ON QUASINORMAL SUBGROUPS II

W. E. DESKINS*

To the Memory of Professor TADASI NAKAYAMA

A subgroup was defined by O. Ore to be quasinormal in a group if it permuted with all subgroups of the group, and he proved [5] that such a subgroup is subnormal (= subinvariant = accessible) in a finite group. Finite groups in which all subgroups are quasinormal were classified by K. Iwasawa [3], and more recently N. Itô and J. Szép [2] and the author [1] proved that a quasinormal subgroup is an extension of a normal subgroup by a nilpotent group. Similar results were obtained by O. Kegel [4] and in [1] for subgroups which permute not necessarily with all subgroups but with those having some special property.

In this note these results are generalized to subgroups which permute with each element of a family $\mathscr{S}$ of subgroups of the group which cover the group in a specified way. The restrictions on $\mathscr{S}$ are slight enough to allow many different realizations in a group, including those studied in the papers indicated above. In the first section a certain normality condition is placed on the elements of $\mathscr{S}$, and in the second section this is replaced by an arithmetic condition which enables counting arguments to be used.

All groups considered here are finite, and the following notation is used: $\mathbf{H} \lhd \mathbf{G}$ ($\mathbf{H} \lhd \lhd \mathbf{G}$) means that $\mathbf{H}$ is a normal (subnormal) subgroup of $\mathbf{G}$; $\langle \mathbf{H}, \mathbf{K} \rangle$ is the subgroup of $\mathbf{G}$ generated by the subsets $\mathbf{H}$ and $\mathbf{K}$; $\mathbf{H}^x = x^{-1}\mathbf{H}x$; $\mathrm{Cor}_{\mathbf{G}}(\mathbf{H})$ denotes the maximal normal subgroup of $\mathbf{G}$ contained in $\mathbf{H}$; $N_{\mathbf{G}}(\mathbf{H})$ is the normalizer of the subgroup $\mathbf{H}$ in $\mathbf{G}$; $|\mathbf{H}|$ denotes the order of the group $\mathbf{H}$; $\pi\mathbf{G}$ denotes the subgroup of $\mathbf{G}$ generated by all p-Sylow subgroups of $\mathbf{G}$ for p in the set π of primes; $\pi\mathbf{G}$ is written $p\mathbf{G}$ when $\pi = \{p\}$; $\mathbf{G}_p$ denotes a p-Sylow subgroup of $\mathbf{G}$; $\mathbf{H}^{\mathbf{G}}$ denotes the normal closure of $\mathbf{H}$ in $\mathbf{G}$; and $\mathbf{G}^p$ is the minimal normal subgroup of $\mathbf{G}$ such that $\mathbf{G}/\mathbf{G}^p$ is a p-group.

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Subgroups **H** and **K** of **G** *permute* if $\mathbf{HK}=\mathbf{KH}$

1. Covers for G. For the subgroup **H** of the group **G** to be quasinormal in **G** it is necessary and sufficient [1] that **H** permute with every subgroup of **G** generated by an element of prime power order in **G**. The collection of those cyclic subgroups is a cover for **G** in the following sense.

DEFINITION 1. A family $\mathscr{S}$ of subgroups of the group **G** is a *cover* for **G** if it satisfies the following conditions:

(1.1) **H** in $\mathscr{S}$ implies $\mathbf{H}^x$ in $\mathscr{S}$, for all x in **G**.

(1.2) If **M** is a maximal subgroup of G and y is an element of **M** of prime power order, then $\mathscr{S}$ contains an element **H** such that y is in **H** and $\mathbf{H}\cap\mathbf{M}\lhd\mathbf{H}$.

An arbitrary group has the following easily identified (but not necessarily distinct in all cases) covers: $\mathscr{A}$, all abelian subgroups of **G**: $\mathscr{M}\mathscr{A}$, all maximal abelian subgroups; $\mathscr{P}\mathscr{A}$, all abelian subgroups of prime power order; $\mathscr{M}\mathscr{P}\mathscr{A}$, the maximal elements of $\mathscr{P}\mathscr{A}$; $\mathscr{C}$, all cyclic subgroups; $\mathscr{M}\mathscr{C}$, the maximal elements of $\mathscr{C}$; $\mathscr{P}\mathscr{C}$, the elements of $\mathscr{C}$ of prime power order; $\mathscr{M}\mathscr{P}\mathscr{C}$, the maximal elments of $\mathscr{P}\mathscr{C}$; $\mathscr{H}$, all hamiltonian subgroups; $\mathscr{M}\mathscr{H}$, all maximal elements of $\mathscr{H}$; $\mathscr{P}\mathscr{H}$, all elements of $\mathscr{H}$ of prime power order; $\mathscr{M}\mathscr{P}\mathscr{H}$, all maximal elements of $\mathscr{P}\mathscr{H}$.

These twelve covers are called the ***elementary covers*** of **G**. An example of a nonelementary cover is provided by $\mathscr{M}$, the set of all maximal subgroups of **G**. The family $\mathscr{N}\mathscr{M}$ of all nonnormal maximal subgroups of **G** is also a cover for **G** provided $\mathscr{N}\mathscr{M}$ is defined to consist of **G** whenever **G** is nilpotent.

DEFINITION 2. A subgroup **K** of the group **G** is *$\mathscr{S}$ quasinormal* if it permutes with each element of the cover $\mathscr{S}$.

Remark 1. The subgroup **K** is quasinormal in **G** if and only if **K** is $\mathscr{S}$-Quasinormal in **G**, where $\mathscr{S}=\mathscr{A}$, $\mathscr{P}\mathscr{A}$, $\mathscr{C}$, $\mathscr{P}\mathscr{C}$, $\mathscr{H}$, or $\mathscr{P}\mathscr{H}$.

This follows from (1.1) in [1].

Remark 2. If **H** and **K** are $\mathscr{S}$-quasinormal subgroups of **G**, then so are $\langle\mathbf{H},\mathbf{K}\rangle$ and $\mathbf{H}^x$, for x in **G**.

The first conclusion is obvious, and the second follows from (1.1).

Thus each $\mathscr{S}$-quasinormal subgroup of **G** is contained in a maximal $\mathscr{S}$-quasinormal subgroup of **G**, and the latter can be proved to be normal.

THEOREM 1. *If $\mathscr{S}$ is a cover for the group* $\mathbf{G}$ *and* $\mathbf{K}$ *is a maximal $\mathscr{S}$-quasinormal subgroup of* $\mathbf{G}$, *then* $\mathbf{K} \lhd \mathbf{G}$.

It follows from Remark 2 that if x is an element of $\mathbf{G}$ then $\langle \mathbf{K}, \mathbf{K}^x \rangle$ is either $\mathbf{K}$ or $\mathbf{G}$. This means that if $\mathbf{H}$ is an elment of $\mathscr{S}$ such that $\mathbf{KH} \neq \mathbf{G}$ then $\mathbf{H} \subseteq N_{\mathbf{G}}(\mathbf{K})$ since $\langle \mathbf{K}, \mathbf{K}^x \rangle \subseteq \mathbf{HK}$ for each x in $\mathbf{H}$. So let $\mathbf{K}^*$ be the subgroup of $\mathbf{G}$ generated by $\mathbf{K}$ and all $\mathbf{H}$ in $\mathscr{S}$ such that $\mathbf{HK} \neq \mathbf{G}$. Then either $\mathbf{K}^* = \mathbf{G}$, in which case $N_{\mathbf{G}}(\mathbf{K}) = \mathbf{G}$, or $\mathbf{K}^* = \mathbf{K}$. The latter follows from the fact that $\mathbf{K}^*$ is $\mathscr{S}$-quasinormal. For either $\mathbf{K}^*$ contains every $\mathbf{H}$ in $\mathscr{S}$, in which case $\mathbf{G}$ is a cyclic group and the statement is certainly true, or $\mathbf{G} = \mathbf{HK}$ for any $\mathbf{H}$ of $\mathscr{S}$ which does not lie in $\mathbf{K}^*$. Hence for any $\mathbf{H}$ in $\mathscr{S}$ either $\mathbf{HK}^* = \mathbf{K}^*$ or $\mathbf{HK}^* = \mathbf{G} = \mathbf{HK} = \mathbf{KH} = \mathbf{K}^*\mathbf{H}$. Since $\mathbf{K}$ is a maximal $\mathscr{S}$-quasinormal subgroup of $\mathbf{G}$, this means that $\mathbf{K} = \mathbf{K}^*$. Moreover, exactly the same argument shows that $\mathbf{K}$ is a maximal subgroup of $\mathbf{G}$ when $\mathbf{K}^* \neq \mathbf{G}$.

Now suppose $\mathbf{K}$ is not normal in $\mathbf{G}$, and let x be an element of $\mathbf{G}$ such that $\mathbf{K} \neq \mathbf{K}^x = \mathbf{K}_1$. Then $\mathbf{K}_1$ contains an element y of prime power order which is not in $\mathbf{K}$. By (1.2) $\mathscr{S}$ contains an element $\mathbf{H}_1$ such that $y \in \mathbf{H}_1$ and $\mathbf{H}_1 \cap \mathbf{K}_1 \lhd \mathbf{H}_1$. Therefore $x = kh$ for some k in $\mathbf{K}$ and h in $\mathbf{H}_1$ so that

$$\mathbf{K}_1 = x^{-1}\mathbf{K}x = h^{-1}k^{-1}\mathbf{K}kh,$$

and

$$h\mathbf{K}_1h^{-1} = \mathbf{K}.$$

But since $y \in \mathbf{H}_1 \cap \mathbf{K}_1$ which is normalized by h, this last equation implies that y lies in $\mathbf{K}$. This impossibility clearly shows that $\mathbf{K}$ is normal in $\mathbf{G}$, and the proof is complete.

COROLLARY 1.1. *A subgroup of the group* $\mathbf{G}$ *which permutes with the elements of $\mathscr{X}$, where $\mathscr{X}$ is any of the 14 families $\mathscr{A}$ through $\mathscr{M}\mathscr{N}$, is contained in a nontrivial normal subgroup of* $\mathbf{G}$.

In order to prove that an $\mathscr{S}$-quasinormal subgroup is subnormal it seems to be necessary to restrict $\mathscr{S}$ somewhat.

DEFINITION 3. A cover $\mathscr{S}$ for the group $\mathbf{G}$ is an *inductive cover* if for each $\mathbf{K} \lhd \lhd \mathbf{G}$ the set $\mathscr{S}_{\mathbf{K}} = \{\mathbf{H} \cap \mathbf{K} : \mathbf{H} \text{ in } \mathscr{S}\}$ is a cover for $\mathbf{K}$.

The following simple lemma enables this concept to be combined with Theorem 1.

LEMMA 1. *If $\mathscr{S}$ is a cover for* $\mathbf{G}$, *if* $\mathbf{K}$ *is a subgroup of* $\mathbf{G}$ *such that*

$\mathscr{S}_{\mathbf{K}} = \{\mathbf{H} \cap \mathbf{K} : \mathbf{H}$ in $\mathscr{S}\}$ is a cover for **K**, *and if the $\mathscr{S}$-quasinormal subgroup* **L** *of* **G** *lies in* **K**, *then* **L** *is $\mathscr{S}_{\mathbf{K}}$-quasinormal in* **K**.

Let **H** be in $\mathscr{S}$; then $\mathbf{HL} = \mathbf{LH}$. Therefore, if x and y are elements of **L** and $\mathbf{K} \cap \mathbf{H}$, respectively, there exist elements x' and y' in **L** and **H**, respectively, such that $xy = y'x'$. Therefore $y' = xy(x')^{-1}$ is also in **K**, so that **L** permutes with the elements of $\mathscr{S}_{\mathbf{K}}$.

COROLLARY 1.2. *If $\mathscr{S}$ is an inductive cover for* **G** *then each $\mathscr{S}$-quasinormal subgroup of* **G** *is subnormal.*

This follows immediately from Theorem 1 and Lemma 1.

COROLLARY 1.3. *If $\mathscr{S}$ is an elementary cover for* **G** *then each $\mathscr{S}$-quasinormal subgroup of* **G** *is subnormal.*

For it can be verified easily that an elementary cover is also inductive.

As noted previously $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ is nilpotent when **K** is quasinormal in **G**, and since quasinormality coincides with $\mathscr{S}$-quasinormality (Remark 1) when $\mathscr{S}$ consists of all abelian, cyclic, or hamiltonian subgroups, this suggests that the structure of $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$, when **K** is $\mathscr{S}$-quasinormal, reflects a common structural property of the elements of $\mathscr{S}$.

THEOREM 2. *If each element of the cover $\mathscr{S}$ for* **G** *is a solvable group and* **K** *is a subnormal $\mathscr{S}$-quasinormal subgroup of* **G**, *then $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ is solvable.*

The well-known fact that each group **G** contains a unique minimal normal subgroup $Q(\mathbf{G})$ having the property that $\mathbf{G}/Q(\mathbf{G})$ is a solvable group, can be used to prove the following result which in turn yields Theorem 2.

PROPOSITION. *If* **H** *and* **K** *are subgroups of* **G** *such that* $\mathbf{K} \lhd \lhd \mathbf{G}$, $\mathbf{G} = \mathbf{HK} = \mathbf{KH}$, *and* **H** *is solvable, then* $Q(\mathbf{G}) = Q(\mathbf{K})$.

Suppose $\mathbf{K} \lhd \mathbf{G}$; then $Q(\mathbf{K}) \lhd \mathbf{G}$ since it is a characteristic subgroup of **K**, and it follows immediately that $Q(\mathbf{K}) \supseteq Q(\mathbf{G})$. Since the image of **K** in $\mathbf{G}/Q(\mathbf{K})$ is solvable it follows that $Q(\mathbf{K}) \subseteq Q(\mathbf{G})$, and so $Q(\mathbf{K}) = Q(\mathbf{G})$. Now proceed by induction on $n(\mathbf{G}, \mathbf{K})$, the length of the shortest normal chain connecting **K** with **G**, and let $\mathbf{G}_1$ be the maximal (proper) subgroup of **G** in such a chain. Then $\mathbf{G}_1 = \mathbf{KH}_1 = \mathbf{H}_1\mathbf{K}$ for $\mathbf{H}_1 = \mathbf{H} \cap \mathbf{G}_1$, and clearly the induction hypothesis implies that $Q(\mathbf{K}) = Q(\mathbf{G}_1)$. Then the above argument implies $Q(\mathbf{G}_1) = Q(\mathbf{G})$ since

$\mathbf{G} = \mathbf{G}_1\mathbf{H}$, and so $Q(\mathbf{K}) = Q(\mathbf{G})$.

Now to prove Theorem 2 let **G** be noncyclic (since the statement is trivially true when **G** is cyclic). Then an element x of prime power order in **G** is contained in some element **L** of $\mathscr{S}$. When the Proposition is applied to the group **KL**, it is seen that x normalizes $Q(\mathbf{K})$. Since each element of **G** is a product of elements of prime power order, this means that $Q(\mathbf{K}) \lhd \mathbf{G}$ and the theorem is proved.

COROLLARY 2.1. *If each element of the inductive cover $\mathscr{S}$ for* **G** *is solvable and if* **K** *is an $\mathscr{S}$-quasinormal subgroup of* **G**, *then* $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ *is solvable.*

This is a consequence of Corollary 1.2 and Theorem 2.

COROLLARY 2.2. *If* **K** *is an $\mathscr{S}$ quasinormal subgroup of* **G**, *for $\mathscr{S}$ an elementary cover for* **G**, then $Q(\mathbf{K}) \lhd \mathbf{G}$.

This follows by Corollary 1.3.

Since it is known that $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ is nilpotent when **K** is $\mathscr{S}$-quasinormal in **G** and $\mathscr{S}$ is one of the six covers in Remark 1, it seems probable that in Corollary 2.2 $Q(\mathbf{K})$ can be replaced by $Q_0(\mathbf{K})$, the minimal normal subgroup of **K** having a nilpotent factor group.

2. Prime Covers. If the elements of a cover for **G** are required to have prime power orders, then counting arguments can be used. In this case the normality condition in (1.2) can be dropped altogether.

DEFINITION 4. A family $\mathscr{P}$ of subgroups having prime power orders of the group **G** is a *prime cover* for **G** if it satisfies the following conditions:

(3.1) **H** in $\mathscr{P}$ implies $\mathbf{H}^x$ in $\mathscr{P}$, for all x in **G**.

(3.2) If x is an element in **G** of prime power order, then $\mathscr{P}$ contains an element **H** of prime power order such that x is in **H**.

Several examples of prime covers for **G** have already been given, viz., $\mathscr{P}\mathscr{A}$, $\mathscr{M}\mathscr{P}\mathscr{A}$, $\mathscr{P}\mathscr{C}$, $\mathscr{M}\mathscr{P}\mathscr{C}$, $\mathscr{P}\mathscr{H}$, and $\mathscr{M}\mathscr{P}\mathscr{H}$, but certainly others exist. The set of all Sylow subgroups of **G** is a prime cover for **G**, and so is the set which consists of the maximal subgroups of the noncyclic Sylow subgroups of **G** and the cyclic Sylow subgroups of **G**.

Note that an element of $\mathscr{P}$ need not be a proper subgroup.

DEFINITION 5. A subgroup **K** of the group **G** is *$\mathscr{P}$-quasinormal* if it permutes with each element of $\mathscr{P}$, a prime cover for **G**.

THEOREM 3. *If* **K** *is a* $\mathscr{P}$*-quasinormal subgroup of* **G**, *then*

(1) $\mathbf{K} \lhd \lhd \mathbf{G}$;

(2) $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ *is nilpotent; and*

(3) *if* q *is a prime factor of* $|\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})|$, *then* **G** *contains a normal subgroup of index* q.

Conclusion (1) is proved by first showing that a maximal $\mathscr{P}$-quasinormal subgroup of **G** is normal in **G**. Suppose **K** is such a subgroup. The proof proceeds exactly as the proof of Theorem 1 until the point is reached where $\mathbf{K} = \mathbf{K}^*$ is a maximal subgroup of **G**. Now let y be an element of prime power order not in **K**. Then $\mathscr{P}$ contains an element **H** of prime power order containing y, and, since $\mathbf{K} = \mathbf{K}^*$, $\mathbf{G} = \mathbf{KH}$ and **K** has index q^m, where q is the prime divisor of $|\mathbf{H}|$. Denote by π the set of all prime divisors of $|\mathbf{G}|$ other than q. Then $\pi\mathbf{G} \subseteq \mathbf{K}$ and $\mathbf{K}/\pi\mathbf{G}$ is a maximal subgroup of the q-group $\mathbf{G}/\pi\mathbf{G}$. Hence $\mathbf{K} \lhd \mathbf{G}$ when **K** is maximal $\mathscr{P}$-quasinormal.

If **T** is a subgroup of **G** then clearly $\mathscr{P}_{\mathbf{T}} = \{\mathbf{H} \cap \mathbf{T} : \mathbf{H} \text{ in } \mathscr{P}\}$ is a prime cover for **T**. Since Lemma 1 extends easily to prime covers, the result above for maximal $\mathscr{P}$-quasinormal subgroups implies (1).

An additional lemma is needed.

LEMMA 2. *If* **K** *is* $\mathscr{P}$*-quasinormal in* **G** *and* p *and* q *are distinct prime factors of* $|\mathbf{G}|$ *then* $p\mathbf{K} \lhd \mathbf{K}(q\mathbf{G})$.

Let x be a q-element of **G** not in **K** and let **H** be an element of $\mathscr{P}$ which contains x and is a q-group. Then **K** has index q^m in **KH**, and each p-Sylow subgroup of **KH** lies in **K**. For otherwise **G** contains a p-element y which is in **KH** but not in **K**. Then $\mathscr{P}$ contains an element $\mathbf{H}_1$ such that y is in $\mathbf{KH}_1 = \mathbf{H}_1\mathbf{K}$ and **K** has index p^n in $\mathbf{KH}_1$. Clearly $p \neq q$ implies $\mathbf{KH} \cap \mathbf{KH}_1 = \mathbf{K}$ which implies that y is in **K**, violating the choice of y. Thus $\mathbf{K}_p^x \subseteq \mathbf{K}$ for all q-elements x in **G**, and the lemma follows.

Now to prove $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ nilpotent set $\mathbf{L} = \bigcap_p (\pi - p)\mathbf{K}$ for all p in π, where π is the set of prime factors of $|\mathbf{G}|$. Fix p and consider an arbitrary $\mathbf{L}_p$; then $\mathbf{L}_p$ is a Sylow subgroup of $(\pi - p)\mathbf{K}$ since the subgroups $(\pi - q)\mathbf{K}$, $q \neq p$ in π, contain all $\mathbf{K}_p$. An arbitrary element g of **G** can be written $g = xy = yx$ for some p-element x and $(\pi - p)$-element y. Now $\mathbf{L}_p^x$ is also a p-Sylow subgroup of $(\pi-p)\mathbf{K}$ since x normalizes $(\pi-p)\mathbf{K}$ by Lemma 2, so that $\mathbf{L}_p^x \subseteq \mathbf{K}$. Also Lemma 2 and the fact that y is a product of q-elements, $q \neq p$ in π, indicate that $\mathbf{T}^y \subseteq \mathbf{K}$ when $\mathbf{T}$ is any p-subgroup of $\mathbf{K}$. Therefore $\mathbf{L}_p^{\mathbf{G}} \subseteq \mathbf{K}$, and this means that $\mathbf{L}^{\mathbf{G}} \subseteq \mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$. Since $\mathbf{K}/\mathbf{L}$ is nilpotent it is clear that $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ is nilpotent also.

Conclusion (3) is a consequence of the following result.

PROPOSITION. *If* $\mathbf{K}$ *is a* $\mathscr{P}$*-quasinormal subgroup of* $\mathbf{G}$, *then* $N_{\mathbf{G}}(\mathbf{K}) \supseteq \bigcup_p \mathbf{G}^p$, *the minimal normal subgroup of* $\mathbf{G}$ *having a nilpotent factor group.*

Now $\mathbf{G}^p = (\pi-p)\mathbf{G}$, so by Lemma 2, $N_{\mathbf{G}}(p\mathbf{K}) \supseteq \mathbf{G}^p$. Since $N_{\mathbf{G}}(\mathbf{K}) = \bigcap_p N_{\mathbf{G}}(p\mathbf{K})$, the Proposition follows.

To prove (3) note that if p is a prime factor of $|\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})|$ then $\mathbf{G} \neq N_{\mathbf{G}}(p\mathbf{K}) \supseteq \mathbf{G}^p$.

Note. In [1] the author remarked that no example was known to him of a group $\mathbf{G}$ having a quasinormal subgroup $\mathbf{K}$ with $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ nonabelian. Recently John Thompson constructed a p-group $\mathbf{G}$ having a quasinormal subgroup $\mathbf{K}$ for which $\mathbf{K}/\mathrm{Cor}_{\mathbf{G}}(\mathbf{K})$ has class 2.

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Michigan State University
East Lansing, Michigan U.S.A.