RINGS WITH ASCENDING CONDITION ON ANNIHILATORS

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Quasi-frobenius ($=QF$) rings have many interesting characterizations. One such, due to Ikeda [17] is that these rings are right (left) artinian and right (left) self-injective. Thus, if R is QF, then R is right (left) noetherian, so each direct sum of injective right R-modules is injective: in particular, each free, hence, each projective, R-module is injective. One object of this paper is to report that this property characterizes QF-rings:

(A) THEOREM. *A ring R is QF if and only if each projective right R-module is injective.*

The symmetrical properties of QF-rings (§2) show that "right" can be replaced by "left" in this statement. The "dual" theorem obtained by the substitutions "projective" $\leftrightarrow$ "injective" is the subject of another paper [7].

The condition that every free module is injective leads naturally to the concept of Σ-injectivity: an injective module is ***Σ-injective*** in case an infinite direct sum of copies is injective. A Σ-injective module M_R with endomorphism ring Λ is characterized by the descending chain condition (d.c.c.) on the lattice of Λ-submodules which are annihilators of subsets of R (Prop. 3.3). If $\hat{R}$ denotes the injective hull of R_R, and if $M=\hat{R}$, this condition implies the ascending chain condition (a.c.c.) on annihilator right ideals ($=$ right annulets) of R, and, in case $M=\hat{R}=R$, this condition is equivalent to the a.c.c. on right annulets (Corollary 3.4 and Theorem 3.5). Thus, the proof of (A) leads to the more general study of the rings of the title, and to the following intrinsic characterization: R satisfies the a.c.c. on right annulets if and only if to each right ideal I there corresponds a finitely generated subideal I_1 having the same left annihilator as I (Prop. 3.1).

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The above results, then, reduce the proof of (A) to the proof of the statement (Theorem 5.2) that a right selfinjective ring which satisfies the a.c.c. on right annulets is *QF*. This is proved by: (1) showing that such a ring R is semiprimary (Theorem 4.1); (2) applying a result of S. U. Chase (Appendix), stating that R then satisfies the d.c.c. on finitely generated right ideals: in conjunction with (3) the characterization of rings satisfying the a.c.c. on right annulets given above. This yields the a.c.c. on left annulets. Since each finitely generated left ideal in a right self-injective ring is a left annulet, we obtain that R is left noetherian, whence *QF*.

In § 6 Σ-injective modules are examined again, the main results being: (1) if R is a semiprime ring having a semisimple classical right quotient ring, then $\hat{R}$ is Σ-injective; (2) if R is an integral domain, then $\hat{R}$ is Σ-injective if and only if R has a right quotient field. More generally, (3) if R is a ring with zero right singular ideal, then $\hat{R}$ is Σ-injective if and only if R satisfies the a.c.c. on complement right ideals. (2) shows that the a.c.c. on right annulets does not suffice for Σ-injectivity of $\hat{R}$.

I wish to take this opportunity to thank Elbert A. Walker for rekindling my old interest in the problem (A), for many stimulating conversations and much encouragement during its solution. To a large extent this work was inspired by the following theorem of Walker (Cf., [15, Theorem 3.5]):

If R is right self-injective, then the ring $R_{(\omega)}$ of row finite matrices over R is right self-injective if and only if R is (right) Σ-injective. Our work shows that this is true if and only if R is *QF*.

0. Notation. We will assume that each ring R will have an identity element, and that all modules are unital. $\mathscr{M}_R$ (respectively ${}_R\mathscr{M}$) denotes the category of all right (respectively left) R-modules.

Let M be a module in $\mathscr{M}_R$ having endomorphism ring Λ. For any subset X of M,

$$X^{\perp} = \{r \in R \mid Xr = 0\}$$

is a right ideal of R. The totality of such right ideals will be denoted by $\mathscr{A}_r(M, R)$. For any subset X of R

$$X^{\perp} = \{m \in M \mid mX = 0\}$$

is a Λ-submodule of ${}_{\Lambda}M$. The totality of such Λ-submodules of M is denoted by $\mathscr{A}_l(M, R)$. It is clear that $\mathscr{A}_r(M, R)$ (respectively $\mathscr{A}_l(M, R)$) is closed under arbitrary intersections, making it a complete lattice. In particular, the ascending chain condition (a.c.c) in $\mathscr{A}_r(M, R)$ (respectively $\mathscr{A}_l(M, R)$) is equivalent to the maximum condition.

Since $I \to I^{\perp}$ is 1-1 and order-inverting between $\mathscr{A}_r(M, R)$ and $\mathscr{A}_l(M, R)$, then one satisfies the a.c.c. if and only if the other satisfies the d.c.c.

In this special case $M = R$, in order to distinguish left from right, we replace $X^{\perp}$ by

$$(X : 0) = \{r \in R \mid Xr = 0\},$$

or

$$(0 : X) = \{r \in R \mid rX = 0\},$$

for any subset X of R. In this case the elements of $\mathscr{A}_r(R, R)$ (respectively $\mathscr{A}_l(R, R)$) are called right (respectively left) annulets of R.

If S is a ring satisfying the a.c.c. on right annulets (that is, on $\mathscr{A}_r(S, S)$), then any subring of S satisfies the a.c.c. on its right annulets. In particular, the a.c.c. on right annulets are inherited by the subrings of S, if S is either left or right artinian, or right noetherian.

1. Properties of Injective Modules. $M \in \mathscr{M}_R$ is injective if and only if, for each module $A \in \mathscr{M}_R$ each map $f : B \to M$ of a submodule B of A can be extended to a map of A into M. A theorem of Baer [1] states that M is injective if and only if each map into M of any right ideal of R can be extended to a map R into M. Thus, M is injective if and only if the following condition, called *Baer's condition*, is satisfied: If $f : I \to M$ is a map of a right ideal I of R into M, then there exists $m \in M$ such that $f(x) = mx$, $\forall x \in I$.

Each $M \in \mathscr{M}_R$ can be embedded in an injective module (Baer [1]), and there exists a unique minimal injective module $\hat{M}$, which is a maximal essential extension of M (Eckmann-Schopf [4]), called the injective hull of M.

Consider the following conditions:

(A) (Baer's Condition). If f is a map of a right ideal I into R, then there exists $a \in R$ such that $f(x) = ax$, $\forall x \in I$.

(B) $(0 : I_1) + (0 : I_2) = (0 : I_1 \cap I_2)$, where I_1, I_2 are right ideals of R.

(C) L is a left annulet, where L is a left ideal of R.

For $X=A$, B, or C, (X^*) denotes the condition (X), when the ideal is finitely generated. Thus (C^*) denotes that each finitely generated left ideal is a left annulet.

Ikeda-Nakayama [11] established the following implications:

$$(A) \Rightarrow [(B),\ (C^*)] \Rightarrow (A^*).$$

Since (A) is Baer's criterion for injectivity of the module R_R, it follows that *if R_R is injective, then each finitely generated left ideal is a left annulet.*

2. Properties of QF-rings. *A* ring R is *quasi-frobenius* ($=QF$) in case each right ideal is a right annulet, each left ideal is a left annulet, and R is right (or left) artinian. Eilenberg-Nakayama [5] proved the equivalence of the following statements:

(1) R is QF;

(2) R is right noetherian, each left ideal is an annulet, and $(0: I_1 \cap I_2) = (0: I_1) + (0: I_2)$ for each pair I_1, I_2 of right ideals;

(3) R is right and left noetherian and left self-injective;

(4) R is right artinian and left self-injective;

(5) R is left noetherian and left self-injective;

(6) The left-right symmetry of any of the preceding conditions.

The only situation not covered by the theorem of Eilenberg and Nakayama is when R is noetherian on one side and self-injective on the other. This gap is removed below.

THEOREM 1. *If R is right or left artinian or noetherian, and if R is right or left self-injective, then R is QF.*

Proof. Let ${}_RR$ be injective, and R_R noetherian.

A theorem of Utumi [13] asserts that $R/\text{rad } R$ is a regular ring. Since $R/\text{rad } R$ is also noetherian, and since any finitely generated right ideal in a regular ring is a direct summand (von Neumann [14]) it follows that each right ideal of $R/\text{rad } R$ is a direct summand, that is, $R/\text{rad } R$ is semisimple. By the method of C. Hopkins [10], in order to prove that R is right artinian, whence QF, it suffices to show that $J=\text{rad } R$ is nilpotent.

By the results on self-injective rings stated in § 1, *each* right ideal of R is a right annulet. Now $(0: J) \subseteq \cdots \subseteq (0: J^n) \subseteq \cdots$ is an ascending sequence of ideals of R, so R_R noetherian implies $(0 : J^n) = (0 : J^{n+1})$, for some n. Since J^n, J^{n+1} are right annulets, this implies that $J^n = J^{n+1}$, which in a noetherian ring implies that $J^n = 0$. This completes the proof.

3. Sigma Injective Modules. If $\{M_a\}$ is a family of right R-modules, indexed by a set A, and if M_a is isomorphic to a fixed right module M, $\forall a \in A$, then set

$$M^A = \prod_{a\in A} M_a \qquad \text{(direct product)},$$

and

$$M^{(A)} = \sum_{a\in A} \oplus M \qquad \text{(direct sum)}.$$

If A is countably infinite, then set $M^\omega = M^A$, $M^{(\omega)} = M^{(A)}$.

If M is injective in $\mathscr{M}_R$, then M^A is injective, for any index set A. M will be said to be *Σ-injective* in case $M^{(A)}$ is injective for any index set A; M is *countably Σ-injective* in case $M^{(\omega)}$ is injective.

PROPOSITION 1. *If $M \in \mathscr{M}_R$, then $\mathscr{A}_r(M, R)$ satisfies the ascending chain condition if and only if to each right ideal I of R there corresponds a finitely generated subideal I_1 such that $I^\perp = I_1^\perp$.*

Proof. Assume a.c.c. for $\mathscr{A}_r(M, R)$, or equivalently, the d.c.c. for $\mathscr{A}_l(M, R)$, let I be a right ideal of R, and let I_1 be a finitely generated subideal such that $I_1^\perp$ is minimal in $\{K^\perp\}$, where K ranges over all finitely generated subideals of I, and $K^\perp$ is taken in M. If $x \in I$, then $Q = I_1 + xR$ is a finitely generated subideal of I satisfying $Q^\perp \subseteq I_1^\perp$. By the choice of I_1, necessarily $Q^\perp = I_1^\perp$, so $I_1^\perp x = 0$. Since this is true $\forall x \in I$, then $I_1^\perp I = 0$, that is, $I_1^\perp \subseteq I^\perp$. But $I_1 \subseteq I$ implies $I_1^\perp \supseteq I^\perp$, so $I_1^\perp = I^\perp$ as asserted.

Conversely, let $I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be a chain of right ideals of R lying in $\mathscr{A}_r(M, R)$, let $X_i = I_i^\perp$, $i = 1, 2, \ldots,$ be the corresponding elements of $\mathscr{A}_l(M, R)$, let $I = \bigcap_{i=1}^{\infty} I_n$, and let J_1 be the finitely generated subideal of I such that $I^\perp = J_1^\perp$. Since $J_1^\perp$ is finitely generated, there is an integer q such that $J_1 \subseteq I_k$, $k \geq q$, that is, $J_1 \supseteq X_k = I_k^\perp$, $k \geq q$. But

$$J_1^\perp = I^\perp = \bigcap_{n=1}^{\infty} X_n,$$

that is, $X_k = J_1^\perp$, $k \geq q$. Then, $I_k = X_k^\perp = I_q$, $k \geq q$, proving the proposition.

In the case $M=R$, $\mathscr{A}_l(M, R)$ (respectively $\mathscr{A}_r(M, R)$) is simply the lattice of left (respectively right) annulets of R, producing the following:

COROLLARY 2. *A ring R satisfies the a.c.c. on right annulets if and only if each right ideal I contains a finitely generated ideal I_1 such that* $(0 : I) = (0 : I_1)$.

We now study Σ-injectivity.

PROPOSITION 3. *The following conditions on an injective module $M \in \mathscr{M}_R$ are equivalent;*

(1) *M is countably Σ-injective.*

(2) *R satisfies the a.c.c. on the ideals in $\mathscr{A}_r(M, R)$.*

(3) *M is Σ-injective.*

Proof. (1) $\Longrightarrow$ (2) (indirect proof). Let $I_1 \subset I_2 \subset \cdots \subset I_m \subset \cdots$ be a strictly ascending sequence of right ideals in $\mathscr{A}_r(M, R)$, let $I = \bigcup_{n=1}^{\infty} I_n$, and let x_n be an element of $I_n^{\perp}$ (taken in M) not in $I_{n+1}^{\perp}$, $n = 1, 2, \ldots$. If $r \in I$, then there exists q such that $r \in I_k$ $\forall k \geq q$, and since $I_q^{\perp} \supset I_k^{\perp}$, $\forall k \geq q$, then $x_k r = 0$ $\forall k \geq q$. Therefore the element $r' = (x_1 r, \ldots, x_n r, \ldots)$ lies in $M^{(\omega)}$, even though $x = (x_1, \ldots, x_n, \ldots)$ lies in M^{ω}. Let f denote the map defined by $f(r) = r'$ $\forall r \in I$. Assuming momentarily that $M^{(\omega)}$ is injective, there is given, by Baer's criterion, an element $y = (y_1, \ldots, y_m, 0, \ldots) \in M^{(\omega)}$ such that

$$f(r) = yr = (y_1 r, \ldots, y_m r, 0, \ldots)$$
$$= (x_1 r, \ldots, x_m r, \ldots), \ \forall r \in I.$$

But this implies that $x_t r = 0$, $\forall t > m$, $\forall r \in I$, that is, $x_t \in I^{\perp} \subseteq I_{t+1}^{\perp}$, contrary to the choice of x_t. Thus, (1) $\Longrightarrow$ (2).

(2) $\Longrightarrow$ (3). Let I be a right ideal of R, and let $I_1 = r_1 R + \cdots + r_n R$ be the finitely generated subideal given by Proposition 3.1 such that $I^{\perp} = I_1^{\perp}$. Let $f : I \to M^{(A)}$ be any map. Since M^A is injective, there exists an element $p \in M^A$ such that $f(r) = pr$ $\forall r \in I$. Since $f(r_i) = pr_i \in M^{(A)}$, $i = 1, \ldots, n$, there exists an element $p' \in M^{(A)}$ such that $p_a r_i = p'_a r_i$ $\forall a \in A$, $i = 1, \ldots, n$, where g_a is the a co-ordinate of any $g \in M^A$. Since $r_1, \ldots, r_n$ generate I_1, this implies that $pr = p'r$ $\forall r \in I_1$, whence $(p_a - p'_a) \in I_1^{\perp}$ $\forall a \in A$. Since $I_1^{\perp} = I^{\perp}$, it follows that $p_a x = p'_a x$, $\forall a \in A$, $\forall x \in I$, that is $px = p'x$, $\forall x \in I$. Thus, $f(x) = p'x$ $\forall x \in I$, with $p' \in M^{(A)}$, so $M^{(A)}$ is injective by Baer's criterion. (A direct proof of (1) $\Longrightarrow$ (3) can be given; see [7].)

Our present main interest in this result is for the case $M = \hat{R}$, the injective hull of R in $\mathscr{M}_R$. In this case, $\mathscr{A}_r(M, R)$ contains the totality of right annulets of R.

COROLLARY 4. *If $\hat{R}$ is countably Σ-injective, then R satisfies the a.c.c. on right annulets.*

THEOREM 5. *The following conditions on a ring R are equivalent:*

(1) *Any countably generated projective module in $\mathscr{M}_R$ is injective;*

(2) *R is right self-injective, and satisfies the a.c.c. on right annulets;*

(3) *Any projective module in $\mathscr{M}_R$ is injective.*

Proof. (1) $\Rightarrow$ (2). R_R is injective, and so is $R^{(\omega)}$. Thus, R is countably Σ-injective, so R satisfies the a.c.c. on right annulets by Corollary 4.

(2) $\Rightarrow$ (3). Since $\mathscr{A}_r(R, R)$ coincides with the set of right annulets of R, Proposition 3 implies that any free, hence, any projective, module in $\mathscr{M}_R$ is injective.

4. Perfect rings with a.c.c. on annulets. A ring R is *semiprimary* in case rad R is nilpotent, and $R/\text{rad } R$ is semisimple. R is said to be (right) *perfect* in case each $M \in \mathscr{M}_R$ has a projective cover. Bass [2] proved the equivalence of the following statements:

(1) R is perfect.

(2) R satisfies the d.c.c. on principal left ideals.

(3) rad R is right T-nilpotent and $R/\text{rad } R$ is semisimple.

(4) Each nonzero left R-module has nonzero socle, and R does not contain infinitely many orthogonal idempotents.

rad R is right T-nilpotent in case each infinite sequence $\{a_n\}$ of elements of rad R satisfies $a_n a_{n-1} \cdots a_2 a_1 = 0$ for some n. Since any nilpotent ideal is left and right T-nilpotent, (3) (and its right-left symmetry) implies that a semiprimary ring is right and left perfect.

PROPOSITION 1. *If R is perfect, and if R satisfies the a.c.c. on right annulets, then R is semiprimary.*

Proof. Since R is perfect, then $R/\text{rad } R$ is a semisimple ring. Let $J = \text{rad } R$, and consider the chain

$$(J : 0) \subseteq (J^2 : 0) \subseteq \cdots \subseteq (J^n : 0) \subseteq \cdots,$$

By hypothesis, $(J^n : 0) = (J^{n+1} : 0)$, for some n. If $R \neq J^n$, then the left module $R/(J^n : 0)$ has nonzero socle which has the form $T/(J^n : 0)$ for some left ideal $T \supset (J^n : 0)$. Since $T/(J^n : 0)$ is semisimple, J annihilates this module, so $JT \subseteq (J^n : 0)$, that is, $T \subseteq (J^{n+1} : 0) = (J^n : 0)$. Thus, $T = (J^n : 0)$, a contradicition proving that $R = (J^n : 0)$. Therefore $J^n = 0$ and R is semiprimary.

S. U. Chase has proved the following important fact about a semiprimary ring R: each $M \in \mathcal{M}_R$ satisfies the d.c.c. on finitely generated submodules. (See Appendix.)

5. Self-injective rings with a.c.c. on annulets. The proof of Theorem A is contained in this section. The lemma below is a restatement of part of Corollary 3.3.

LEMMA 1. *If R is a ring, then each projective module in $\mathcal{M}_R$ is injective if and only if R is a right self-injective ring satisfying the a.c.c. on right annulets.*

Together with this lemma, the following theorem completes the proof of Theorem A.

THEOREM 2. *A ring R is QF if and only if R is a right self-injective ring satisfying the a.c.c. on right annulets.*

Proof. If R is *QF*, then R_R is injective, and noetherian, by the results stated in § 2. Then R satisfies the a.c.c. on right annulets.

Conversely, suppose R_R is injective, and R satisfies the a.c.c. on right, that is, the d.c.c. on left annulets. Injectivity of R_R implies that each finitely generated left ideal is a left annulet. Thus, R satisfies the d.c.c. on finitely generated, hence principal, left ideals. Then R is perfect, and Proposition 4.1 implies that R is semiprimary.

By Chase's theorem (see Appendix), R satisfies the d.c.c. on finitely generated right ideals. Let $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$ be any descending sequence of right annulets of R. By the lemma below, there is a corresponding sequence $A'_1 \supseteq \cdots \supseteq A'_n \supseteq \cdots$ of finitely generated right ideals such that $(0 : A'_i) = (0 : A_i)$, $i = 1, 2, \ldots$. Consequently, there is an integer n such that $A'_n = A'_m$, $\forall m \geq n$. Then $(0 : A_n) = (0 : A_m)$ $\forall m \geq n$, proving that R satisfies the d.c.c. (respectively a.c.c.) on right (respectively left) annulets. Since each finitely generated left ideal is a left annulet, R satisfies the a.c.c. on finitely generated left ideals.

Consequently each left ideal is finitely generated. Then R is QF by Theorem 2.1.

LEMMA 3. *If R is right self-injective, and if R satisfies the a.c.c. on right annulets, and if A, B are right ideals such that $A \supseteq B$, then there exist finitely generated subideals $A' \supseteq B'$ such that $(0 : A) = (0 : A')$ and $(0 : B) = (0 : B')$.*

Proof. Since R_R is injective, by a result stated in § 2, $(0 : A \cap B) = (0 : A) + (0 : B)$ for any two right ideals A, B. Let $A \supseteq B$, and let A' be the finitely generated subideal of A, given by Proposition 3.1, such that $(0 : A) = (0 : A')$. Then

$$(0 : A' \cap B) = (0 : A) + (0 : B) = (0 : B).$$

The last equality holds since $(0 : A) \subseteq (0 : B)$. Hence, by Proposition 3.1 again, there exists a finitely generated subideal B' of $A' \cap B$ such that $(0 : B') = (0 : B)$.

COROLLARY 4. *If R is right or left self-injective, and satisfies the a.c.c. on right annulets, then R is QF.*

Proof. By Theorem 2, we may assume that ${}_RR$ is injective. Then, each finitely generated right ideal of R is a right annulet, so R satisfies the a.c.c. on finitely generated right ideals, that is, R is right noetherian. Then R is QF by Theorem 2.1.

8. **Characterizations of Σ-injectivity.** In § 3, Σ-injectivity of $\hat{R}$ was characterized by the a.c.c. on $\mathscr{A}_r(\hat{R}, R)$. However, there is no intrinsic test for the ideals of R which lie in $\mathscr{A}_r(\hat{R}, R)$. By restricting R to be a ring with zero right singular ideal, we are able to characterize Σ-injectivity of $\hat{R}$ by the a.c.c. on complement right ideals of R. Since a complement right ideal I of R is a right ideal which is maximal in a set of right ideals having zero intersection with a given fixed right ideal K, this amounts to an intrinsic characterization of Σ-injectivity in this case.

PROPOSITION 1. *Let R be a ring with zero right singular ideal. Then $\hat{R}$ is Σ-injective if and only if R satisfies the a.c.c. on complement right ideals.*

Proof. The right singular ideal $Z_r(R)$ consists of all those $x \in R$ for which $(x : 0)$ is an essential right ideal of R, that is, for which $(x : 0)$ meets each nonzero right ideal of R. When $Z_r(R) = 0$, then $\hat{R}$ is a regular ring which contains R as a subring (Johnson [12], Wong-Johnson [16]). (This theorem is discussed from this point of view in [6].) In this case, if $\mathscr{C}_r(T)$ denotes the set of complement right ideals of a ring T, then $\mathscr{C}_r(\hat{R})$ consists of the direct summands of $\hat{R}$, and the contraction mapping $\varphi: Q \to Q \cap R$ is 1-1 and inclusion-preserving between $\mathscr{C}_r(\hat{R})$ and $\mathscr{C}_r(R)$; if $I \in \mathscr{C}_r(R)$, then $\varphi^{-1}I$ is the unique injective hull of I contained in $\hat{R}$. Henceforth, denote $\varphi^{-1}I$ by $\hat{I}$.

A result of Goldie [8] states that R satisfies the a.c.c. on complement right ideals if and only if each collection of independent right ideals is finite. If $I_1, I_2, \ldots, I_n, \ldots$ is a collection of independent right ideals, then $\hat{I}_1, \ldots, \hat{I}_n, \ldots$ is a corresponding collection of independent right ideals of $\hat{R}$. Since $\hat{I}_n$ is a direct summand of $\hat{R}$, if $\hat{R}$ is Σ-injective, then $K = \sum_{n=1}^{\infty} \oplus \hat{I}_n$, being a direct summand of $\hat{R}^{(\omega)}$, is injective. Let $P_n = I_1 + \cdots + I_n$ (= the sum of $I_1, \ldots, I_n$ in R), let $Q_n = \hat{I}_1 + \cdots \hat{I}_n$, (= the sum of $\hat{I}_1, \ldots, \hat{I}_n$ in $\hat{R}$), $n = 1, 2, \ldots$, set $P = \sum_{n=1}^{\infty} P_n$, and set $Q = \sum_{n=1}^{\infty} Q_n$. Since the ideals $\{\hat{I}_i \mid i = 1, 2, \ldots\}$ are independent, then $Q \cong K$, as a right R-module, so Q is injective in $\mathscr{M}_R$. This means that $Q \in \mathscr{C}_r(\hat{R})$, a direct summand of $\hat{R}$. If $e = e^2 \in \hat{R}$ is such that $Q = e\hat{R}$, then $e \in Q_n$ for some n. Since Q_n is a right ideal of $\hat{R}$, $Q = e\hat{R} \subseteq Q_n$. It follows that $\hat{I}_k = 0$, hence $I_k = 0$, $\forall k > n$, proving that R satisfies the a.c.c. on complement right ideals.

Conversely, if R satisfies the a.c.c. on complement right ideals, then $\hat{R}$ satisfies the a.c.c. on direct summands. Since $\hat{R}$ is a regular ring, each finitely generated right ideal is a direct summand; consequently $\hat{R}$ is right noetherian. A regular noetherian ring is artinian and semisimple. The fact we need is that $\hat{R}$ is left artinian ring. Since $\Lambda = \mathrm{Hom}_R(\hat{R}, \hat{R})$ is naturally ring-isomorphic to $\hat{R}$ (see [6, §8]), then $\mathscr{A}_l(\hat{R}, R)$ satisfies the d.c.c., or equivalently $\mathscr{A}_r(\hat{R}, R)$ satisfies the a.c.c. Then Proposition 3.3 implies that $\hat{R}$ is Σ-injective.

Lemma 2. *The following conditions are equivalent:*

(1) *R is a semiprime ring satisfying the a.c.c. on complement right ideals, and $Z_r(R) = 0$.*

(2) *R is a semiprime ring satisfying the a.c.c. on complement right ideals, and the a.c.c. on right annulets.*

(3) *R has a classical right quotient ring S which is a semisimple ring.*

Proof. The equivalence of (2) and (3) is Goldie's theorem [9], whereas that of (1) and (2) is [6, § 9, Theorem 7].

Using this lemma, Proposition 1 implies,

Corollary 3. *If R is a semiprime ring satisfying the a.c.c. on complement right ideals, and the a.c.c. on right annulets, then $\hat{R}$ is Σ-injective.*

An integral domain R is a *right Ore domain* in case R has a right quotient field K (and then $K = \hat{R}$). (Any commutative integral domain is an Ore domain.)

Corollary 4. *If R is an integral domain, then $\hat{R}$ is Σ-injective if and only if R is a right Ore domain.*

Proof. If R is Σ-injective, then the proposition implies that R satisfies (1), hence (3), of the lemma. Thus, R is right Ore in this case. The converse is similarly proved.

Thus, the a.c.c. on right annulets of R does not imply Σ-injectivity of $\hat{R}$.

Remark. It is known that each injective right R-module is Σ-injective if and only if R is right noetherian (see [7]).

Appendix

The following theorem is proved by S. U. Chase [3]. For the convenience of readers, we include a proof here.

Theorem (S. U. Chase). *If R is semiprimary, then each module in $\mathscr{M}_R$, and each module in ${}_R\mathscr{M}$, satisfies the d.c.c. on finitely generated submodules.*

Proof. (S. U. Chase). The proof is by induction on the index n of nilpotency of the radical J of R. If $n = 1$, then $J = 0$ and R is semisimple. Then every module in $\mathscr{M}_R$ is semisimple, and the theorem is true in this case.

Hence, assume the theorem for all semiprimary rings whose radical has index of nilpotency $< n$. If $M \in \mathscr{M}_R$, let $S(M)$ denote the socle of M (= the sum of all the simple submodules of M), and let $\bar{A}$ denote the image of any submodule A of M under the natural homomorphism $\varphi : M \to \bar{M}$, where $\bar{M} = M/S$, $S = S(M)$. Since $S = (0 : J)$, and since $MJ^{n-1} \subseteq S$, $\bar{M} = M/S$ is an R/J^{n-1}-module. Since $\operatorname{rad}(R/J^{n-1}) = J/J^{n-1}$ has index of nilpotency equal to $n-1$, we may assume that $\bar{M}$ satisfies the d.c.c. on finitely generated submodules.

Now let $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_t \supseteq \cdots$ be any descending chain of finitely generated submodules of M. Since $\bar{I}_i$ is also finitely generated, $i = 1, 2, \ldots$, it follows that there exists an integer k such that $\bar{I}_k = \bar{I}_q$, $\forall q \geq k$. Thus, $I_k + S = I_q + S$, $\forall q \geq k$, where $S = S(M)$. Since $SJ = 0$, this implies that $I_kJ = I_qJ$, $\forall q \geq k$. Denote this submodule of M by K, and let $I'_i = I_i/K$, $i = 1, 2, \ldots$. Then

$$I'_1 \supseteq I'_2 \supseteq \cdots \supseteq I'_t \supseteq \cdots$$

is a descending chain of finitely generated modules. Since $I'_iJ = 0$, $i = 1, 2, \ldots$, these are R/J-modules. Hence, by the $n = 1$ case of the theorem, there exists $p \geq k$ such that $I'_t = I'_p$, $\forall t \geq p$, whence $I_t = I_p$, $\forall t \geq p$.

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