ON THE EXPLICITE DEFINING RELATIONS OF ABELIAN SCHEMES OF LEVEL THREE

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Dedicated to the memory of Professor TADASI NAKAYAMA

It is known classically that abelian varieties of dimension one over the field of complex numbers may be expressed by non-singular *Hesse's canonical cubic plane curves*, $X_0^3+X_1^3+X_{-1}^3-6\gamma X_0X_1X_{-1}=0$. The purpose of the present paper is to generalize this idea to higher dimensional case.

Let $\mathbf{Z}(3)$ be the residue group of the additive group $\mathbf{Z}$ of integers modulo $3\mathbf{Z}$ and $\mathbf{Z}(3)^r$ be the r-times direct sum of $\mathbf{Z}(3)$. We mean by $\mathbf{Z}(3)^{+r}$ the subset of $\mathbf{Z}(3)^r$ consisting of all the elements $(a_1^+,\ldots,a_r^+)$ such that $a_i^+=0$ or 1 $(1\leq i\leq r)$. Then, roughly speaking, our result may be expressed as follows: a generic abelian variety with a positive divisor U such that $l(U)=1$[1] is defined by relations of the following type

$$(*)\qquad \Delta_2 Y_{a+b}Y_{-a+b}Y_b-\sum_{c\in \mathbf{Z}(3)^r}\gamma_{a,c}Y_{c+b}^3=0$$

$$(**)\qquad \Delta_1 Y_{a+b}Y_{-a+b}-\sum_{c^+\in \mathbf{Z}(3)^{+r}}\beta_{a,c^+}Y_{c^++b}Y_{-c^++b}=0,\qquad (a,b\in \mathbf{Z}(3)^r).$$

§ 1. Formal theta functions of level n and the scheme $A(r,n)$ associated with them

1.1. We mean by $\mathbf{Z}$ and $\mathbf{Q}$ the ring of intergers and the field of rational numbers. We mean by $\mathbf{Z}^r$ the r-times direct sum of the $\mathbf{Z}$-module $\mathbf{Z}$ and by $\mathbf{Q}^r$ the r-times direct sum of the $\mathbf{Q}$-module $\mathbf{Q}$. Let $\{W(i;\alpha),\ W(j,l;\beta)\mid 1\leq i,j,l\leq r;\ \alpha,\beta\in\mathbf{Q}\}$ be a system of indeterminates on which rational numbers operate such that $W(i;\alpha)^\gamma=W(i;\alpha\gamma)$, $W(j,l;\beta)^\gamma=W(j,l;\beta\gamma)$. We denote by I the ideal in the polynomial ring $\mathbf{Z}[\{W(i;\alpha),\ W(f,l;\beta)\}]$ generated by

$$W(i;0)-1,\ W(j,l;0)-1,\ W(i;n\alpha)-\overbrace{W(i,\alpha)\cdots W(i;\alpha)}^{n},$$
$$W(j,l;n\beta)-\overbrace{W(j,l,\beta)\cdots W(j,l;\beta)}^{n}$$

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[1] $l(U)$ means the rank of the module of the multiples of $-U$.

$W(i;\alpha)W(i;\beta)-W(i;\alpha+\beta)$, $W(j,l;\alpha)W(j,l;\beta)-W(i,l;\alpha+\beta)$,
$W(i,l;\alpha)-W(l,j;\alpha)$,

$$(1\leq i,j,l\leq r;,\ \alpha,\beta\in\mathbf{Q},n=1,2,\dots)$$

We mean by U_j^α and Q_{jl}^β the images of $W(i;\alpha)$ and $W(j,l;\beta)$ in the residue ring $B=\mathbf{Z}[\{W(i;\alpha),W(j,l;\beta)\}]/I$. Then it follows the relations:

$$U_i^{n\alpha}=\overbrace{U_i^\alpha\cdots U_i^\alpha}^{n},\quad Q_{il}^{n\beta}=\overbrace{Q_{il}^\beta\cdots Q_{il}^\beta}^{n},\quad U_i^0=1,\quad Q_{li}^0=1, \tag{1}$$

$$U_i^\alpha U_i^\beta=U_i^{\alpha+\beta},\quad Q_{lj}^\alpha Q_{lj}^\beta=Q_{jl}^{\alpha+\beta} \tag{2}$$

$$(U_i^\alpha)^\gamma=U_i^{\alpha\gamma},\quad (Q_{lj}^\beta)^\gamma=Q_{jl}^{\beta\gamma}, \tag{3}$$

$$Q_{jl}^\alpha=Q_{jl}^\alpha, \tag{4}$$

$$(1\leq i,j,l,\leq r;\alpha,\beta,\gamma\in\mathbf{Q};n=1,2,3,\dots).$$

We shall use the following brief notations:

$$U(\alpha)=\prod_{i=1}^r U_i^{\alpha_i}, \tag{5}$$

$$Q(\alpha,\beta)=\prod_{i,j=1}^r Q_{ij}^{\alpha_i\beta_j}, \tag{6}$$

$$(\alpha=(\alpha_1,\dots,\alpha_r),\ \beta=(\beta_1,\dots,\beta_r)\in\mathbf{Q}^r).$$

Then if follows

$$U(\alpha)U(\beta)=U(\alpha+\beta), \tag{7}$$

$$Q(\alpha+\beta,\gamma+\delta)=Q(\alpha,\gamma)Q(\alpha,\delta)Q(\beta,\gamma)Q(\beta,\delta), \tag{8}$$

$$(\alpha,\beta,\gamma,\delta\in\mathbf{Q}^r).$$

1.2. We mean by Hom $(\mathbf{Z}^r,G_m)$ the functor of the category of commutative rings into the category of ablian groups such that Hom $(\mathbf{Z}^r,G_m)(A)$ means the group of all the homomorphisms of the additive group $\mathbf{Z}^r$ into the multiplicative group of the units in A. By virtue of (7) U may be considered as an element in Hom $(\mathbf{Z}^r,G_m)(B)$. For each α in $\mathbf{Q}^r$ we may construct an element $Q(\alpha)$ in Hom $(\mathbf{Z}^r,G_m)(B)$ given by

$$Q(\alpha)(m)=Q(\alpha,m),\ (m\in\mathbf{Z}^r)^{2)}. \tag{9}$$

We mean by $Q(\alpha)U$ the product of $Q(\alpha)$ and U in Hom $(\mathbf{Z}^r,G_m)$.

Definition 1. We mean by a formal rational power series in Q of restricted

[2] We change the notation $Q(\alpha)$ slightly. In [1] and [2] we defined $Q(\alpha)$ by $Q(\alpha)(m)=Q(\alpha,m)^2$.

type a formal rational power series $\sum_{(\alpha_{ij})} \lambda_{(\alpha_{ij})} \prod_{i\leq j} Q_{ij}{}^{\alpha_{ij}}$ satisfying the conditions: 1° there exists a positive integer m such that $m\alpha_{ij} \in \mathbf{Z}(1 \leq i, j \leq r)$ for (α_{ij}) satisfying $\lambda_{(\alpha_{ij})} \neq 0$, 2° for any positive integer n there exist only a finite number of terms $\lambda_{(\alpha_{ij})} \prod_{i\leq j} Q_{ij}{}^{\alpha_{ij}}$ such that $\lambda_{(\alpha_{ij})} \neq 0$ and $\alpha_{ii} \leq n (1 \leq i \leq r)$.

All the formal rational power series in Q of restricted type with coefficients in $\mathbf{Z}$ form a commutative integral domain. We denote it by $\mathbf{Z}[[Q]]$.

We shall now give the definition of formal theta functions:

Definition 2. We mean by a formal theta functions of level n with coefficients in $\mathbf{Z}[[Q]]$ a formal power series in U_i^2, $U_i^{-2} (1 \leq i \leq r)$

$$\varphi(U) = \sum_{m\in\mathbf{Z}^r} \lambda_m U(m)^2$$

with coefficients λ_m *in* $\mathbf{Z}[[Q]]$ *such that*

$$\varphi(Q(m)U) = Q(m, m)^{-n} U(m)^{-2n} \varphi(U), \ (m \in \mathbf{Z}^r). \tag{10}$$

1.2. We denote by $\mathbf{Z}(n)$ the residue group of $\mathbf{Z}$ modulo $n\mathbf{Z}$. We denote by $0, 1, 2, \ldots, n-1$ sometimes the elments in $\mathbf{Z}(n)$ and sometimes integers $0, 1, 2, \ldots, n-1$ in $\mathbf{Z}$ so that $a/n (a \in \mathbf{Z}(n))$ makes sense. We denote by $\mathbf{Z}(n)^r$ the r-times direct sum of $\mathbf{Z}(n)$. By virtue of the difference equation (10) the coefficients λ_m of a formal theta function $\sum_{m\in\mathbf{Z}^r} \lambda_m U(m)^2$ of level n are given by

$$\lambda_{nm+g} = \lambda_g Q\left(\frac{g}{n}, \frac{g}{n}\right)^{-n} Q\left(m + \frac{g}{n}, m + \frac{g}{n}\right)^n, \quad (g \in \mathbf{Z}(n)^r, \ m \in \mathbf{Z}^r). \tag{11}$$

We shall introduce the canonical system of formal theta functions of level n:

$$X_{n,g}(Q \mid U) = \sum_{m\in\mathbf{Z}^r} Q\left(m + \frac{g}{n}, m + \frac{g}{n}\right)^n U\left(m + \frac{g}{n}\right)^{2n}, \quad (g \in \mathbf{Z}(n)^r). \tag{12}$$

Then it follows the formulae

$$X_{n,g}(Q \mid U^{-1}) = X_{n,-g}(Q \mid U) \tag{13}$$

$$X_{n,g}\left(Q \mid Q\left(\frac{h}{n}\right) U\right) = Q\left(\frac{h}{n}, \frac{h}{n}\right)^{-n} U\left(\frac{h}{n}\right)^{-2n} X_{n,g+h}(Q \mid U), \quad (g, h \in \mathbf{Z}(n)^r). \tag{14}$$

From (11) it follows that a formal theta function $\sum_{m\in\mathbf{Z}^r} \lambda_m U(m)^2$ is a linear combination of $X_{n,g}(Q \mid U) (g \in \mathbf{Z}(n)^r)$ as follows:

(15) $$\sum_{m\in Z^r}\lambda_m U(m)^2=\sum_{g\in Z(n)^r}\lambda_g Q\left(\frac{g}{n}\cdot\frac{g}{n}\right)^{-n}X_{n,g}(Q\mid U).$$

Putting $U=1$, we have a system of elements in $\mathbf{Z}[[Q]]$:

(16) $$T_{n,g}(Q)=X_{n,g}(Q\mid 1),\ (g\in \mathbf{Z}(n)^r),$$

From (13) it follows

(17) $$T_{n,-g}(Q)=T_{n,g}(Q),\ (g\in\mathbf{Z}(n)^r).$$

We denote by $R(r,n)$ the graded ring $\mathbf{Z}[(T_{n,g}(Q))_{g\in \mathbf{Z}(n)^r}]$[3] and by $S(r,n)$ the projective scheme Proj $(R(r,n))$ of the graded ring $R(r,n)$, where deg $T_{n,g}(Q)=n(g\in\mathbf{Z}(n)^r)$. Since $R(r,n)$ is a commutative noetherian integral domain, the scheme $S(r,n)$ is irreducible, reduced and noetherian.

By virtue of (12) and (15) it follows the fundamental property of theta functions of level n:

Proposition 1. *The formal theta functions of level n $X_{n,g}(Q\mid U)(g\in\mathbf{Z}(n)^r)$ form a base of the formal theta functions of level n over any field containing $R(r,n)$.*

We denote by $O_{S(r,n)}$ the structure sheaf of $S(r,n)$ and by $O_{S(r,n)}[(X_{n,g}(Q\mid U))_{g\in\mathbf{Z}(n)^r}]$ the graded $O_{S(r,n)}$-algebra induced by $R(r,n)[(X_{n,g}(Q\mid U))_{g\in\mathbf{Z}(n)^r}]$. We mean by $A(r,n)$ the projective $S(r,n)$-scheme $\mathrm{Proj}_{S(r,n)}(O_{S(r,n)}[(X_{n,g}(Q\mid U))_{g\in\mathbf{Z}(n)^r}])$. Since the ring $R(r,n)[(X_{n,g}(Q\mid U))_{g\in\mathbf{Z}(n)^r}]$ is a noetherian integral domain, $A(r,n)$ is also a noetherian irreducible reduced scheme.

§ 2. Formal theta functions of level three

2.1. We shall use the following notations;

$\mathbf{Z}(3)$: the residue group of $\mathbf{Z}$ modulo $3\mathbf{Z}$, we denote by $0, 1, -1$, sometimes the elements in $\mathbf{Z}(3)$ and sometimes integers $0, 1, -1$ in $\mathbf{Z}$ so that $a/3$ $(a\in(3))$ makes sense,

$\mathbf{Z}(3)^+$: the subset $\{0, 1\}$ in $\mathbf{Z}(3)$,

$\mathbf{Z}(3)^r$: the r-times direct sum,

$\mathbf{Z}(3)^{+r}$: the subset in $\mathbf{Z}(3)^r$ consisting of all the elements $(a_1^+,\dots,a_r^+)$ such that $a_i^+=0$ or 1, $(1\le i\le r)$,

[3] Since $T_{n,g}(g\in Z(n)^r)$ are modular forms of degree r for certain congruence subgroup, $Z[T_{n,g}(Q))_{g\in Z(n)^r}]$ forms a graded ring.

$a, b, c, \ldots$: the elements in $\mathbf{Z}(3)^r$,

$a^+, b^+, c^+, \ldots$: the elements in $\mathbf{Z}(3)^{+r}$,

$$T_a = T_{3,a}(Q) = \sum_{m\in Z^r} Q\left(m+\frac{a}{3}, m+\frac{a}{3}\right)^3 \qquad (a\in\mathbf{Z}(3)^r),$$

$(T_{a^++b^+}T_{-a^++b^+})$: the $2^r\times 2^r$-matrix of which (a^+, b^+)-component is

$$T_{a^++b^+}T_{-a^++b^+},$$

$(T_{a+b}T_{-a+b})$: the $3^r\times 2^r$-matrix of which (a, b^+)-component is

$$T_{a+b^+}T_{-a+b^+},$$

$(T_{a+b}T_{-a+b}T_b)$: the $3^r\times 3^r$-matrix of which (a, b)-component is

$$T_{a+b}T_{-a+b}T_b,$$

(T^3_{a+b}): the $3^r\times 3^r$-matrix of which (a, b)-component is T^3_{a+b},

$$\begin{aligned}
&\Delta_1(T) = \det\,(T_{a^++b^+}T_{-a^++b^+}),\\
&\Delta_2(T) = \det\,(T^3_{a+b}),\\
&\alpha_{a^+,b^+}(T)) = \Delta_1(T)(T_{a^++b^+}T_{-a^++b^+})^{-1},\\
&(\beta_{a,b^+}(T)) = \Delta_1(T)(T_{a+b^+}T_{-a+b^+})(T_{a^++b^+}T_{-a^++b^+})^{-1}\\
&\qquad\qquad = (T_{a+b^+}T_{-a^++b^+})(\alpha_{+,b^+}(T)),\\
&(\gamma_{a,c}(T)) = \Delta_2(T)(T_{a+b}T_{-a+b}T_b)(T^3_{a+b})^{-1},\\
&\qquad\qquad\qquad\qquad\qquad\qquad (a^+, b^+\in\mathbf{Z}(3)^{+r};\ a, b\in\mathbf{Z}(3)^r).
\end{aligned}$$

We shall first show some typical relations between

$$X_{3,a}(Q\mid U) = \sum_{m\in Z^r} Q\left(m+\frac{a}{3}, m+\frac{a}{3}\right)^3 U\left(m+\frac{a}{3}\right)^6 \qquad (a\in\mathbf{Z}(3)^r)$$

with coefficients in $R(r, 3) = \mathbf{Z}[(T_a)_{a\in Z(3)^r}]$:

Proposition 2.

$$\begin{aligned}
(18)\qquad &\Delta_1(T)X_{3,a+b}(Q\mid U)X_{3,-a+b}(Q\mid U)\\
&= \sum_{c^+\in Z(3)^{+r}} \beta_{a,c}(T)X_{3,c^++b}(Q\mid U)X_{3,-c^++b}(Q\mid U),\\
(19)\qquad &\Delta_2(T)X_{3,a+b}(Q\mid U)X_{3,-a+b}(Q\mid U)X_{3,b}(Q\mid U)\\
&= \sum_{c\in Z(3)^r} \gamma_{a,c}(T)X_{3,c+b}(Q\mid U)^3, \qquad (a, b\in\mathbf{Z}(3)^r).
\end{aligned}$$

Proof. As we shall see in § 3 the determinants $\Delta_1(T)$ and $\Delta_2(T)$ are not zero. By virtue of (14) it follows

$$\begin{aligned}\Delta_1(T) &= \det(T_{a^++b^+}T_{-a^++b^+}) = \det(X_{3,a^++b^+}(Q\,|\,1)X_{3,-a^++b^+}(Q\,|\,1))\\ &= \det\Big(Q\Big(\frac{b^+}{3},\frac{b^+}{3}\Big)^6 X_{3,a^+}\Big(Q\,|\,Q\Big(\frac{b^+}{3}\Big)\Big)X_{3,-a^+}\Big(Q\,|\,Q\Big(\frac{b^+}{3}\Big)\Big)\Big)\\ &= \prod_{b^+}\Big(Q\Big(\frac{b^+}{3},\frac{b^+}{3}\Big)^6 \det\Big(X_{3,a^+}\Big(Q\,|\,Q\Big(\frac{b^+}{3}\Big)\Big)X_{3,-a^+}\Big(Q\,|\,Q\Big(\frac{b^+}{3}\Big)\Big)\\ &\neq 0,\end{aligned}$$

$$\begin{aligned}\Delta_2(T) &= \det(T^3_{a+b}) = \det(X_{3,a+b}(Q\,|\,1)^3)\\ &= \det\Big(Q\Big(\frac{b}{3},\frac{b}{3}\Big)^9 X_{3,a}\Big(Q\,|\,Q\Big(\frac{b}{3}\Big)\Big)^3\Big)\\ &= \prod_{b}\Big(Q\Big(\frac{b}{3},\frac{b}{3}\Big)^9 \det\Big(X_{3,a}\Big(Q\,|\,Q\Big(\frac{b}{3}\Big)\Big)^3\Big)\\ &\neq 0.\end{aligned}$$

Since $R(r,3)$ is an integral domain, it follows that $\{X_{3,c}(Q\,|\,U)X_{3,-c}(Q\,|\,U)c^+\in \mathbf{Z}(3)^{+r}\}$ and $\{X_{3,a}(Q\,|\,U)^3\,|\,a\in\mathbf{Z}(3)^r\}$ are sets of linearly independent formal theta functions of level 6 and 9, respectively. By virtue of (15) formal theta functions of level 6 (resp. level 9) $\sum_{m\in Z}\lambda_m U(m)^2$ such that $\lambda_{3m+1}=\lambda_{3m-1}=0$ $(m\in\mathbf{Z}^r)$ form a vector space of dimension 6 (resp. dimension 9) over the quotient field of $R(r,3)$. Therefore we may put

$$X_{3,a}(Q\,|\,U)X_{3,-a}(Q|U) = \sum_{c^+\in Z(3)^{+r}}\lambda_{a,c}X_{3,c^+}(Q\,|\,U)X_{3,-c^+}(Q\,|\,U),$$

$$X_{3,a}(Q\,|\,U)X_{3,-a}(Q\,|\,U)X_{3,0}(Q\,|\,U) = \sum_{c\in Z(3)^r}\mu_{a,c}X_{3,c}(Q\,|\,U)^3.$$

Putting $U=Q\Big(\frac{c}{3}\Big)(c\in\mathbf{Z}(3)^r)$ we have

$$\lambda_{a,c}=\Delta_1(T)^{-1}\beta_{a,c}(T),\ \mu_{a,c}=\Delta_2(T)^{-1}\gamma_{a,c}(T),\ (a,c\in\mathbf{Z}(3)^r,\ c^+\in\mathbf{Z}(3)^{+r}).$$

For every $b\in\mathbf{Z}(3)^r$ by virtue of (15) it follows

$$\Delta_1(T)X_{3,a+b}(Q\,|\,U)X_{3,-a+b}(Q\,|\,U) = \sum_{c^+\in\mathbf{Z}(3)^{+r}}\beta_{a,c^+}(T)X_{3,c^++b}(Q\,|\,U)X_{3,-c^++b}(Q\,|\,U),$$

$$\Delta_2(T)X_{3,a+b}(Q\,|\,U)X_{3,-a+b}(Q\,|\,U)X_{3,b}(Q\,|\,U) = \sum_{c\in Z(3)^r}\gamma_{a,c}(T)X_{3,c+b}(Q\,|\,U)^3.$$

§ 3. The explicite defining relations of abelian schemes of level three

3.1. Let $(Y_a)_{a\in\mathbf{Z}(3)^r}$ be a system of indeterminates and $R(r,3)[Y]$ be the graded ring $R(r,3)[(Y_a)_{a\in\mathbf{Z}(3)^r}]$. Let I_v be the homogeneous ideal in $R(r,3)[Y]$ generated by the following homogeneous elements

$$\{k_{l,a}(T)Y_{a+b}Y_{-a+b} - \sum_{c^+\in\mathbf{Z}(3)^r}h_{3,a\,c^+}(T)Y_{c^++b}Y_{-c^++b}\,|\,h_{3,a,c^+}(T)h_{l,a}(T)^{-1} = \beta_{a,c^+}(T)\Delta_1(T)^{-1};\ a,b\in\mathbf{Z}(3)^r,\ c^+\in\mathbf{Z}(3)^{+r}\}$$

and

$$\{h_{2,a}(T)Y_{a+b}Y_{-a+b}Y_b - \sum_{c\in Z(3)^r} h_{4,a,c}(T)Y^3_{c+b} \mid h_{4,a,c}(T)h_{2,a}(T)^{-1} = \gamma_{a,c}(T)\Delta_2(T)^{-1};\ a, b, c \in \mathbf{Z}(3)^r\}.$$

Since $\Delta_1(T)$, $\Delta_2(T)$, $\beta_{a,c^+}(T)$, $\gamma_{a,c}(T)$ are homogeneous elements in $R(r, 3)$ such that $\deg \Delta_1(T) = \deg \beta_{a,c^+} = 2^r$ and $\deg \Delta_2(T) = \deg \gamma_{a,c}(T)$ the homogeneous ideal I_V of $R(r, 3)[Y]$ induces an indeal $\mathfrak{J}_V$ of the $O_{S(r,3)}$-algebra $O_{S(r,3)}[Y]$. We denote by $V(r, 3)$ the $S(r, 3)$-projective scheme $\mathrm{Proj}_{S(r,3)}(O_{S(r,3)}[Y]/\mathfrak{J}_V)$ of the graded $O_{S(r,3)}$ algebra $O_{S(r,3)}[Y]/\mathfrak{J}_V$.

We mean by $(X_a)_{a\in\mathbf{Z}(3)^r}$ the image of $(Y_a)_{a\in\mathbf{Z}(3)^r}$ in the residue ring $R(r,3)[Y]/I_V$ and by $R[X]$ the graded ring $R(r,3)[(X_a)_{a\in\mathbf{Z}(3)^r}]$. The elements $X_a (a \in \mathbf{Z}(3)^r)$ are characterized by the relations:

(20) $$h_{1,a}(T)X_{a+b}X_{-a+b} = \sum_{c^+\in\mathbf{Z}(3)^{+r}} h_{3,a,c^+}(T)X_{c^++b}X_{-c^++b},$$

$$(h_{3,a,c^+}(T)h_{1,a}(T)^{-1} = \beta_{a,c^+}(T)\Delta_1(T)^{-1},\ a, b \in \mathbf{Z}(3)^r,\ c^+ \in \mathbf{Z}(3)^{+r})$$

(11) $$h_{2,a}(T)X_{a+b}X_{-a+b}X_b = \sum_{c\in\mathbf{Z}(3)^r} h_{4,a,c}(T)X^3_{c+b},$$

$$(h_{4,a,c}(T)h_{2,a}(T)^{-1} = \gamma_{a,c}(T)\Delta_2(T)^{-1},\ a, b, c \in \mathbf{Z}(3)^r).$$

By virtue of Proposition 2 the formal theta functions $X_{3,a}(Q \mid U)$ $(a \in \mathbf{Z}(3)^r)$ satisfy the relations (20) and (21). Hence the map: $X_a \to X_{3,a}(Q \mid U)$ $(a \in \mathbf{Z}(3)^r)$ may be extended to an $O_{S(r,3)}$-morphism ρ of $O_{S(r,3)}[Y]/\mathfrak{J}_V$ onto $O_{S(r,3)}[(X_{3,a}(Q \mid U))_{a\in\mathbf{Z}(3)^r}]$. The dual ρ^* of ρ gives the injection morphism of $A(r, 3)$ into $V(r, 3)$.

3.2. When $r = 1$, the relation (21) is reduced to a single relation

(22) $$(T_0^3 + 2T_1^3)X_0X_1X_{-1} = T_0T_1^2(X_0^3 + X_1^3 + X_{-1}^3)$$

and the relation (20) is trivial. We shall express $\Delta_1(T)$, $\Delta_2(T)$, $\beta_{a,c^+}(T)$, $\gamma_{a,c}(T)$:

(23) $$\Delta_1(T) = \det\begin{pmatrix} T_0T_0, & T_1T_1 \\ T_1T_{-1}, & T_{-1}T_0 \end{pmatrix} = T_1(T_0^3 - T_1^3) \neq 0$$

(24) $$\Delta_2(T) = \det\begin{pmatrix} T_0^3, & T_1^3, & T_{-1}^3 \\ T_1^3, & T_{-1}^3, & T_0^3 \\ T_{-1}^3, & T_0^3, & T_1^3 \end{pmatrix} = 3T_0^3T_1^6 - (T_0^9 + 2T_1^9) \neq 0$$

(25) $$\beta_{01}(T) = \beta_{\pm0,1}(T) = 0,\ \beta_{00}(T) = \beta_{\pm1,1}(T) = \Delta_1(T)$$

(26) $$\gamma_{0,\pm1}(T) = 0,\ \gamma_{0,0}(T) = \Delta_2(T),\ \gamma_{\pm1,0}(T) = \gamma_{\pm1,1} = \gamma_{\pm1,-1}.$$

We denote by $T_a^{(i)}$ the power series

$$T_a^{(i)} = X_a^{(i)}(Q_{ii}\,|\,1) \sum_{m\in\mathbf{Z}(3)} Q_{ii}^{3(m+a/3)^2}, \quad (1\leq i\leq r;\ a\in\mathbf{Z}(3)),$$

and denote by $X_a^{(i)}$ $(1\leq i\leq r;\ a\in(3))$ the quantities defined by the relation:

$$(T_0^{(i)3}+2\,T_1^{(i)2})X_0^{(i)}X_1^{(i)}X_{-1}^{(i)} = T_0^{(i)}T_1^{(i)2}(X_0^{(i)3}+X_1^{(i)3}+X_{-1}^{(i)3})$$
$$(1\leq i\leq r).$$

We denote by $R_i^*(1,3)$ the subring $Z[(T_a^{(i)})_{a\in(3)}]_{(\Delta_1(T^{(i)})\Delta_2(T^{(i)}))}$ of degree zero in the quotient ring of $Z[(T_a^{(i)})_{a\in\mathbf{Z}(3)}]$ with respect to $\Delta_1(T^{(i)})\Delta_2(T^{(i)})$, $(1\leq i\leq r)$ and by $R^*(r,3)$ the subring $R_{(\Delta_1(T)\Delta_2(T))}$ of degree zero in the quotient ring of R with respect to $\Delta_1(T)\Delta_2(T)$. We denote by $S^*(1,3)$ and $S^*(r,3)$ the affine scheme $\mathrm{Spec}(R_1^*(1,3))$ and $\mathrm{Spec}(R^*(r,3))$, respectively

Since Q_{ii} $(1\leq i\leq r)$ are indetermenates and $T_a^{(i)} = T_{3,a}^{(i)}(Q)$ $(a\in\mathbf{Z}(3))$ are formal power series in Q_{ii}, the map:

$$T_{a_1}^{(1)}\otimes\cdots\otimes T_{a_r}^{(r)}\to T_{a_1}^{(1)}\cdots T_{a_r}^{(r)} \quad (a_1,\ldots,a_r\in\mathbf{Z}(3))$$

induces an isomorphism of $R_1^*(1,3)\otimes\cdots\otimes R_r^*(1,3)$ onto the subring R^{**} of degree zero in the quotient ring of $\mathbf{Z}[(T_{a_1}^{(1)}\cdots T_{a_r}^{(r)})_{a_1,\ldots,a_r\in\mathbf{Z}(3)}]$ with respect to $\prod_{i=1}^{r}\Delta_1(T^{(i)})\Delta_2(T^{(i)})$. For the sake of simplicity we shall identify $T_{a_1}^{(1)}\otimes\cdots\otimes T_{a_r}^{(r)}$ with $T_{a_1}^{(1)}\cdots T_{a_r}^{(r)}$, $(a,\ldots,a_r\in\mathbf{Z}(3))$. Let $F(Y)$ be an element in $\mathbf{Z}[Y]$. Then $F(T)=0$ means that $F(T(Q))$ is zero as a formal rational power series. Therefore an equality $F(T)=0$ implies $F(T^{(1)}\otimes\cdots\otimes T^{(r)})=0$, because, replacing $Q_{i,j}$ $(i\neq j)$ by 1 in $F(T(Q))$, we have $F(T^{(1)}\otimes\cdots\otimes T^{(r)})$. This shows that the map:

$$T_{(a_1,\ldots,a_r)}\to T_{a_1}^{(1)}\otimes\cdots\otimes T_{a_r}^{(r)} = T_{a_1}^{(1)}\cdots T_{a_r}^{(r)}, \quad (a_1,\ldots,a_r\in\mathbf{Z}(3))$$

induces a surjective morphism λ: $R^*(r,3)\to R^{**} = R_1^*(1,3)\otimes\cdots\otimes R_r^*(1.3)$.

The dual λ^* of λ is the injection morphism:

$$S^{**} = S^*(1,3)\times\overset{r}{\cdots}\times S^*(1,3)\to S^*(r,3),$$

where S^{**} is considered as the affine scheme $\mathrm{Spec}(R^{**})$.

We mean by I^{**} the homogeneus ideal in $R^{**}[Y]$ generated by

$$\lambda(\Delta_1(T))Y_{a+b}Y_{-a+b} - \sum_{c^+\in\mathbf{Z}(3)^{+r}}\lambda(\beta_{a,c^+}(T))Y_{c^++b}Y_{-c^++b},$$
$$\lambda(\Delta_2(T))Y_{a+b}Y_{-a+b}Y_b - \sum_{c\in\mathbf{Z}(3)^r}\lambda(\gamma_{a,c}(T))Y_{c+b}^3, \quad (a,b\in\mathbf{Z}(3)^r).$$

We mean by $(Z_a)_{a\in\mathbf{Z}(3)^r}$ the image of $(Y_a)_{a\in\mathbf{Z}(3)^r}$ in the residue ring $R^{**}[Y]/I^{**}$. Then $(Z_a)_{a\in\mathbf{Z}(3)^r}$ satisfies the relations:

$$Z_{a+b}Z_{-a+b} = (\Delta_1(T))^{-1} \sum_{c^+\in \mathbf{Z}(3)^r} \lambda(\beta_{a,c^+}(T))Z_{c^++b}Z_{-c^++b},$$

$$Z_{a+b}Z_{-a+b}Z_b = \lambda(\Delta_2(T))^{-1} \sum_{c\in \mathbf{Z}(3)^r} \lambda(\gamma_{a,c}(T))Z^3_{c+b}, \quad (a, b \in \mathbf{Z}(3)^r).$$

We shall prove the isomorphism:

$$(V(1,3)\times \cdots \times V(1,3))\times_{S(1,3)\times\cdots\times S(1,3)} S^{**} = V(r,3)\times_{S(r,3)} S^{**}.$$

LEMMA 1.

$$\lambda(\Delta_1(T)) = \Delta_1(T^{(1)})\cdots \Delta_1(T^{(r)}),$$

$$\lambda(\Delta_2(T)) = \Delta_2(T^{(1)})\cdots \Delta_2(T^{(r)}),$$

$$\lambda(\beta_{(a_1,\ldots,a_r),(b_1^+,\ldots,b_r^+)}(T)) = \begin{cases} (\Delta^1(T^{(1)})\cdots \Delta_1(T^{(r)}), \textit{ for } (a_1,\ldots,a_r)\times \\ (b_1^+,\ldots,b_r^+) \textit{ such that } |a_i| = b_i^+ \ (1\leq i\leq r), \\ 0 \textit{ otherwise,}\end{cases}$$

$$\lambda(\gamma_{(a_1,\ldots,a_r),(b_1,\ldots,b_r)}(T)) = \gamma_{a_1,b_1}(T^{(1)})\cdots \gamma_{r,b_r}(T^{(r)}),$$

$$(a_1,\ldots,a_r, b_1,\ldots,b_r \in \mathbf{Z}(3)),$$

where we mean

$$|a| = \begin{cases} a & \textit{for } a = 0, 1, \\ -a & \textit{for } a = -1.\end{cases}$$

Proof. From the definition it follows:

$$\lambda(\Delta_i(T)) = \Delta_i(\lambda(T)) = \Delta_i(T^{(1)}\otimes\cdots\otimes T^{(r)})$$

$$= \Delta_i(T^{(1)})\otimes\cdots\otimes \Delta_i(T^{(r)}) = \Delta_i(T^{(1)})\cdots \Delta_i(T^{(r)}),$$

$$\lambda(\beta_{a(_1,\ldots,a_r),(b_1^+,\ldots,b_r^+)}(T)) = \beta_{(a_1,\ldots,a_r),(b_1^+,\ldots,b_r^+)}(T^{(1)}\cdots T^{(r)})$$

$$= \beta_{a_1,b_1^+}(T^{(1)})\cdots \beta_{a_r,b_r^+}(T^{(r)})$$

$$= \begin{cases} \Delta_1(T)^{(1)})\cdots \Delta_1(T^{(r)}), \text{ for } (a_1,\ldots,a_r)\times \\ (b_1^+,\ldots,b_r^+) \text{ such that } |a_i| = b_i^+ \ (1\leq i\leq r), \\ 0 \text{ otherwise,}\end{cases}$$

$$\lambda(\gamma_{(a_1,\ldots,a_r),(b_1,\ldots,b_r)}(T)) = \gamma_{(a_1,\ldots,a_r),(b_1,\ldots,b_r)}(T^{(1)}\cdots T^{(r)})$$

$$= \gamma_{a_1,b_r}(T^{(1)})\cdots \gamma_{a_r,b_r}(T^{(r)}),$$

$$(a_1,\ldots,a_r, b_1,\ldots,b_r \in \mathbf{Z}(3)).$$

LEMMA 2. *It follows the relations:*

$$Z_{(a_1+b_1,\ldots,a_r+b_r)}Z_{(-a_1+b_1,\ldots,-a_r+b_r)} = Z_{(|a_1|+b_1,\ldots,|a_r|+b_r)}Z_{(-|a_1|+b_1,\ldots,-|a_r|+b_r)},$$

$$Z_{(b_1,\ldots,b_{i-1},a+b_i,b_i+1,\ldots,b_r)}Z_{(b_1,\ldots,b_{i-1},a+b_i,b_i+1,\ldots,b_r)}Z_{(b_1,\ldots,b_r)}$$

$$= \Delta(T^{(i)})^{-1}\sum_{c\in \mathbf{Z}(3)} \gamma_{,c}(T^{(i)})Z^3_{(b_1,\ldots,b_{i-1},c+b_i,b_i+1,\ldots,b_r)},$$

$$(1\leq i\leq r;\ a_1,\ldots,a_r,b_1,\ldots,b_r \in \mathbf{Z}(3)).$$

This is a consequence of the definition of $(Z_a)_{a\in\mathbf{Z}(3)^r}$ and Lemma 1.

LEMMA 3. *Let $(c_1, \dots, c_r)$ be a fixed element in $\mathbf{Z}(3)^r$ and put*

$$Z_a^{(i)} = Z_{(c_1 \dots, c_{i-1}, i, c_{i+1} \dots, c_r)} \qquad (1 \leq i \leq r;\ a \in \mathbf{Z}(3)).$$

Then it follows

$$Z_{(1 \dots r)} Z_{(c_1, \dots, c_r)}^{r-1} = Z_{a_1}^{(1)} \cdots Z_{a_r}^{(r)}, \qquad (a_1, \dots a_r \in \mathbf{Z}(3)).$$

Proof. From Lemma 2 it follows

$$Z_{(a_1, \dots, a_r)} Z_{(c_2, \dots c_r)} = Z_{(a_1, c_2, \dots, c_r)} Z_{(c_1, a_2, \dots, a_r)}$$
$$Z_{(c_1, a_2, \dots, a_r)} Z_{(c_1, \dots c_r)} = Z_{(c_1, a_2, c_3, \dots, c_r)} Z_{(c_1, c_2, a_3, \dots, a_r)},$$
$$\cdots$$
$$Z_{(c_1, \dots, c_{r-2}, a_1, a_2)} Z_{(c_1, \dots, c_r)} = Z_{(c_1, \dots, c_{r-2}, a_{r-1}, c_r)} Z_{(c_1, \dots, c_{r-1}, a_r)}$$

Hence, making the product of the both sides of these equations, we have

$$Z_{(a_1, \dots, a_r)} Z_{(c_1, \dots, c_r)} = Z_{a_1}^{(1)} \cdots Z_{a_r}^{(r)}, \qquad (a_1, \dots, a_r \in \mathbf{Z}(3)).$$

LEMMA 4. $(V(1,3) \times \cdots \times V(1,3)) \times_{S(1,3) \times \cdots \times S(1,3)} S^{**} = V(r,3) \times_{S(r,3)} S^{**}$ *(as $S(r,3)$-scheme).*

Proof. We denote by $D_+(Z_c)$ the affine open subscheme $\{z \in V(r,3) \times_{S(r,3)} S^{**} \mid Z_c(z) \neq 0\}$ and by $D_+(X_{c_1}^{(1)} \cdots X_{c_r}^{(r)})$ the affine open subscheme $\{u_1 \times \cdots \times u_r \in (V(1,3) \times \cdots \times V(1,3)) \times_{S(1,3) \times \dots \times S(1,3)} S^{**} \mid X_{c_1}^{(1)}(u_1) \cdots X_{c_r}^{(r)}(u_r) \neq 0\}$, where $(X_a^{(i)})_{1 \leq i \leq r, a \in \mathbf{Z}(3)}$ is a system of quantities defined by

$$\Delta_1(T^{(i)}) X_0^{(i)} X_1^{(i)} X_{-1}^{(i)} - \gamma(T^{(i)})(X_0^{(i)3} + X_3^{(i)3} + X_{-1}^{(i)3}) = 0 \qquad (1 \leq i \leq r).$$

From Lemma 1 it follows

$$\lambda(\Delta_1(T))(X_{a_1+b_1}^{(1)} \otimes \cdots \otimes X_{a_r+b_r}^{(r)})(X_{-a_1+b_1}^{(1)} \otimes \cdots \otimes X_{-a_r+b_r}^{(r)})$$
$$- \sum_{c_1^+, \dots, c_1^+} \lambda(\beta_{(a_1, \dots, a_r), (c_1^+, \dots c_1^+, \dots, c_1^+)}(T))(X_{c_1^+ + b_1}^{(1)} \otimes \cdots \otimes X_{c_r^+ + b_r}^{(r)})$$
$$(X_{-c^+ + b_1}^{(1)} \otimes \cdots \otimes X_{-c_r^+ + b_r}^{(1)})$$
$$= [\Delta_1(T^{(1)}) X_{a_1+b_1}^{(1)} X_{-a_1+b_1}^{(1)} - \sum_{c_1^+} \beta_{a_1, c_1^+}(T^{(r)}) X_{c_1^+ + b_1}^{(1)} X_{-c_1^+ + b_1}^{(1)}] \otimes \dots \otimes$$
$$[\Delta_1(T^{(r)}) X_{a_r+b_r}^{(r)} X_{-a_r+b_r}^{(r)} - \sum_{c_r^+} \beta_{a_r, c_r^+}(T^{(r)}) X_{c_r^+ + b_r}^{(r)} X_{-c_r^+ + b_r}^{(r)}] = 0,$$

$$\lambda(\Delta_2(T))(X_{a_1+b_1}^{(1)} \cdots X_{a_r+b_r}^{(r)}) - \sum_{(c_1, \dots, c_r)} \lambda(\gamma_{(a),(c)}(T)) X_{c_1+b_1}^{(1)} \otimes \cdots \otimes X_{c_r+b_r}^{(r)}$$
$$= [\Delta_2(T^{(1)}) X_{a_1+b_1}^{(1)} X_{-a_1+b_1}^{(1)} X_{b_1}^{(1)} - \sum_{c_1} \gamma_{a_1, c_1}(T^{(1)}) X_{c_1+b_1}^{(1)3}] \otimes \cdots \otimes$$
$$[\Delta_2(T^{(r)}) X_{a_r+b_r}^{(r)} X_{-a_r+b_r}^{(r)} X_{b_r}^{(r)} - \sum_{c_r} \gamma_{a_r, c_r}(T^{(r)}) X_{c_r+b_r}^{(r)3}] = 0.$$

This shows that the map: $Z_{(a_1,\dots,a_r)} \to X_{a_1}^{(1)} \cdots X_{a_r}^{(r)} (a_1, \dots, a_r \in Z(3))$ induces the injection morphism ψ of $(V(1,3) \times \cdots \times V(1,3)) \times_{S(1,3) \times \cdots \times S(1,3)} S^{**}$ into $V(r,3) \times_{S(r,3)} S^{**}$ such that $\psi(D_+(X_{c_1}^{(1)} \cdots X_{c_r}^{(r)})) \subseteq D_+(Z_{(c_1,\dots,c_r)})$. We shall construct the inverse morphism of ψ. Put $Z_a^{(i)} = Z_{(c_1,\dots,c_{i-1},a,c_{i+1},\dots,c_r)}$ $(1 \le i \le r;$ $a \in (3))$. Then by virtue of Lemma 1 and 2 it follows

$$\begin{aligned}
&Z_{a+b}^{(i)} Z_{-a+b}^{(i)} - Z_{|a|+b}^{(i)} Z_{-|a|+b}^{(i)} \\
&\quad = Z_{(c_1,\dots,c_{i-1},a+b,c_{i+1},\dots,c_r)} Z_{(c_1,\dots,c_{i-1},-a\pm b,c_{i+1},\dots,c_r)} \\
&\quad - Z_{(c_1,\dots,c_{i-1},|a|+b,c_{i+1},\dots,c_r)} Z_{(c_1,\dots,c_{i-1},-|a|+b,c_{i+1},\dots,c_r)} = 0, \\
&Z_{a+b}^{(i)} Z_{-a+b}^{(i)} Z_b^{(i)} - \Delta_2(T^{(i)})^{-1} \sum_c \gamma_{a,c}(T^{(i)}) Z_{c+b}^{(i)3} \\
&\quad = Z_{(c_1,\dots,c_{i-1},a+b,c_{i+1},\dots,c_r)} Z_{(c_1,\dots,c_{i-1},-a+b,c_{i+1},\dots,c_r)} Z_{c_1,\dots,c_{i-1},b_{i+1},\dots,c_r)} \\
&\quad - \Delta_2(T^{(1)}) \cdots \Delta_2(T^{(r)}) \sum \gamma_{(c_1,\dots,c_{i-1},a,c_{i+1}\cdots c_r),(c_1,\dots,c_{i-1},d,c_{i+1},\dots,c_r)} \\
&\quad (T^{(1)} \oplus \cdots \otimes T^{(r)}) Z^3_{(c_1,\dots,c_{i-1},b+d,c_{i+1},\dots,c_r)} = 0.
\end{aligned}$$

Therefore by virtue of Lemma 3 it follows that the map:

$$X_{a_1}^{(1)} \cdots X_{a_r}^{(r)} \to Z_{a_1}^{(1)} \cdots Z_{a_r}^{(r)} = Z_{(1,\dots,a_r)} Z_{(c_1,\dots,a_r)}^{-1} \ (a_1, \dots, a_r \in \mathbf{Z}(3)^r)$$

induces the injective morphism $\psi'_{(c_1,\dots,c_r)}$ of $D_+(Z_{(c_1,\dots,c_r)})$ into $D_+(X_{c_1}^{(1)}, \dots, X_{c_r}^{(1)})$. These morphisms ψ and $\psi'_{(c,\dots,c_r)}$ are the inverse each other as $S(r,3)$-morphisms between $D_+(X_{c_1}^{(1)} \cdots X_{c_r}^{(1)})$ and $D(Z_{(c_1,\dots,c_r)})$. Since ψ is defined on $(V(1,3) \times \cdots \times V(1,3)) \times_{S(1,3) \times \cdots \times S(1,3)} S^{**}$, there exists an $S(r,3)$-morphism ψ' such that $\psi' | D(Z_{(c_1,\dots,c_r)}) = \psi'_{(c_1,\dots,c_r)}$. This completes the proof of Lemma 4.

3.3. We denote by $M_{R^*(r,3)}(Y, m)$ the $R^*(r,3)$-submodule in $R^*(r,3)[Y]$ consisting of all the elements of degree m, by $M_{R^*(r,3)}(X, m)$ the $R^*(r,3)$-submodule in $R^*(r,3)[X]$ consisting of all the elements of degree m and by $I_{R(r,3)}(m)$ the $R^*(r,3)$-submodule in the kernel of $R^*(r,3)[Y]$ onto $R^*(r,3)[X]$ consisting of all the elements of degree m. For a point x in $S = S(r,3)$ we mean by $M_{O_{S,x}}(X, m)$, $M_{C_{S,x}}(Y, m)$, $I_{O_{S,x}}(m)$, $M_{k_x}(Y, m)$, $M_{k_x}(Y, m)$, $I_{k_x}(m)$ the tensor products

$$\begin{gathered}
M_{R^*(r,3)}(X, m) \otimes_{R^*(r,3)} O_{S,x}, \quad M_{R^*(r,3)}(X, m) \otimes_{R^*(r,3)} O_{S,x}, \quad I_{R(r,3)} \otimes_{R^*(r,3)} O_{S,x}, \\
M_{O_{S,x}}(Y, m) \otimes_{O_{S,x}} k_x, \quad M_{O_{S,x}}(X, m) \otimes_{O_{S,x}} k_x, \quad I_{O_{S,x}}(m) \otimes_{O_{S,x}} k_x,
\end{gathered}$$

respectively. Then it follows the exact sequence

$$0 \to I_{R(r,3)}(m) \to M_{R^*(r,3)}(Y, m) \to M_{R^*(r,3)}(Y, m) \to 0$$

$$0 \to I_{O_{S,x}}(m) \to M_{O_{S,x}}(Y, m) \to M_{O_{S,x}}(X, m) \to 0 \tag{27}$$

$$I_{k_x} \to M_{k_x}(Y, m) \to M_{k_x}(X, m) \to 0 \tag{28}$$

LEMMA 4. *Let x_0 be the generic point on $S(r, 3)$ and y be any point on S^* $S^*(r, 3)$. Then it follons:*

$$\operatorname{rank}_{k_{x_0}} M_{k_{x_0}}(X, m) \leqq \operatorname{rank}_{k_y} M_{k_y}(X, m) \qquad (m = 1, 2, 3, \ldots).$$

Proof. Since O_{S, x_0} is the quotient field of $R^*(r, 3)$, it follows $k_{x_0} = O_{S, x_0}$ and the exact sequence

$$0 \to I_{k_{x_0}}(m) \to M_{k_{x_0}}(Y, m) \to M_{k_{x_0}}(X, m) \to 0.$$

Since $Y_a (a \in Z(3)^r)$ are indeterminates, it follows

$$\operatorname{rank}_{k_y} M_{k_0}(Y, m) = \operatorname{rank}_{k_{x_0}} M_{k_{x_0}}(Y, m).$$

Then it is sufficient to prove the inequality

$$\operatorname{rank}_{k_{x_0}} I_{k_{x_0}}(m) \geq \operatorname{rank} I_{k_y}(m) \qquad (m = 1, 2, 3, \ldots).$$

Let $L_1, L_2, \ldots, L_{N(m)}$ be all the monomials of degree m in $Y_a (a \in Z(3)^r)$. Then there exists a matrix with coefficients in $R^*(r, 3)$:

$$\Omega^{(m)} = (\omega_{i,j}^{(m)}), \qquad (1 \leq i \leq \lambda(m),\ 1 \leq j \leq N(m))$$

such that

$$\sum_{j=1}^{N(m)} \omega_{i,j}^{(m)} L_j \qquad (1 \leq i \leq \lambda(m))$$

generates $I_{R^*(r,3)}(m)$. Let $\mathfrak{p}_y$ be the prime ideal in $R^*(r, 3)$ corresponding to a point y in $S^*(r, 3)$. Then it follows

$$\operatorname{rank}_{k_{x_0}} I_{k_{x_0}}(m) = \operatorname{rank}_{k_{x_0}} \Omega^{(m)},\ \operatorname{rank}_{k_y} I_{k_y}(m) = \operatorname{rank}_{k_y}(\Omega^{(m)} \bmod \mathfrak{g}_y),$$
$$(m = 1, 2, 3, \ldots).$$

This implies

$$\operatorname{rank}_{k_0} I_{k_0}(m) \geq \operatorname{rank}_{k_y} I_{k_y}(m), \qquad (m = 1, 2, 3, \ldots).$$

PROPOSITION 3. *Let x_0 be the generic point on $S(r, 3)$. Then it follows*

$$V(r, 3) \times_{S(r,3)} x_0 \simeq A(r, 3) \times_{S(r,3)} x_0.$$

Proof. Let $z_0^{(1)}$ be the generic point in $S(1, 3)$. Then $V(1, 3) \times_{S(1,3)} z_0^{(1)}$ is defined by the equation $(T_0^3 + 2T_1^3) X_0 X_1 X_{-1} - (T_0 T_1^2)(X_0^3 + X_1^3 + X_{-1}^3) = 0$. On the other hand the scheme $A(1, 3) \times_{S(1,3)} z_0^{(1)}$is also defined by a cubic equation. This shows that Proposition 3 is true for $r = 1$. Let z_0 be the generic point in S^{**} $= S^*(1, 3) \overbrace{\times \cdots \times}^{r} S^*(1, 3)$. Then by virtue of Lemma 3 we have

$$
\begin{aligned}
V(r,3)\times_{S(r,3)}z_0 &= (\overbrace{V(1,3)\times\cdots\times V(1,3)}^{r})\times_{S(1,3)\times\cdots\times S(1,3)}z_0\\
&= (\overbrace{V(1,3)\times_{S(1,3)}z_0^{(1')})\times\cdots\times(V(1,3)\times_{S(1,3)}z_0^{(1)}}^{r})\\
&= (\overbrace{A(1,3)\times_{S(1,3)}z_0^{(1)})\times\cdots\times(A(1,3)\times_{S(1,3)}z_0^{(1)}}^{r})\\
&= (\overbrace{A(1,3)\times\cdots\times A(1,3))\times_{S(1,3)\times\cdots\times S(1,3)}}^{r}z_0\\
&= (A(1,3)\times\cdots\times A(1,3))\times_{S(r,3)}z_0\\
&= A(r,3)\times_{S(r,3)}z_0.
\end{aligned}
$$

From (22) it follows

$$\mathrm{rank}_{k_{z_0^{(1)}}}M_{k_{z_0^{(1)}}}(X^{(1)},m)=3\,m,\qquad (m=1,2,3,\ldots).$$

From the above relation we have

$$\mathrm{rank}_{k_{z_0}}M_{k_{z_0}}(X,m)=3^r m^r,\qquad (m=1,2,3,\ldots).$$

Therefore from Lemma 4 it follows

$$\mathrm{rank}_{k_{x_0}}M_{k_{x_0}}(X,m)\leq \mathrm{rank}_{k_{z_0}}M_{k_{z_0}}(X,m)\leq 3^r m^r,\qquad (m=1,2,3,\ldots).$$

On the other hand by virtue of Proposition 2 there exists an injection ρ^* of $A(r,3)$ into $V(r,3)$ as $S(r,3)$-scheme. From Proposition 1 this implies

$$\mathrm{rank}_{k_{x_0}}M_{k_{x_0}}(X,m)\geq 3^r m^r,\qquad (m=1,2,3,\ldots).$$

Hence we have the relations

$$\mathrm{rank}_{k_{x_0}}M_{k_{x_0}}(X,m)=3^r m^r,\ \mathrm{rank}_{k_{x_0}}I_{k_{x_0}}(X,m)=N(m)-3^r m^r \qquad (m=1,2,3,\ldots).$$

Let J_{k_0} be the ideal in $k_{x_0}[Y]$ corresponding to the closed scheme $\rho^*(A(r,3))\times_{S(r,3)}x_0$ and $J_{k_0}(m)$ the k_{x_0}-submodule in J_{k_0} consisting of all the elements of degree m. Then by virtue of Proposition 1 and 2 it follows

$$\mathrm{rank}\,J_{k_{x_0}}(m)=N(m)-3^r m^r,\ \mathrm{rank}\,J_{k_{x_0}}(m)\leq \mathrm{rank}\,I_{k_{x_0}}(m),\qquad (m=1,2,3,\ldots).$$

This implies $J_{k_{x_0}}=I_{k_{x_0}}$. Namely we have the isomorphism between scheme $V(r,3)\times_{S(r,3)}x_0$ and $A(r,3)\times_{S(r,3)}x_0$.

Finally we shall state the main theorem:

THEOREM 1. *There exists an open subscheme $U(r,3)$ in $S(r,3)$ such that the $U(r,3)$-scheme*

$$V(r, 3) \times_{S(r,3)} U(r, 3)$$

is an abelian scheme, and that $U(r,3) \times_{\mathbf{Z}} GF(p)$ *is a non-empty open set in* $S(r,3) \times_{\mathbf{Z}} GF(p)$ *for every finite prime* p[4].

§ 4. The explicite addition formula for theta functions of level theree

4.1. The addition formula for theta functions of level three is simple and beautiful. First we shall introduce theta functions in $U \otimes_R \cdots \otimes_R U$, where $R = R(r, 3)$. We mean by a theta function of level n in $U \otimes_R \cdots \otimes_R U$ with period $Q \otimes_R \cdots \otimes Q_R$ a power series

$$\varphi(U \otimes_R \cdots \otimes_R U) = \sum_{m_1, \ldots, m_l \in \mathbf{Z}^r} \lambda_{(m_1, \ldots, m_l)} U(m_1)^2 \otimes_R \cdots \otimes_R U(m_e)^2$$

such that

$$\begin{aligned} &\varphi(Q(m_1) \otimes_R \cdots \otimes_R Q(m_l))(U \otimes_R \cdots \otimes_R U)) \\ &= Q(m_1, m_1)^{-n} \cdots Q(m_l, m_l)^{-n} (U(m_1)^{-2n} \otimes \cdots \otimes U(m_l)^{-2n})(U \otimes_R \cdots \otimes_R U), \\ &\qquad (m_1, \ldots, m_l \in \mathbf{Z}^r). \end{aligned}$$

Then, similarly as Proposition *l*, the tensor products $X_{3,a_1}(Q \mid U) \otimes_R \cdots \otimes_R X_{3,a_l}(Q \mid U)$ $(a_1, \ldots, a_l \in \mathbf{Z}(3)^r)$ form a base of theta functions of level three in $U \otimes_R \cdots \otimes_R U$ over the quotient field of R. Similarly as Proposition 2 the tensor products $(X_{3,c^+ + a}(Q \mid U) X_{3,-c^+ + a}(Q \mid U)) \otimes_R (X_{3,d^+ + b}(Q \mid U) X_{3,-d^+ + b}(Q \mid U))$ $(c^+, d^+ \in \mathbf{Z}(3)^{+r},\ a, b \in \mathbf{Z}(3)^r)$ form a base of theta functions of level six in $U \otimes_R U$.

We mean by $X_{3,a}(Q \mid U \otimes_R U)$ and $X_{3,a}(Q \mid U \otimes U^{-1})$ the power series

$$\sum_{m \ni \mathbf{Z}^r} Q\left(m + \frac{a}{3}, m + \frac{a}{3}\right)^3 U\left(m + \frac{a}{3}\right)^6 \otimes_R U\left(m + \frac{a}{n}\right)^6$$

and

$$\sum_{m \in \mathbf{Z}^r} Q\left(m + \frac{a}{3}, m + \frac{a}{3}\right)^3 U\left(m + \frac{a}{n}\right)^6 \otimes_R U\left(m + \frac{a}{n}\right)^{-6}, \qquad (a \in \mathbf{Z}(3)^r).$$

In these notation the addition formula is expressed as follows:

THEOREM 2. (*The addition formula*)

---

[4] Starting with $R_p = GF(p)[(T_a)_{a \in Z(3)^r}]$ and $R_p[(X_a(T \mid U))_{a \in Z(3)^r}]$ we can get the result over $GF(p)$ similar as that over $\mathbf{Z}$ in § 3 (by the same method). This shows that $U(r, 3) \times_{\mathbf{Z}} GF(p)$ is a non-empty open set.

$$\text{(29)} \qquad \begin{aligned} &\Delta_1(T)X_{3,a+b}(Q\,|\,U\otimes_R U)X_{3,a-b}(Q\,|\,U\otimes_R U^{-1}) \\ &= \sum_{c^+,d^+\in \mathbf{Z}(3)^{+r}} \alpha_{c^+,d^+}(T)(X_{3,c^++a}(Q\,|\,U)X_{3,-c^++a}(Q\,|\,U)) \\ &\otimes_R (X_{3,d^++b}(Q\,|\,U)X_{3,-d^++b}(Q\,|\,U)), \qquad (a,b\in \mathbf{Z}(3)^r). \end{aligned}$$

Proof. From the definitions it follows the relations

$$\begin{aligned} &X_{3,a+b}(Q\,|\,U\otimes_R U)X_{3,a-b}(Q\,|\,U\otimes_R U^{-1}) \\ &= \sum_{m',m''\in \mathbf{Z}^r} Q\Big(m'+\frac{a+b}{3},\ m'+\frac{a+b}{3}\Big)^3 Q\Big(m''+\frac{a-b}{3},\ m''+\frac{a-b}{3}\Big)^6 \\ &\Big(U\Big(m'+\frac{a+b}{3}\Big)^3\otimes_R U\Big(m'+\frac{a+b}{3}\Big)^6\Big)\Big(U\Big(m''+\frac{a-b}{3}\Big)^6\otimes_R U\Big(m''+\frac{a-b}{3}\Big)^{-6}\Big) \\ &= \sum_{m',m''\in \mathbf{Z}^r} Q\Big(m'+\frac{a+b}{3},\ m'+\frac{a+b}{3}\Big)^3 Q\Big(m''+\frac{a-b}{3},\ m''+\frac{a-b}{3}\Big)^3 \\ &\qquad\qquad U(3(m'+m'')+2a)^2\otimes_R U(3(m'-m'')+2b)^2. \end{aligned}$$

This shows that $X_{3,a+b}(Q\,|\,U\otimes_R U)X_{3,-a-b}(Q\,|\,U\otimes_R U^{-1})$ has an expansion $\sum_{m',m''}\lambda_{3m'+2a,3m''+2b}U(3m'+2a)\otimes_R U(3m''+2b)$. On the other hand from the difference equation (10) it follows

$$\begin{aligned} &X_{3,a+b}(Q\,|\,Q(m')\otimes_R Q(m''))(U\otimes_R U))X_{3,a-b}(Q\,|\,(Q(m')\otimes_R Q(m'')^{-1})(U\otimes_R U^{-1})) \\ &= Q(m',m')^{-6}Q(m'',m'')^{-6}U(m')^{-12}\otimes_R U(m)^{-12}X_{3,a+b}(Q\,|\,U\otimes_R U)X_{3,a-b})Q\,|\,U\otimes_R U^{-1}). \end{aligned}$$

Therefore we have

$$\begin{aligned} &X_{3,a+b}(Q\,|\,U\otimes_R U)X_{3,a-b}(Q\,|\,U\otimes_R U^{-1}) \\ &= \sum_{c^+,d^+\in \mathbf{Z}(3)^{+r}} \lambda_{c^+,d^+}(X_{3,c^++a}(Q\,|\,U)X_{3,-c^++a}(Q\,|\,U))\otimes_R(X_{3,d^++b}(Q\,|\,U)X_{3,-d^++}(Q\,|\,U)). \end{aligned}$$

Putting $U=Q\Big(\frac{c}{3}\Big)$ $(c\in \mathbf{Z}(3)^r)$, we have

$$\lambda_{c^+,d^+} = \Delta_1(T)^{-1}\alpha_{c^+,d^+}(T), \qquad (c^+,d^+\in \mathbf{Z}(3)^{+r}).$$

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