ON BLOCK IDEMPOTENTS OF MODULAR GROUP RINGS

MASARU OSIMA

To the memory of TADASI NAKAYAMA

We consider a group G of finite order $g = p^a g'$, where p is a prime number and $(p, g') = 1$. Let Ω be the algebraic number field which contains the g-th roots of unity. Let $K_1, K_2, \ldots, K_n$ be the classes of conjugate elements in G and the first $m(\leqq n)$ classes be p-regular. There exist n distinct (absolutely) irreducible characters $\chi_1, \chi_2, \ldots, \chi_n$ of G. Let $\mathfrak{o}$ be the ring of all algebraic integers of Ω and let $\mathfrak{p}$ be a prime ideal of $\mathfrak{o}$ dividing p. If we denote by $\mathfrak{o}^*$ the ring of all $\mathfrak{p}$-integers of Ω, then $\mathfrak{p}$ generates an ideal $\mathfrak{p}^*$ of $\mathfrak{o}^*$ and we have

$$\Omega^* = \mathfrak{o}^*/\mathfrak{p}^* \cong \mathfrak{o}/\mathfrak{p}$$

for the residue class field. The residue class map of $\mathfrak{o}^*$ onto Ω^* will be denoted by an asterisk; $\alpha \to \alpha^*$.

Let $\Gamma = \Gamma(G)$ be the modular group ring of G over Ω^* and let

$$Z = Z_1 \oplus Z_2 \oplus \cdots \oplus Z_s$$

be the decomposition of the center $Z = Z(G)$ of Γ into indecomposable ideals Z_σ. Then the ordinary irreducible characters χ_i and the modular irreducible characters φ_κ of G (for p) are distributed into s blocks $B_1, B_2, \ldots, B_s$, each χ_i and φ_κ belonging to exactly one block B_σ. We determined in [6] explicitly the primitive orthogonal idempotents δ_σ of Z corresponding to B_σ in the following way. We set

$$b_\alpha = \sum_{\chi_i \in B_\sigma} z_i \chi_i(a_\alpha^{-1})/g \qquad (a_\alpha \in K_\alpha)$$

where $z_i = \chi_i(1)$. Let U_κ be the indecomposable constituent of the regular representation of G corresponding to the modular irreducible representation F_κ and denote by u_κ its degree. We see that $b_\alpha = \sum_{\varphi_\kappa \in B_\sigma} u_\kappa \varphi_\kappa(a_\alpha^{-1})/g \in \mathfrak{o}^*$ for p-regular

Received July 13, 1965.

classes K_α since $p^a | u_\kappa$ ($\kappa = 1, 2, \ldots, m$). On the other hand $b_\alpha = 0$ for $m < \alpha \leqq n$. Then we have

$$(1) \qquad \delta_\sigma = \sum_{\alpha=1}^{m} b_\alpha^* K_\alpha$$

where the sum of the elements of K_α is also denoted by K_α. In what follows we shall call δ_σ the block idempotents of Γ associated with B_σ or simply the block idempotents of B_σ. Let B_σ be a block of defect d with defect group D. Then $b_\alpha^* = 0$ if the defect group D_α of K_α is not contained in any conjugate of D ([6], Theorem 4, see also [5]). Hence we obtain

$$(2) \qquad \delta_\sigma = \sum_{D_\alpha \subseteq D} b_\alpha^* K_\alpha \qquad (1 \leqq \alpha \leqq m).$$

Here the notation $D_\alpha \subseteq D$ means that D_α is contained in some conjugate of D. In the special case where $p \nmid g$, there exist n modular irreducible characters of G. Further each χ_i forms a block B_σ of its own. Hence

$$(3) \qquad \delta_i = \sum_{\alpha=1}^{n} (z_i \chi_i(a_\alpha^{-1})/g)^* K_\alpha.$$

We consider the fixed block $B = B_\sigma$ of defect d with defect group D. If we define $\nu(s)$ by $p^{\nu(s)} \| s$ for a rational integer s, then there exist characters $\chi_k \in B$ such that $\nu(z_k) = a - d$. We shall first prove that $l = \sum_{\alpha=1}^{m} \chi_k(a_\alpha^{-1}) \, \omega_k(K_\alpha) \not\equiv 0 \pmod{\mathfrak{p}}$ where $\omega_k(K_\alpha) = g_\alpha \chi_k(a_\alpha)/z_k$ and g_α denotes the number of elements of K_α. The main purpose of this short note is to prove the following

THEOREM 1. *Let δ be the block idempotent of B and let $\varepsilon = \sum_{\alpha=1}^{m} c_\alpha^* K_\alpha$ be an element of Z where $c_\alpha = \chi_k(a_\alpha^{-1})/l$. Then $\delta - \varepsilon$ belongs to the radical of Z.*

In the case where $p \nmid g$ we see easily that this fact coincides with the formula (3) since $l = g/z_k$ for every χ_k and rad $Z = 0$.

Let χ_i be any character of B and λ_i be the height of χ_i, that is, $\nu(z_i) = a - d + \lambda_i$ ($\lambda_i \geqq 0$). Let K_β be p-regular classes with defect group $D_\beta = D$. Then $\omega_k(K_\beta) \equiv \omega_i(K_\beta) \pmod{\mathfrak{p}}$ and hence $g_\beta \chi_k(a_\beta)/z_k \equiv g_\beta \chi_i(a_\beta)/z_i \pmod{\mathfrak{p}}$. Then it follows from $g_\beta/z_k \not\equiv 0 \pmod{\mathfrak{p}}$ that

$$(4) \qquad \chi_i(a_\beta) \equiv (z_i/z_k) \chi_k(a_\beta) \qquad (\bmod\, p^{\lambda_i} \mathfrak{p}).$$

Since the modular irreducible characters of B can be expressed by the ordinary irreducible characters of B (restricted to p-regular elements) with integral coefficients, we have for $\varphi_\kappa \in B$

$$\varphi_\kappa = \sum_{\chi_i \in B} r_{\kappa i} \chi_i.$$

Hence, by (4)

$$\varphi_\kappa(a_\beta) \equiv \sum_{\chi_i \in B} (r_{\kappa i} z_i / z_k) \chi_k(a_\beta) \qquad (\text{mod } \mathfrak{p})$$

and consequently

$$\varphi_\kappa(a_\beta) \equiv (f_\kappa / z_k) \chi_k(a_\beta) \qquad (\text{mod } \mathfrak{p}) \tag{5}$$

where $f_\kappa = \varphi_\kappa(1)$.

LEMMA 1. *Let $\chi_k \in B$ be the character of height* 0. *Then* $\sum_{\alpha=1}^{m} \chi_k(a_\alpha^{-1})\, \omega_k(K_\alpha) \not\equiv 0$ (mod $\mathfrak{p}$).

Proof. It follows from (5) that

$$\begin{aligned} b_\beta &= \sum_{\varphi_\kappa \in B} u_\kappa \varphi_\kappa(a_\beta^{-1}) / g \\ &\equiv \sum_{\varphi_\kappa \in B} (u_\kappa f_\kappa / g z_k) \chi_k(a_\beta^{-1}) \qquad (\text{mod } \mathfrak{p}) \end{aligned}$$

and hence

$$b_\beta \equiv \Big(\sum_{\chi_i \in B} z_i^2 / g z_k\Big) \chi_k(a_\beta^{-1}) \qquad (\text{mod } \mathfrak{p}) \tag{6}$$

for p-regular classes K_β with defect group $D_\beta = D$. Since there exist p-regular classes K_γ with defect group $D_\gamma = D$ such that $b_\gamma \not\equiv 0$ (mod $\mathfrak{p}$) and $\chi_k(a_\gamma^{-1}) \not\equiv 0$ (mod $\mathfrak{p}$), we obtain from (6)

$$h = \sum_{\chi_i \in B} z_i^2 / g z_k \not\equiv 0 \qquad (\text{mod } \mathfrak{p}). \tag{7}$$

It follows from (2) that

$$\sum_{D_\beta = D} b_\beta \omega_k(K_\beta) \equiv 1 \qquad (\text{mod } \mathfrak{p})$$

since $\omega_k(K_\alpha) \equiv 0$ (mod $\mathfrak{p}$) for p-regular classes K_α with defect group D_α properly contained in some conjugate of D. Then we have by (6) and (7)

$$h \sum_{D_\beta = D} \chi_k(a_\beta^{-1}) \omega_k(K_\beta) \equiv 1 \qquad (\text{mod } \mathfrak{p}). \tag{8}$$

Hence we see

$$\sum_{D_\beta = D} \chi_k(a_\beta^{-1}) \omega_k(K_\beta) \not\equiv 0 \qquad (\text{mod } \mathfrak{p}).$$

If $\omega_k(K_\alpha) \not\equiv 0$ (mod $\mathfrak{p}$), then $D \subseteq D_\alpha$ and if D is properly contained in some conjugate of D_α, then $\chi_k(a_\alpha)\equiv 0 \pmod{\mathfrak{p}}$. Hence

$$\sum_{\alpha=1}^{m}\chi(a_\alpha^{-1})\omega_k(K_\alpha)\equiv\sum_{D_\beta=D}\chi_k(a_\beta^{-1})\omega_k(K_\beta)\qquad \pmod{\mathfrak{p}}$$

which proves the lemma.

We set $l=\sum_{\alpha=1}^{m}\chi_k(a_\alpha^{-1})\omega_k(K_\alpha)$ and $c_\alpha=\chi_k(a_\alpha^{-1})/l$ and consider the element $\xi=\sum_{\alpha=1}^{m}c_\alpha K_\alpha$ of the center of the ordinary group ring of G. Then

$$\omega_k(\xi)=\sum_{\alpha=1}^{m}\chi(a_\alpha^{-1})\omega_k(K_\alpha)/l=1$$

and hence for any $\chi_i\in B$ we have $\omega_i(\xi)\equiv 1 \pmod{\mathfrak{p}}$. On the other hand, for any $\chi_j\notin B$

$$\omega_j(\xi)=\sum_{\alpha=1}^{m}\chi_k(a_\alpha^{-1})\,\omega_j(K_\alpha)/l=0$$

because $\sum_{\alpha=1}^{m}g_\alpha\chi_k(a_\alpha^{-1})\chi_j(a_\alpha)=0$. This implies that if we set $\varepsilon=\sum_{\alpha=1}^{m}c^*K_\alpha$, then $\delta-\varepsilon\in\operatorname{rad}Z$. This completes the proof of Theorem 1.

If $d_\alpha>d$ where d_α denotes the defect of K_α, then $\chi_k(a_\alpha)\equiv 0 \pmod{\mathfrak{p}}$ and hence $c_\alpha^*=0$. Further if $d_\alpha=d$ and D_α is not conjugate to D, then $\omega(K_\alpha)\equiv 0$ $(\text{mod }\mathfrak{p})$ and $\chi_k(a_\alpha)\equiv 0 \pmod{\mathfrak{p}}$. Thus we have also $c_\alpha^*=0$. It follows from (6), (7) and (8) that $b_\beta^*=c_\beta^*$ for all p-regular classes K_β with defect group $D_\beta=D$.

LEMMA 2. *Let Q be the normal p-subgroup of G. Then the block idempotent δ of B with defect group D is given by*

$$\delta=\sum_{Q\subseteq D_\alpha\subseteq D}b_\alpha^*K_\alpha\qquad(1\leqq\alpha\leqq m).$$

Proof. We see that $b_\alpha^*=0$ for p-regular classes K_α such that Q is not contained in D_α ([6]). This, combined with (2) proves the lemma.

THEOREM 2. *Let B be the block of G with normal defect group D. Then*

$$\varepsilon=\sum_{D_\beta=D}c_\beta^*K_\beta\qquad(1\leqq\beta\leqq m)$$

is the block idempotent of B where $c_\beta=\chi_k(a_\beta^{-1})/l$ and $l=\sum_{D_\beta=D}\chi_k(a_\beta^{-1})\omega_k(K_\beta)$.

Proof. It follows from Lemma 2 that $\delta=\sum_{D_\beta=D}b_\beta^*K_\beta$. Then $\delta=\varepsilon$ since b_β^*

$=c_\beta^*$ for all p-regular classes K_β with defect group $D_\beta=D$.

Now let B_1 be the principal block of G which contains the principal character $\chi_1=1$ and let δ_1 be its block idempotent. Obviously we may choose χ_1 as the character χ_k in Theorem 1. We then have $l=v$ where v denotes the number of p-regular elements in G. If Q is a p-Sylow subgroup of G, then $v\equiv u \pmod p$ where u denotes the number of p-regular elements in the centralizer $C_G(Q)$. Hence

$$\varepsilon_1=(1/v)^*\sum_{\alpha=1}^{m}K_\alpha=(1/u)^*\sum_{\alpha=1}^{m}K_\alpha.$$

If Q is normal in G, then we see by Theorem 2 that

$$\varepsilon_1=(1/u)^*\sum_{D_\beta=Q}K_\beta \qquad (1\leqq\beta\leqq m) \tag{9}$$

is the block idempotent δ_1 of B_1 ([7]).

Some applications of our results will be presented elsewhere.

References

[1] R. Brauer, Zur Darstellungstheorie der Gruppen endlicher Ordnung, I, Math. Zeits. **63** (1956), 406-444.

[2] R. Brauer, Zur Darstellungstheorie der Gruppen endlicher Ordnung, II, Math. Zeits. **72** (1959), 25-46.

[3] R. Brauer, Some applications of the theory of blocks of characters of finite groups, I, J. Alg. **1** (1964), 152-167.

[4] R. Brauer and W. Feit, On the number of irreducible characters of finite groups in a given block. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 361-365.

[5] C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Interscience, New York, London, 1962.

[6] M. Osima, Notes on blocks of group characters, Math. J. Okayama Univ. **4** (1955), 175-188.

[7] M. Osima, On a theorem of Brauer, Proc. Japan Acad. **40** (1964), 795-798.

Osaka University