INVOLUTIVE PROPERTY OF RESOLUTIONS OF DIFFERENTIAL OPERATORS

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Dedicated to the Memory of Professor TADASI NAKAYAMA

§ 0. Introduction

Let E and E' be C^∞ vector bundles over a C^∞ manifold M. Denote by $\Gamma(E)$ (resp. by $\Gamma(E')$) the vector space of C^∞ cross-sections of E (resp. of E') over M. Take a linear differential operator of the first order $D: \Gamma(E) \to \Gamma(E')$ induced by a vector bundle mapping $\sigma(D): J^1(E) \to E'$, where $J^k(E)$ denotes the vector bundle of k-jets of cross-sections of E. Take an integer $l \geq 0$. Then $\sigma(D)$ induces a vector bundle mapping $\sigma^l(D): J^l(J^1(E)) \to J^l(E')$. Now $J^{l+1}(E)$ can be canonically considered as a vector sub-bundle of $J^l(J^1(E))$. Denote by $B_{(l)}$ the image of $J^{l+1}(E)$ by $\sigma^l(D)$. In the case $B_{(l)}$ is a sub-bundle of $J^l(E')$ denote by $E''_{(l)}$ the quotient vector bundle $J^l(E')/B_{(l)}$. Then the canonical projection $J^l(E') \to E''_{(l)}$ induces a linear differential operator $D'_{(l)}: \Gamma(E') \to \Gamma(E''_{(l)})$. We set $E'' = E''_{(1)}$ and $D' = D'_{(1)}$ when they are defined.

We say that a differential operator D is involutive when the equations $Du = 0$ ($u \in \Gamma(E)$) is involutive. It is shown in [5] that if D is involutive then $B_{(l)}$ is a sub-bundle and $D'_{(l)}$ is involutive for sufficiently large l. The purpose of the present note is to show that B is a sub-bundle and D' is involutive.

The reason why such problem is considered is the following: If we consider in the category of real analyticity in stead of in the category of infinitely differentiability and if we assume that $B_{(l)}$ is a sub-bundle and that $D'_{(l)}$ is involutive, then the sequence

$$\Gamma_\omega(E) \to \Gamma_\omega(E') \to \Gamma_\omega(E''_{(l)})$$

is exact, where Γ_ω indicates the sheaf of germs of real analytic cross-sections and the first (resp. the second) arrow denotes D (resp. $D'_{(l)}$) (cf. [5]). Thus our result shows that a linear involutive differential operator D of the first order

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can be resolved by an exact sequence of involutive linear differential operators of the first order : $\Gamma_\omega(E)\to\Gamma_\omega(E')\to\Gamma_\omega(E'')\to\cdots$ provided we stay in the category of real analyticity.

The main tool in the proof is the cohomological treatment of the condition of involutiveness developed by I. M. Singer and S. Sternberg ([7] and [3]). The above question of resolution is also treated in somewhat different or restricted form in [1], [2], [4], [6], and [8].

The treatment of the prolongation of linear differential equation in §2 was taken from [1].

§1. Involutive differential operators

We recall in this section known properties of involutive differential operators and prove a lemma which will be used in §3. Let V be a vector space. Denote by $\otimes_k V$ a tensor product of k copies of V. Denote by $S^k(V)$ (resp. $\Lambda^k(V)$) the vector subspace of elements in $\otimes_k V$ which are symmetric (resp. anti-symmetric) in k components. If A and B are vector subspace of $\otimes_k V$ and $\otimes_l V$, respectively, then we automatically consider $A\otimes B$ as a vector subspace of $\otimes_{k+l}V$. Take a vector space W and denote by g a vector subspace of $W\otimes V$. Set

$$p^l(g)=g\otimes S^l(V)\cap W\otimes S^{l+1}(V), \tag{1.1}$$

$p^l(g)$ is called the l-th prolongation of g. By convention, we set $p^0(g)=g$. Consider a mapping

$$W\otimes S^{l+1}(V)\otimes\Lambda^r(V)\to W\otimes S^l(V)\otimes\Lambda^{r+1}(V)$$

which is induced by operating anti-symmetrization in the last $r+1$ components in $W\otimes(\otimes_{l+r+1}V)$. When $r=0$, we regard the left hand side as $W\otimes S^{l+1}(V)$. Similar convention will be used without mentioning it. Thus this mapping is defined for $l\geq 0$ and $r\geq 0$. This mapping sends $p^l(g)\otimes\Lambda^r(V)$ into $p^{l-1}(g)\otimes\Lambda^{r+1}(V)$, where we set $p^{-1}(g)=W$. Hence the above mapping induces

$$\delta_{l,r}: p^l(g)\otimes\Lambda^r(V)\to p^{l-1}(g)\otimes\Lambda^{r+1}(V)\ (l,\ r\geq 0). \tag{1.2}$$

We set $H^{l,r}(g)=$ (the kernel of $\delta_{l,r}$)/(the image of $\delta_{l+1,r-1}$) for $l\geq 0$ and $r\geq 1$. We say that g is an involutive subspace of $W\otimes V$ when $H^{l,r}(g)=0$ for all $l\geq 0$ and $r\geq 1$.

Let g be an involutive subspace of $W\otimes V$. Then we claim that, for all $l\geq 2$ and $l-1\geq r\geq 1$, the following inclusion holds:

$$(1.3)_{l,r}\qquad \begin{aligned} & g\otimes S^1\otimes S^{l-1}\cap(W\otimes S^2\otimes S^{l-1}+W\otimes S^1\otimes S^l) \\ & \subseteq g\otimes S^{r+1}\otimes S^{l-r-1}+p(g)\otimes S^1\otimes S^{l-2}+p^2(g)\otimes S^1\otimes S^{l-3}+ \\ & +\cdots+p^r(g)\otimes S^1\otimes S^{l-r-1}, \end{aligned}$$

where we abbreviate $S^l(V)$ by S^l. To prove this, take a base x_t (resp. e^λ) of V (resp. W) and choose a base $a_\lambda^{i\rho}e^\lambda\otimes x_i$ of g. We denote by J_r (resp. K_l) general symmetric index $j_1\cdots j_r$ (resp. $k_1\cdots k_l$). Consider first $(1.3)_{2,1}$. Take $a_\lambda^{i\rho}u_\rho^{j,k}e^\lambda\otimes x_i\otimes x_j\otimes x_k$ in the left hand side. Then

$$(a_\lambda^{i\rho}u_\rho^{j,k}-a_\lambda^{i\rho}u_\rho^{k,j})e^\lambda\otimes x_i\otimes x_j\otimes x_k$$

is in the kernel of $\delta_{0,2}$. Since $H^{0,2}(g)=0$, the above element can be written as $a_\lambda^{i\rho}\,(v_\rho^{j,k}-v_\rho^{k,i})e^\lambda\otimes x_i\otimes x_j\otimes x_k$, where $a_\lambda^{i\rho}v_\rho^{j,k}e^\lambda\otimes x_i\otimes x_j$ is in $p(g)$ for all k. This means that $a_\lambda^{i\rho}\,(u_\rho^{j,k}-v_\rho^{j,k})$ is symmetric in j, k. Hence we have $(1.3)_{2.1}$. For $l>2$, we prove $(1.3)_{l,r}$ by induction of r. The case $r=1$ is an immediate consequence of $(1.3)_{2,1}$.

For $r>1$, take ω in the left hand side of $(1,1)_{l,r}$. We can write by the induction assumption $\omega=\Theta+\omega_1+\cdots+\omega_{r-1}$, where $\Theta\in g\otimes S^r\otimes S^{l-r}$ and $\omega_t\in p^t(g)\otimes S\otimes S^{l-t-1}$. Set $\Theta=a_\lambda^{i\rho}u_\rho^{J_r,K_{l-r}}e^\lambda\otimes x_{J_r}\otimes x_{K_{l-r}}$. Then $a_\lambda^{i\rho}(u_\rho^{J_{r-1}j,kK_{l-r-1}}-u_\rho^{J_{r-1}k,jK_{l-r-1}})e^\lambda\otimes x_{J_{r-1}}\otimes x_j\otimes x_k\otimes x_{K_{l-r-1}}$ is in $p^{r-1}(g)\otimes\Lambda^2\otimes S^{l-r-1}$ because of the condition for ω and of the fact that $\omega_t\varepsilon p^t(g)\otimes S\otimes S^{l-t-1}$. Since $H^{r-1,2}(g)=0$, the above element is in the image of $\delta_{r,1}$. Then $(1.3)_{l,r}$ follows easily. This completes the proof of $(1.3)_{l,r}$. By applying the symmetrization to the last $l-1$ components, we deduce from. $(1.3)_{l,l-1}$ the following:

LEMMA 1. *Assume that g is an involutive submodule of $W\otimes V$. Then for $l\geq 2$*

$$g\otimes S\otimes S^{l-1}\cap(W\otimes S^2\otimes S^{l-1}+W\otimes S\otimes S^l)=p(g)\otimes S^{l-1}+g\otimes S^l.$$

Let $D:\Gamma(E)\to\Gamma(E')$ be a differential operator of the first order induced by a vector bundle mapping $\sigma(D):J^1(E)\to E'$. Assume that the kernel A of $\sigma(D)$ is a vector sub-bundle of $J^1(E)$. Then we have an exact sequence

$$(1.4)\qquad 0\to A\to J^1(E)\to E'\to 0.$$

This induces an exact sequence for $l\geq 0$

(1.5) $$0 \to J^l(A) \to J^l(J^1(E)) \to J^l(E') \to 0.$$

Now $J^{l+1}(E)$ is a vector sub-bundle of $J^l(J^1(E))$. Set

(1.6) $$p^l(A) = J^{l+1}(E) \cap J^l(A).$$

$p^l(A)$ may not be a vector sub-bundle. On the other hand we have a canonical exact sequence for $l \geq 1$

(1.7) $$0 \to E \otimes S^l(T^*) \to J^l(E) \to J^{l-1}(E) \to 0$$

where the last arrow is the canonical projection, T^* is the dual bundle of tangent vector bundle of M. Set

(1.8) $$g = A \cap E \otimes S^1(T^*),$$

(1.9) $$p^l(g) = p^l(A) \cap E \otimes S^{l+1}(T^*).$$

g is called the homogenous part of A. Take a point x in M. For a subset B of a vector bundle over M denote by B_x the intersection of B with the fiber over x. The $p^l(g)_x$ is a subspace of $E_x \otimes S^{l+1}(T_x^*)$. On the other hand, g_x is a subspace of $E_x \otimes T_x^*$. Hence we have $p^l(g_x)$ as a subspace of $E_x \otimes S^{l+1}(T_x^*)$ (cf. (1.1)). We have

(1.10) $$p^l(g)_x = p^l(g_x).$$

By definition D is involutive if and only if the following two conditions are satisfied:

1°) $p^1(A)$ is a vector sub-bundle of $J^2(E)$ and the canonical projection $p^1(A) \to A$ (induced by $J^2(E) \to J^1(E)$) is surjective.

2°) g_x is an involutive subspace of $E_x \otimes T_x^*$ for any x in M.

If D is involutive, it follows that $p^l(A)$ and $p^l(g)$ are sub-bundles for all $l \geq 0$ and the canonical projection $p^{l+1}(A) \to p^l(A)$ is surjective.

§ 2. Observation on exact sequences

We state in this section some trivial manipulation on commutative diagrams of exact sequences of homomorphism of vector spaces. All arrows in this section are supposed to be homomorphism of vector spaces. By a diagram of exact sequences we mean a diagram in which every horizontal as well as vertical sequence in the diagram is exact. Notations in this section is independent of previous sections.

Assume that we have a commutative diagram of exact sequences of the type:

$$\text{(I)}\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & A & \to & A'' & \to & 0 \end{array}$$

Take a commutative diagram of exact sequences

$$\text{(II)}\qquad \begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P' & \to & P & \to & P'' & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

Denote by Q, Q', Q'' the image of B, B', B'' by the vertical arrows in (II), respectively. Then it is easy to see that we have a commutative diagram of exact sequences

$$\text{(III)}\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Q' & \to & Q & \to & Q'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P' & \to & P & \to & P'' & \to & 0 \end{array}$$

(III) will be called the image of (I) by (II).

Take another commutative diagram of exact sequences

$$\text{(IV)}\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' & \to & C & \to & C'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & A & \to & A'' & \to & 0. \end{array}$$

Then the following comutative diagram is exact

$$\text{(V)}\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B'\cap C' & \to & B\cap C & \to & B''\cap C'' & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \end{array}$$

Moreover, it is easy to verify that $B\cap C\to B''\cap C''\to 0$ is exact provided

$$(B+C)\cap A' = B'+C' \tag{2.1}$$

(V) will be called the cut of (IV) by (I).

§ 3. Involutiveness of D'

Take an involutive differential operator $D: \Gamma(E) \to \Gamma(E')$ and we use the notation in § 0. We abbreviate $S^l(T^*)$ by S^l, and $J^l(E)$ by J^l. We have commutative diagrams of exact sequences,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E\otimes S^{l+1} & \to & J^{l+1} & \to & J^l & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J^1\otimes S^l & \to & J^l(J^1) & \to & J^{l-1}(J^1) & \to & 0, \end{array} \tag{3.1}$$

and

$$\begin{array}{ccccccccc} 0 & \to & J^1\otimes S^l & \to & J^l(J^1) & \to & J^{l-1}(J^1) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E'\otimes S^l & \to & J^l(E') & \to & J^{l-1}(E') & \to & 0 \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array} \tag{3.2}$$

where (3.2) is induced by $\sigma(D): J^1 \to E'$. Taking the image of (3.1) by (3.2), we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & h^l & \to & B^l & \to & B^{l-1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & E'\otimes S^l & \to & J^l(E') & \to & J^{l-1}(E') & \to & 0 \end{array} \tag{3.3}$$

By definition we have exact sequences

$$0 \to p^l(A) \to J^{l+1} \to B^l \to 0 \qquad (l \geq 1), \tag{3.4}$$

$$0 \to p^l(g) \to E\otimes S^{l+1} \to h^l \to 0 \qquad (l \geq 1). \tag{3.5}$$

Since $p^l(A)$ and $p^l(g)$ are vector sub-bundles by the involutiveness of D it follows then that B^l and h^l are vector sub-bundles, (3.4) shows that $B^1 = B$, and (3.3) shows that h^1 is the homogeneous part of B. We set $h^1 = h$. We are going to show that $B^{l+1} = p^l(B)$ and $h^{l+1} = p^l(h)$.

By (3.4) for $l=1$, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J^l(p(A)) & \to & J^l(J^2) & \to & J^l(B) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J^l(J^1(A)) & \to & J^l(J^1(J^1)) & \to & J^l(J^1(E')) & \to & 0 \end{array} \tag{3.6}$$

We have also

$$
(3.7)\qquad
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & J^{l+1}(A) & \to & J^{l+1}(J^1) & \to & J^{l+1}(E') & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & J^l(J^1(A)) & \to & J^l(J^1(J^1)) & \to & J^l(J^1(E')) & \to 0
\end{array}
$$

By taking the cut of (3.6) by (3.7), we have a commutative diagram of exact sequences

$$
(3.8)\qquad
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & p^{l+1}(A) & \to & J^{l+2} & \to & p^l(B) & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & J^{l+1}(A) & \to & J^{l+1}(J^1) & \to & J^{l+1}(E') & \to 0
\end{array}
$$

By (3.5) and (3.8), we find that $B^{l+1}\subseteq p^l(B)$. Hence by (3.3), we have a commutative diagram of exact sequences

$$
(3.9)\qquad
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & h^{l+1} & \to & B^{l+1} & \to & B^l & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & p^l(h) & \to & p^l(B) & \to & p^{l-1}(B) & \\
 & \cap\!| & & \cap\!| & & \cap\!| & \\
0\to & E'\otimes S^{l+1} & \to & J^{l+1}(E') & \to & J^l(E') & \to 0
\end{array}
$$

where sequence containing only arrows are exact. We prove the equations $B^{l+1}=p^l(B)$ and $h^{l+1}=p^l(h)$ by induction of l. When $l=0$, these equalities hold by definition. Assume the case for $l-1$. Then by (3.9) it is enough to show that $h^{l+1}=p^l(h)$. To prove this, consider the following commutative diagrams of exact sequences (cf. (3.5))

$$
(3.10)\qquad
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & p(g)\otimes S^l & \to & E\otimes S^2\otimes S^l & \to & h\otimes S^l & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & g\otimes S\otimes S^l & \to & E\otimes S\otimes S\otimes S^l & \to & E_1'\otimes S\otimes S^l & \to 0
\end{array}
$$

where $E_1'=(E\otimes S)/g\subseteq E'$, and

$$
(3.11)\qquad
\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & g\otimes S^{l+1} & \to & E\otimes S\otimes S^{l+1} & \to & E_1'\otimes S^{l+1} & \to 0\\
 & \downarrow & & \downarrow & & \downarrow & \\
0\to & g\otimes S\otimes S^l & \to & E\otimes S\otimes S\otimes S^l & \to & E_1'\otimes S\otimes S^l & \to 0
\end{array}
$$

Take the cut of (3.11) by (3.10). Then we obtain the following exact sequence as the upper sequence in the cut

$$0 \to p^{l+1}(g) \to E \otimes S^{l+2} \to p^{l}(h).$$

The last arrow in the above sequence is surjective, as remarked in §2, if

$$g \otimes S \otimes S^{l} \cap (E \otimes S^{2} \otimes S^{l} + E \otimes S \otimes S^{l+1})$$
$$= p(g) \otimes S^{l} + g \otimes S^{l+1}.$$

But this equality holds by Lemma 1. Hence the sequence

$$(3.12) \qquad 0 \to p^{l+1}(g) \to E \otimes S^{l+2} \to p^{l}(h) \to 0 \qquad (l \geq 0)$$

is exact. Hence by (3.5) and (3.9), $p^{l}(h) = h^{l+1}$. Thus we proved that

$$(3.13) \qquad p^{l}(B) = B^{l+1} \qquad (l \geq 0).$$

Writing down the cut of (3.11) by (3.10) in full, we have a commutative diagram of exact sequence

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 & \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \to & p^{l+1}(g) & \to & E \otimes S^{l+2} & \to & p^{l}(h) & \to 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \to & g \otimes S^{l+1} & \to & E \otimes S \otimes S^{l+1} & \to & E_1' \otimes S^{l+1} & \to 0 \end{array}$$

Hence by abbreviating $\Lambda^{r}(T^{*})$ by Λ^{r} we have a commutative diagram of exact sequences.

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ & & p^{l+1}(g) \otimes \Lambda^{r} & \to & p^{l}(g) \otimes \Lambda^{r+1} \\ & & \downarrow & & \downarrow \\ p^{l+1}(E \otimes S) \otimes \Lambda^{r} & = & E \otimes S^{l+2} \otimes \Lambda^{r} & \to & E \otimes S^{l+1} \otimes \Lambda^{r+1} = p^{l}(E \otimes S) \otimes \Lambda^{r+1} \\ & & \downarrow & & \downarrow \\ & & p^{l}(h) \otimes \Lambda^{r} & \to & p^{l-1}(h) \otimes \Lambda^{r-1} \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

where the horizontal arrows denote, from the top, $\delta_{l+1,r}\delta_{l+1,r}$ and $\delta_{l,r}$, respectively (cf. (1.2)). Hence, for each x in M, we have an exact sequence

$$\cdots \to H^{l+1,r}(g_x) \to H^{l+1,r}(E_x \otimes T_x^{*}) \to H^{l,r}(h_x) \to H^{l,r+1}(g_x) \to \cdots$$

Since g_x is an involutive subspace of $E_x \otimes T_x^{*}$, $H^{l,r}(g_x) = 0$. It is easy to see that $H^{l+1,r}(E_x \otimes T_x^{*}) = 0$. Hence $H^{l,r}(h_x) = 0$ for all $l \geq 0$ and $r \geq 1$. Thus h_x is an involutive subspace of $E_x' \otimes T_x^{*}$.

The differential operator $D' : \Gamma(E') \rightarrow \Gamma(E'')$ is induced by the mapping $J^1(E') \rightarrow E''$, kernel of which is B. By (3.13) $p(B) = B^0$ which is a vector sub-bundle and $p(B) \rightarrow B$ is surjective by (3.13) and (3.9). Moreover, we showed that h is the homogeneous part of B and h_x is an involutive subspace of $E'_x \otimes T^*_x$ for all x in M. Therefore D' is involutive.

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