is transitive.

A lattice with 0 in which all intervals $[0, x]$ are complemented is called *section-complemented*. For a lattice L without 0 we shall define L to be section-complemented when every element of L is contained in a section-complemented principal dual ideal. If a lattice L is section-complemented, then any triple $\{x, y, z\}$ is contained in a section-complemented dual ideal $S = [a)$, in which the condition in Theorem 4.2 holds; hence by Theorem 2.3 we infer

COROLLARY 1. *In a section-complemented lattice every quasi-congruence is transitive.*

Further, by Theorem 2.5 we can assert the following propositions in our previous paper [1].

COROLLARY 2. *If all intervals of a lattice L containing a modular element m are complemented, then congruence relations on L are permutable.*

COROLLARY 3. *On a section-complemented lattice congruence relations are permutable.*

Next we shall inquire into the structure of the lattice $Q(L)$ of quasi-congruences on a lattice L. It is well-known that congruence relations on a

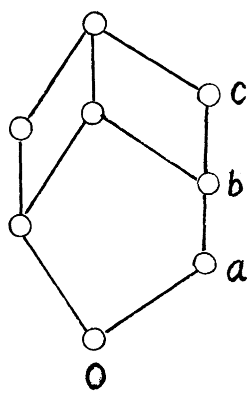

FIG. 1

lattice form a distributive lattice. However the lattice $Q(L)$ is not necessarily modular. Indeed if we set in the lattice of Fig. 1

$$\theta=\theta(0, b), \ \varphi=\theta(b, c) \text{ and } \psi=\theta(a, c),$$

then $\varphi\leqq\psi$ and $a\to c((\varphi\cup\theta)\cap\psi)$ holds nevertheless $a\to c(\varphi\cup(\theta\cap\psi))$ does not hold.

LEMMA 4.2. *If we define in a lattice* L $a\omega b$ *to mean* $a\leqq b$, *then* ω *is a quasi-congruence on* L *and a lower distributive element in* $Q(L)$: $\omega\cap(\varphi\cup\psi)=(\omega\cap\varphi)\cup(\omega\cap\psi)$ *for all* $\varphi, \psi\in Q(L)$.

Proof. Put $\rho=\omega\cap(\varphi\cup\psi)$, $\varphi_0=\omega\cap\varphi$, $\psi_0=\omega\cap\psi$ and $\sigma=\varphi_0\cup\psi_0$. It suffices to show $\rho\leqq\sigma$. As is mentioned in § 1, $x\rho y$ implies that a lattice polynomial p exists such that

$$x=p(a_1, \dots, a_l, s_1, \dots, s_m, u_1, \dots, u_n),$$
$$y=p(a_1, \dots, a_l, t_1, \dots, t_m, v_1, \dots, v_n)$$

and $x\leqq y$, $s_i\varphi t_i$, $u_j\psi v_j$. Then since $s_i\varphi s_i\cup t_i$ and $u_j\psi u_j\cup v_j$, we get $s_i\varphi_0 s_i\cup t_i$ and $u_j\psi_0 u_j\cup v_j$. Hence if we put

$$z=p(a_1, \dots, a_l, s_1\cup t_1, \dots, s_m\cup t_m, u_1\cup v_1, \dots, u_n\cup v_n),$$

then we get $x\leqq y\leqq z$, $x\sigma z$ and $x=x\cap y\sigma z\cap y=y$, proving $\rho\leqq\sigma$.

Dually we define $a\omega' b$ to mean $a\geqq b$. Then we can show

LEMMA 4.3. *If* $\theta\cap(\varphi\cap\psi)=(\theta\cap\varphi)\cup(\theta\cap\psi)$ *holds for the cases* $\theta, \varphi, \psi\leqq\omega$ *and* $\theta, \varphi, \psi\leqq\omega'$ *in* $Q(L)$, *then* $Q(L)$ *is distributive.*

Proof. Let θ, φ and ψ be any quasi-congruences on L and put $\rho=\theta\cap(\varphi\cup\psi)$, $\sigma=(\theta\cap\varphi)\cup(\theta\cap\psi)$. Then by Lemma 4.2 we get $\omega\cap\rho=(\omega\cap\theta)\cap((\omega\cap\varphi)\cup(\omega\cap\psi))$, and by the assumption $\omega\cap\rho=(\omega\cap\theta\cap\varphi)\cup(\omega\cap\theta\cap\psi)\leqq\sigma$. Hence $x\rho y$ implies $x\cap y\rho y$, $x\cap y(\omega\cap\rho)y$ and $x\cap y\sigma y$. Dually we can show that $x\rho y$ implies $x\sigma x\cap y$. Then we have $(x\cap y)\cup x\sigma y\cup(x\cap y)$, $x\sigma y$ and thus $\rho\leqq\sigma$.

THEOREM 4.3. *If all quasi-congruences on a lattice are transitive, then they form a distributive lattice.*

Proof. By Lemma 4.3, it is sufficient to prove $\theta\cap(\varphi\cup\psi)=(\theta\cap\varphi)\cup(\theta\cap\psi)$ for $\theta, \varphi, \psi\leqq\omega$. Put $\rho=\theta\cap(\varphi\cup\psi)$ and $\sigma=(\theta\cap\varphi)\cup(\theta\cap\psi)$. Since σ is transitive, we can write $\sigma=(\theta\cap\varphi)(\theta\cap\psi)$ by Theorem 2.5. If $x\rho y$, then we have

$$x=p(a_1, \dots, a_l, s_1, \dots, s_m, u_1, \dots, u_n),$$

$$y = p(a_1, \ldots, a_l, t_1, \ldots, t_m, v_1, \ldots, v_n)$$

with $s_i \varphi t_i$, $u_j \psi v_j$. If we put

$$z = p(a_1, \ldots, a_l, t_1, \ldots, t_m, u_1, \ldots, u_n),$$

then $x\varphi z$, $z\psi y$ and $x \leqq z \leqq y$, since φ, $\psi \leqq \omega$. Since $x\theta y$, $x = x \cap z\theta y \cap z = z$ and $z = x \cup z\theta y \cup z = y$. Hence we have $x(\theta \cap \varphi)z$, $z(\theta \cap \psi)y$ and $x(\theta \cap \varphi)(\theta \cap \psi)y$; namely $x\sigma y$. Thus $\theta \cap (\varphi \cup \psi) = (\theta \cap \varphi) \cup (\theta \cap \psi)$.

COROLLARY. *The lattice of quasi-congruences on a section-complemented lattice is distributive.*

THEOREM 4.4. *The lattice of quasi-congruences on a distributive lattice is distributive.*

Proof. Put $\rho = \theta \cap (\varphi \cup \psi)$ and $\sigma = (\theta \cap \varphi) \cup (\theta \cap \psi)$ for quasi-congruences θ, φ, $\psi \leqq \omega$, and assume that $x\rho y$. Then we can write

$$x = p(a, s, u) = p(a_1, \ldots, a_l, s_1, \ldots, s_m, u_1, \ldots, u_n),$$
$$y = p(a, t, v) = p(a_1, \ldots, a_l, t_1, \ldots, t_m, v_1, \ldots, v_n)$$

with $s_i \varphi t_i$, $u_j \psi v_j$. We define two weights $w_1(p)$ and $w_2(p)$ of the polynomial p by $w_1(p) = m + n$ and $w_2(p) = l + m + n$. We shall prove $x\sigma y$ by induction on $w_1(p)$ and $w_2(p)$. If $w_1(p) \geqq 2$, we can write either $p = p_1 \cap p_2$ or $p = p_1 \cup p_2$ with $w_1(p) = w_1(p_1) + w_1(p_2)$, $w_2(p) = w_2(p_1) + w_2(p_2)$, $w_1(p_i) \geqq 0$ and $w_2(p_i) \geqq 1$. We may deal only with the case $p = p_1 \cap p_2$.

Case 1. $w_1(p_1) < w_1(p)$, $w_1(p_2) < w_1(p)$. Since $x\rho y$ and

$$x \leqq y \cap p_1(a, s, u) \leqq y \cap p_1(a, t, v) = y,$$

we get $y \cap p_1(a, s, u)\rho y \cap p_1(a, t, v)$. Since $w_1(y \cap p_1) = w_1(p_1) < w(p)$, we get $y \cap p_1(a, s, u)\sigma y \cap p_1(a, t, v) = y$, by the hypothesis of induction, and similarely $y \cap p_2(a, s, u)\sigma y$. Then

$$x = (y \cap p_1(a, s, u)) \cap (y \cap p_2(a, s, u))\sigma y.$$

Case 2. $w_1(p_1) = w_1(p)$, $w_1(p_2) = 0$. If we put $b = p_2(a)$, then $x = p_1(a, s, u) \cap b$, $y = p_1(a, t, v) \cap b$ and hence $x = p_1(a, s, u) \cap y$, $y = p_1(a, t, v) \cap y$. We can write either $p_1 = p_3 \cap p_4$ or $p_1 = p_3 \cup p_4$ in the same manner as above. If $p_1 = p_3 \cap p_4$, then by regarding p_3 and $p_4 \cap b$ as p_1 and p_2 we can reduce to either Case 1 or the case $p_1 = p_3 \cup p_4$. Hence we may assume that $p_1 = p_3 \cup p_4$.

Case 2.1. $w_1(p_3) < w_1(p_1)$, $w_1(p_4) < w_1(p_1)$. Since $x\rho y$ and

$$x=(p_3(a, s, u)\cap y)\cup x\leqq(p_3(a, t, v)\cap y)\cup x\leqq y,$$

we get $(p_3(a, s, u)\cap y)\cup x\rho(p_3(a, t, v)\cap y)\cup x$ and $w_1'((p_3\cap y)\cup x)=w_1'(p_3)<w_1(p)$. Hence we have $x\sigma(p_3(a, t, v)\cap y)\cup x$, $x\sigma(p_4(a, t, v)\cap y)\cup x$ and $x\sigma(p_3(a, t, v)\cap y)\cup x\cup(p_4(a, t, v)\cap y)\cup x=(p_1(a, t, v)\cap y)\cup x=y$ by the distributivity.

Case 2.2. $w_1(p_3)=w_1(p_1)$, $w_1(p_4)=0$. Then we can write, putting $p_4(a)=c$,

$$x=(p_3(a, s, u)\cup c)\cap y=(p_3(a, s, u)\cap y)\cup(c\cap y),$$
$$y=(p_3(a, t, v)\cup c)\cap y=(p_3(a, t, v)\cap y)\cup(c\cap y)$$

and $x=(p_3(a, s, u)\cap y)\cup x$, $y=(p_3(a, t, v)\cap y)\cup x$, since $c\cap y\leqq x$. We may assume $p_3=p_5\cap p_6$ without loss of generality. Then since $x\rho y$ and

$$x\leqq(p_5(a, s, u)\cap y)\cup x\leqq(p_5(a, t, v)\cap y)\cup x=y,$$

we have $(p_5(a, s, u)\cap y)\cup x\rho(p_5(a, t, v)\cap y)\cup x$. Since $w_2((p_5\cap y)\cup x)=w_2(p_5)+2$ and $w_2(p_5)<w_2(p_3)<w_2(p_1)<w_2(p)$, $w_2((p_5\cap y)\cup x)<w_2(p)$. Hence we can infer $(p_5(a, s, u)\cap y)\cup x\sigma(p_5(a, t, v)\cap y)\cup x=y$, by the hypothesis of induction, and $(p_6(a, s, u)\cap y)\cup x\sigma y$. Then

$$\begin{aligned}x&=(p_5(a, s, u)\cap p_6(a, s, u)\cap y)\cup x\\&=((p_5(a, s, u)\cap y)\cup x)\cap((p_6(a, s, u)\cap y)\cup x)\sigma y,\end{aligned}$$

completing the proof.

It seems the distributivity of $Q(L)$ may be deduced from more weaker conditions on L. For instance we guess that $Q(L)$ may be distributive for a modular lattice L. Further we intend to inquire into the structure of a lattice L by the investigation of $Q(L)$ but we have obtained no useful result on it.

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