GEOMETRY OF GROUP REPRESENTATIONS

G. DE B. ROBINSON

To the memory of TADASI NAKAYAMA

The many unanswerable questions (1) which arise in the study of finite groups have lead to a review of fundamental ideas, e.g. the Theorem of Burnside (3, p. 299 ; 2, 6) that *if λ be any faithful irreducible representation of G over a field K, then every irreducible representation of G over K is contained in some tensor power of λ.*

If we take K to be the complex field and write the inner tensor product in question $\lambda \times \lambda \times \cdots$ (n factors) as $\lambda \times^n$, we recall Schur's result that this representation of G splits according to the formula (7, p. 129)

1.1 $$\lambda \times^n = \sum f_\nu \lambda \otimes [\nu]$$

where $\lambda \otimes [\nu]$ is the *symmetrized* inner product associated with the irreducible representation $[\nu]$ of degree f_ν of the symmetric group S_n. For a finite group G, $\lambda \otimes [\nu]$ is in general reducible, while for the full linear group and certain of its subgroups this representation is irreducible.

These symmetrized tensor products are hard to handle, though their degrees δ^ν are given by the formula (5, p. 60)

1.2 $$\delta^\nu(f_\lambda) = G^\nu(f_\lambda)/H^\nu$$

where f_λ is the degree of λ. If we denote the Young diagram associated with the irreducible representation ν of S_n by $[\nu]$, then H^ν is the product of hook length of $[\nu]$ and $G^\nu(f_\lambda) = \prod_{i,j} (f_\lambda + j - i)$, taken over $[\nu]$. It follows immediately that for $n \leq f_\lambda$, all these symmearized products are defined.

It would be interesting if Burnside's theorem could be refined so as to relate the apperances of the different irreducible representations of G to these symmetrized components of $\lambda \times^n$, but the difficulties seem insurmountable at present.

Received May 14, 1965.

2. Another application of these tensor products is of interest. In Chapter XII of (3) Burnside studies at some length the permutation representation g_i of G induced by the identity representation of a subgroup H_i $(i=1, 2, \ldots, r)$ of orders h_i. It is natural to arrange the H_i so that $H_1 = I$ and g_1 is the regular representation of G, $h_i \leq h_{i+1} \leq h_r$ with $H_r = G$ so that g_r is the identity representation of G. If we suppose g_i to be represented on the variables x_u and g_j on the variables y_v, the tensor product $g_i \times g_j$ is represented on the variables $x_u y_v$ and

2.1 $$g_i \times g_j = \sum a_{ijk} g_k.$$

If $j=i$, we obtain the symmetrized components for $n=2$ on the variables (5, p. 57).

$$x_1y_1,\ x_2y_2,\ \ldots,\ \frac{1}{2}(x_u y_v + x_v y_u)\ ;\ \ldots\ \frac{1}{2}(x_u y_v - x_v y_u)$$

by setting $y_u = x_u$. It follows, as in the case of $g_i \times g_j$, that $g_i \otimes [2]$ *is also a permutation representation of* G, while $g_i \otimes [1^2]$ is not. The argument is quite general so that 2.1 becomes

2.2 $$g_i \times^n = \sum_j a_{ij}^n g_j,$$

and we have

2.3 $$g_i \otimes [n] = \sum_j b_{ij}^n g_j,$$

where the a_{ij}^n, b_{ij}^n are rational integers.

3. What is of interest here is that 2.1-2.3 can be interpreted in a natural way relative to the geometry of the irreducible representations λ of G. A start was made on this many years ago (4). For purposes of illustration, we reproduce two tables which set the stage for this interpretation in the case of S_4. Here we write

$$g_i = \sum_\nu m_i^\nu [\nu]$$

and Table 2 gives the values of the m_i^ν. For completeness, it would have been desirable to list all the solutions of 2.1, but this has been omitted in favour of Table 3 which gives the solutions of 2.2 and 2.3 for $n=2, 3$. Since there are *five* irreducible representations of S_4, we have the following linear relations between the g_i:

TABLE 1

H	sub-group	h
H_1	1	1
H_2	1, (12)	2
H_3	1, (12)(34)	2
H_4	1, (123), (132)	3
H_5	1, (1234), (13)(24), (1432)	4
H_6	1, (12)(34), (14)(23), (13)(24)	4
H_7	1, (12), (34), (12)(34)	4
H_8	1, (12), (13), (23), (123), (132)	6
H_9	1, (12), (34), (12)(34), (14)(23), (13)(24), (1324), (1423)	8
H_{10}	A_4	12
H_{11}	S_4	24

TABLE 2

	$[1^4]$	$[2, 1^2]$	$[2^2]$	$[3.1]$	$[4]$
g_1	1	3	2	3	1
g_2	•	1	1	2	1
g_3	1	1	2	1	1
g_4	1	1	•	1	1
g_5	•	1	1	•	1
g_6	1	•	2	•	1
g_7	•	•	1	1	1
g_8	•	•	•	1	1
g_9	•	•	1	•	1
g_{10}	1	•	•	•	1
g_{11}	•	•	•	•	1
			m_i^ν		

TABLE 3

	$\times^2$	$\times^3$	$\otimes[2]$	$\otimes[3]$
g_1	$24\,g_1$	$576\,g_1$	$8\,g_1+6\,g_2+3\,g_3$	$17\,g_1+4\,g_4$
g_2	$5\,g_1+2\,g_2$	$70\,g_1+4\,g_2$	$g_1+4\,g_2+g_6$	$11\,g_1+7\,g_2+2\,g_4$
g_3	$4\,g_1+4\,g_3$	$64\,g_1+16\,g_3$	$3\,g_2+3\,g_3+g_5$	$10\,g_1+9\,g_3+2\,g_4$
g_4	$2\,g_1+2\,g_4$	$20\,g_1+4\,g_4$	$g_2+g_3+g_4+g_8$	$4\,g_1+3\,g_4$
g_5	$g_1+2\,g_5$	$8\,g_1+4\,g_5$	$g_2+g_5+g_9$	$g_1+g_3+g_4+g_5$
g_6	$6\,g_6$	$36\,g_6$	$3\,g_6+g_9$	$9\,g_6+g_{10}$
g_7	$g_1+2\,g_7$	$8\,g_1+4\,g_7$	$g_2+g_7+g_9$	$g_1+g_3+2\,g_7+2\,g_8$
g_8	g_2+g_8	$g_1+3\,g_2+g_8$	g_7+g_8	$g_2+2\,g_8$
g_9	g_6+g_9	$4\,g_6+g_9$	$2\,g_9$	$g_6+g_9+g_{11}$
g_{10}	$2\,g_{10}$	$4\,g_{10}$	$g_{10}+g_{11}$	
g_{11}	g_{11}	g_{11}		

$$2\,g_6+g_1=3\,g_3 \qquad 2\,g_9+g_1=g_2+g_3+g_5$$
$$2\,g_7+g_1=2\,g_2+g_3 \qquad 2\,g_{10}+g_1=g_3+2\,g_4$$
$$2\,g_8+g_1=2\,g_2+g_4 \qquad 2\,g_{11}+g_1=g_2+g_4+g_5$$

Consider, in particular the irreducible representation [3, 1] whose invariant configuration is a regular tetrahedron. Since $H_4\subset H_8$, the groups of stability of the vertices are H_8 and its conjugates. Taking the bi-vector defined by two such vertices, we have from Table 3,

$$g_8\times{}^2=g_8+g_2$$

which indicates that the group of stability of the corresponding edge is H_2 with $m_2^{[3,1]}=2$. However, this does not take into account the extra symmetry arising by interchanging the two vertices. For this we go to

$$g_8\otimes[2]=g_8+g_7,$$

and the group of stability of the mid-edge point is H_7. As already mentioned, the component

$$g_8\otimes[1^2]=[3,\ 1]+[2,\ 1^2]$$

has no geometrical significance.

We may study the geometry of the representation $[2,\ 1^2]$ in a similar fashion, noting from Table 2 that only the vertices of the fundamental region are well defined; since $H_3\subset H_5$, the groups of stability are H_2, H_4 and H_5 and their conjugates. It may be verified that

$$g_2\times g_4=4\,g_1,\ g_2\times g_5=3\,g_1,\ g_4\times g_5=2\,g_1$$

and from Table 3

$$g_2\times{}^2=5\,g_1+2\,g_2,\ g_4\times{}^2=2\,g_1+g_4,\ g_5\times{}^2=g_1+2\,g_5.$$

Moreover, these inner products and the $g_i\otimes[2]$ and $g_i\otimes[3]$ $(i=2,\ 4,\ 5)$ interpreted relative to $[2,\ 1^2]$, describe the familiar arrangement of the vertices, mid-edge and mid-face points, of the octahedron, since the rotation group of the octahedron is isomorphic to the representation $[2,\ 1^2]$ of S_4.

4. Thus it appears that the geometry of the fundamental region of a real irreducible λ can be completely described in terms of $g_i\times g_j$ and $g_i\otimes[n]$. In order to clarify further these ideas, consider the relation

$$g_7 \otimes [2] = g_7 + g_9 + g_2$$

which is more interesting than $g_3 \otimes [2] = g_3 + g_7$, since the octahedron is centrally symmetrical. Denoting the mid-point of the edge ij of the tetrahedron by P_{ij}, we have three possibilities: i) pairing P_{12} with P_{12} yields g_7; ii) pairing P_{12} with P_{34} allows an extra symmetry, since H_7 is invariant under (1324), which yields g_9; iii) pairing P_{12} with P_{13} yields a point on the edge of the fundamental region and so g_2. Since no point is invariant under H_7 and also (1324), g_9 does not register in either [3, 1] or [2, 1^2].

In particular, if H_i is a group of stability with $m_i^\lambda = 1$, considerations of linear dependence imply that

4.1 *$g_i \otimes [n]$ yields every g_j with $m_j^\lambda = 1$, for n sufficiently large.*

The geometry of the octahedron suggests immediately that $g_5 \otimes [3]$ yields g_4 but we must go to $g_2 \otimes [4]$ and $g_4 \otimes [4]$ to obtain g_5, as may readily be verified.

These ideas may be extended to apply to complex λ but we shall not consider such a genralization here.

References

[1] R. Brauer, *On finite groups and their characters.* Bull. Amer. Math. Soc. **69** (1936), 125-130.

[2] R. Brauer, *A note on theorems of Burnside and Blichfeldt.* Proc. Amer. Math. Soc. **15** (1964), 31-34.

[3] W. Burnside, *Theory of groups of finite order*, 2nd. ed. (Cambridge, 1911).

[4] G. de B. Robinson, *On the fundamental region of an orthogonal representation of a finite group.* Proc. London Math. Soc. **43** (1937), 289-301.

[5] G. de B. Robinson, *Representation theory of S_n*, (Toronto, 1961).

[6] R. Steinberg, *Complete sets of representations of algebras.* Proc. Amer. Math. Soc. **13** (1962), 746-747.

[7] H. Weyl, *Classical Groups*, (Princeton, 1946).

University of Toronto