THE NOTION OF RESTRICTED IDÈLES WITH APPLICATION TO SOME EXTENSION FIELDS

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Dedicated to the memory of Professor TADASI NAKAYAMA

Let k be an algebraic number field of finite degree, K be its normal extension of degree n, and $\hat{s}$ be the set of those primes of K which have degree 1. Using this set $\hat{s}$ instead of the set of all primes of K, we define an $\hat{s}$-restricted idèle of K by the same way as ordinary idèles. It is known by Bauer that the normal extension of an algebraic number field is determined by the set of all primes of the ground field which are decomposed completely in the extension field. This suggests that if we treat abelian extensions over K which are normal over k, the class field theory is expressed by means of the $\hat{s}$-restricted idèles (theorem 2). When $K=k$, $\hat{s}$ is the set of all primes of K, and we have the ordinary class field theory.

In fact, the $\mathfrak{P}$-completion $K_{\mathfrak{P}}$ is isomorphic to the $\mathfrak{p}$-completion $k_{\mathfrak{p}}$, when $\mathfrak{P}$ belongs to $\hat{s}$ and divides a prime $\mathfrak{p}$ of k. Therefore the group of all $\hat{s}$-restsicted idèles of K is isomorphic to the direct product of n-fold of a group of restricted idèles of k. This means that the abelian extensions over K which are normal over k are determined by some *objects* ($\hat{s}$*-admissible* subgroups of the $\hat{s}$-restricted idèle group) in the ***ground*** field k. In this paper we shall characterize this object by the connected component of the unity in the idèle class group of K (theorem 3) and by the principal idèle group of K (theorem 4). This result is a generalization of our preceeding paper [8] (§8).

Throughout this paper the following notations will be used.

F	an algebraic number field of finite degree.
$F^{\times}$	the multiplicative group of all non zero elements of F, which is identified with the principal idèle group of F.
$F_{\mathfrak{p}}$	the $\mathfrak{p}$-adic completion of F, where $\mathfrak{p}$ is a finite or infinite prime of F.
$U_{\mathfrak{p}}$	the $\mathfrak{p}$-adic unit group of F.

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$J=J_F$	the idèle group of F.
J_∞	the subgroup of all elements of J which have components 1 at all finite primes.
$\mathfrak{C}=\mathfrak{C}_F=J/F^\times$	the idèle class group of F.
$\mathfrak{D}$	the connected component of the unity of $\mathfrak{C}$.
$G(K/F)$	the Galois group of a Galois extension K/F.
$S(F)$	the set of all finite or infinite primes of F.
$S(K/F)$	the set of those primes of F which have at least a prime divisor of degree 1 in a finite extension K/k.
$\hat{S}(K/F)$	the set of those primes of K which are of degree 1 in a finite extension K/F.
$(A\ ;\ B)$	the index of the subgroup B in the group A or the degree of an extension field A/B.
$\overline{A}$	the closure of A in J if A is a subset of J,
$A\cdot B$	the set of all products of elements from A and B, if A and B are subsets of a group.

§ 1. Projection and restriction of idèle groups

Let $S=S(F)$ be the set of all primes of F, s be a subset of S containing infinitely many elements, and s' be its complement in $S: s'=S-s$. We denote by J_s the restricted direct product (as topological group) of $\mathfrak{p}$-adic completions $F_\mathfrak{p}$ over $\mathfrak{p}$-adic unit groups $U_\mathfrak{p}$ of F where $\mathfrak{p}$ runs over s, that is, the group of all elements of the direct product $\prod_{\mathfrak{p}\in s} F_\mathfrak{p}$ whose $\mathfrak{p}$-components belong to $U_\mathfrak{p}$ for almost all $\mathfrak{p}$ of s. Denote also by U_s the unit group $\prod_{\mathfrak{p}\in s} U_\mathfrak{p}$ of J_s. If $s=S$, then $J_s=J_S$ is the ordinary idèle group of F. We shall call an element of J_s resp. of U_s a *s-restricted idèle* resp. *s-restricted unit idèle*.

We have algebraically and topologically

$$J\cong J_s\times J_{s'} \quad \text{(direct)}. \tag{1}$$

Throughout this paper we shall fix this isomorphism, and the notation $\times$ is used for this direct product. Denote by π_s the projection of J to J_s, which is a multiplicative and open continuous map and is called *s-projection*. For any $\mathfrak{o}\in J$ we set always $\mathfrak{o}_s=\pi_s(\mathfrak{o})$. Put $F_s=\pi_s(F^\times)$, $C_s=J_s/F_s$. By $\iota_\mathfrak{p}$, $\iota_{\mathfrak{p},s}$ and ι_s we mean the natural injections of $F_\mathfrak{p}$ to J, $F_\mathfrak{p}$ to J, $F_\mathfrak{p}$ to J_s, and J_s to J, respectively, These notations are omitted frequently when it can be done without danger of misunderstanding.

We define the s-*restrtction* ρ_s by

$$\rho_s(A) = \pi_s(A \cap \iota_s J_s) \tag{2}$$

for any subset of A of J. If an idèle $\mathfrak{a} \in J$ is not contained in $\iota_s J_s$, we write $\rho_s(\mathfrak{a}) = \phi$. Then $\rho_s(\mathfrak{a}\mathfrak{b}) = \rho_s(\mathfrak{a})\rho_s(\mathfrak{b})$ for $\mathfrak{a}, \mathfrak{b} \in J$ if $\rho_s(\mathfrak{a}) \neq \phi$ and $\rho_s(\mathfrak{b}) \neq \phi$.

LEMMA 1. *Let A be any subgroup of J.*

Then $$(\iota_s J_s)A = J_s \times \pi_{s'}(A) \quad (\textit{direct}).$$

Proof. Let $\mathfrak{a}\mathfrak{b} \in A(\iota_s J_s)$, $\mathfrak{a} \in A$, $\mathfrak{b} \in \iota_s J_s$. Since $\mathfrak{a} = \mathfrak{a}_s \times \mathfrak{a}_{s'}$ and $\mathfrak{b} = \mathfrak{b}_s \times 1$, we have $\mathfrak{a}\mathfrak{b} = (\mathfrak{a}\mathfrak{b})_s \times \mathfrak{a}_{s'} \in J_s \times \pi_{s'}(A)$. Conversely let $\mathfrak{b}_s \in J_s$ and $\mathfrak{a} \in A$. Then $\mathfrak{b}_s \times \mathfrak{a}_{s'} = (\mathfrak{a}_s \times \mathfrak{a}_{s'})(\mathfrak{b}_s \mathfrak{a}_s^{-1} \times 1)A(\iota_s J_s)$. This lemma implies algebraically

$$\frac{J_s}{\rho_s(A)} = \frac{\iota_s J_s}{A \cap \iota_s J_s} \cong \frac{(\iota_s J_s)A}{A} = \frac{J_s \times \pi_{s'}(A)}{A}. \tag{3}$$

If $(J; A)$ is finite, then $(J_s; \rho_s(A)) = (J_s \times \pi_{s'}(A); A)$

$$= \frac{(J; A)}{(J; J_s \times \pi_{s'}(A))} = \frac{(J; A)}{(J_{s'}; \pi_{s'}(A))}.$$ Hence we have

LEMMA 2. *Let A be a subgroup of J of finite index. Then*

$$(J; A) = (J_s; \pi_s(A))(J_{s'}; \rho_{s'}(A))$$
$$= (J_s; \rho_s(A))(J_{s'}; \pi_{s'}(A)).$$

LEMMA 3. *Let A be any subgroup of J and B be any subgroup of J_s. Then*

$$\rho_s(A \cdot (\iota_s B)) = \rho_s(A) \cdot B.$$

Proof. Let $\mathfrak{a} \in A$ and $\mathfrak{b} \in \iota_s B$, then $\mathfrak{a}\mathfrak{b} = (\mathfrak{a}_s \times \mathfrak{a}_{s'})(\mathfrak{b}_s \times 1) = \mathfrak{a}_s \mathfrak{b}_s \times \mathfrak{a}_{s'}$. Hence $\rho_s(\mathfrak{a}\mathfrak{b}) = \mathfrak{a}_s \mathfrak{b}_s$ or $= \phi$ according to $\mathfrak{a}_{s'} = 1$ or $\neq 1$, that is, $\rho_s(\mathfrak{a}) = \mathfrak{a}_s$ or $= \phi$. Hence $\rho_s(A(\iota_s B)) \subset \rho_s(A)B$. Conversely let $\mathfrak{a} \in A$, $\mathfrak{b} \in B$ and $\rho_s(\mathfrak{a}) \neq \phi$. Then $\rho_s(\mathfrak{a})\mathfrak{b} = \rho_s(\mathfrak{a})\rho_s(\iota_s \mathfrak{b}) = \rho_s(\mathfrak{a}(\iota_s \mathfrak{b})) \in \rho_s(A(\iota_s B))$.

LEMMA 4. *Let A be a subgroup of J, B' be a subgroup of $S_{s'}$, and suppose $A \supset \iota_{s'} B'$. Then $\rho_s(A) \supset \pi_s(A \cap (J_s \times B'))$.*

Proof. Let $\mathfrak{a} \times \mathfrak{b} \in A \cap (J_s \times B')$, where $\mathfrak{a} \in J_s$ and $\mathfrak{b} \in B'$. Then $\pi_s(\mathfrak{a} \times \mathfrak{b}) = \mathfrak{a} = \pi_s((\mathfrak{a} \times \mathfrak{b})(1 \times \mathfrak{a}^{-1}))$. Since $\mathfrak{a} \times \mathfrak{b} \in A$ and $1 \times \mathfrak{b}^{-1} \in A$, we have $\mathfrak{a} \in \pi_s(A \cap \iota_s J_s) = \rho_s(A)$.

§ 2. Restricted idèles and the class field theory

Let k be any algebraic number field of finite degree, K/k be a normal extension of finite degree, and M be an abelian extension over K which is normal over k. Let us characterize such M by the restricted idèles attached to K/k.

The following propositions are well known.

PROPOSITION 1. (*Bauer*[1]) *Let K be any normal extension of k of finite degree and Ω be any extension of k of finite degree. Then $S(K/k)$ contains $S(\Omega/k)$ if and only if K is contained in Ω.*

We have immediately

PROPOSITION 1′. *Let K_1 and K_2 be normal extensions of finite degree. Then $S(K_1/k) = S(K_2/k)$ if and only if $K_1 = K_2$.*

PROPOSITION 2. (*Whaples*[2]). *Let A be any abelian extension of F of finite degree, and H be the subgroup of the idèle group $J = J_F$ corresponding to A by the class field theory. Let $\mathfrak{p}$ be a prime ideal of F. Then $\mathfrak{p}$ is not ramified in A if and only if $\iota_{\mathfrak{p}} U_{\mathfrak{p}} \subset H$, and $\mathfrak{p}$ decomposes completely in A if and only if $\iota_{\mathfrak{p}} F_{\mathfrak{p}} \subset H$.*

Now we prove the following

PROPOSITION 3. *Let K be a normal extension of k of finite degree and put $\hat{s} = \hat{S}(K/k)$. Let M_1 and M_2 be abelian extensions of K which are normal and of finite degree over k. Let further H_1 and H_2 be subgroups of the idèle group of K corresponding to M_1 and M_2, respectively, by the class field theory. Then we have $\rho_{\hat{s}}(H_1) = \rho_{\hat{s}}(H_2)$ if and only if $H_1 = H_2$, namely if and only if $M_1 = M_2$.*

Proof. It is obvious that $\mathfrak{p} \in S(M_1/k)$ if and only if $\mathfrak{p} \in S(K/k)$ and $\mathfrak{p} \in S(M_1/K)$ for a prime divisor $\mathfrak{P}$ of $\mathfrak{p}$ in K. On the other hand $\mathfrak{P} \in S(M_1/K)$ if and only if $\iota_{\mathfrak{P}} K_{\mathfrak{P}} \subset H_1$ by prop. 2. Assume $\mathfrak{p} \in S(K/k)$, then $\iota_{\mathfrak{P}} K_{\mathfrak{P}} \subset H_1$ if and only if $\iota_{\mathfrak{P}} K_{\mathfrak{P}} \subset \rho_{\hat{s}}(H_1)$ for any prime divisor $\mathfrak{P}$ of $\mathfrak{p}$ in K. Hence $S(M_1/k)$ consists of all $\mathfrak{p} \in S(K/k)$ such that $\iota_{\mathfrak{P}} K_{\mathfrak{P}} \subset \rho_{\hat{s}}(H_1)$ for any $\mathfrak{P}$ diving $\mathfrak{p}$. Therefore $\rho_{\hat{s}}(H_1) = \rho_{\hat{s}}(H_2)$ implies $S(M_1/k) = S(M_2/k)$, and this implies $M_1 = M_2$, hence $H_1 = H_2$, by prop. 1′. Conversely if $H_1 = H_2$, obviousely $\rho_{\hat{s}}(H_1) = \rho_{\hat{s}}(H_2)$. Thus the proposition is proved.

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1) See M. Bauer [3], H. Hasse [10], § 25, and also M. Deuring [4].

2) See G. Whaples [13], also E. Artin and J. Tate [2], Ch. 8.

Now let F be any algebraic number field of finite degree, J be the ordinary idèle group of F, and $\mathfrak{C} = J/F^{\times}$ be its idèle class group. Denote by $\mathfrak{D}$ the connected component of the unity of $\mathfrak{C}$, and by $\mathfrak{D}$ the complete inverse image of $\mathfrak{D}$ by the natural homomorphism of J to $\mathfrak{C}$. Let s be an infinite set of primes of F and J_s be the s-restricted idèle group of F. We shall call a subgroup H_s of J_s *s-admissible* if $H_s = \rho_s(\overline{H_s D})$. If H_s is s-admissible, then it is closed in J_s. In the case where s is equal to the set $S = S(F)$ of all primes of F, we call simply admissible instead of S-admissible. There is a one to one correspondence between the set of admissible subgroups of J and the set of class fields over F.

Let again K/k be a normal extension of finite degree with the Galois group $G(K/k)$, and put $\hat{s} = \hat{S}(K/k)$. For each $\hat{s}$-restricted idèle $\mathfrak{a} \in J_{\hat{s}}$ and each $\sigma \in G(K/k)$, we define $\mathfrak{a}^{\sigma}$ as the $\hat{s}$-restricted idèle whose $\mathfrak{p}^{\sigma}$-component is determined by $(\mathfrak{a}^{\sigma})_{\mathfrak{p}\sigma} = (\mathfrak{a}_{\mathfrak{p}})^{\sigma}$, where $\mathfrak{p}$ is a prime of K.

THEOREM 1. *Let K be a normal extension of k of finite degree and put $G = G(K/k)$, and $\hat{s} = \hat{S}(K/k)$. Then the $\hat{s}$-restriction $\rho_{\hat{s}}$ gives a one to one correspondence between the set of all G-invariant admissible subgroups H of J of finite index and the set of all G-invariant $\hat{s}$-admissible subgroups $H_{\hat{s}}$ of $J_{\hat{s}}$ of finite index. Futhermore if $\rho_{\hat{s}}(H) = H_{\hat{s}}$, then we have G-isomorphism $J/H \cong J_{\hat{s}}/H_{\hat{s}}$.*

Proof. Let H be a G-invariant admissible subgroup of J of finite index. Then $\rho_{\hat{s}}(H)$ is obviousely G-invariant. Since H contains both $\rho_{\hat{s}}(H)$ and D, and is closed, we see $H \supset \overline{\rho_{\hat{s}}(H) \cdot D}$. Hence $\rho_{\hat{s}}(H) \supset \rho_{\hat{s}}(\overline{\rho_{\hat{s}}(H)D})$. On the other hand obviousely $\rho_{\hat{s}}(H) \subset \rho_{\hat{s}}(\overline{\rho_{\hat{s}}(H)D})$. Thus $\rho_{\hat{s}}(H)$ is $\hat{s}$-admissible. Moreover we have $(J; H) = (J_{\hat{s}}; \rho_{\hat{s}}(H))(J_{\hat{s}'}; \pi_{\hat{s}'}(H))$ by lemma 2, and this implies that $\rho_{\hat{s}}(H)$ is of finite index as a subgroup of $J_{\hat{s}}$. Conversely let $H_{\hat{s}}$ be a G-invariant $\hat{s}$-admissible subgroup of $J_{\hat{s}}$ of finite index. Put $H = \overline{H_{\hat{s}} \cdot D}$. Then H is G-invariant and admissible. Since $H_{\hat{s}}$ is $\hat{s}$-admissible, $\rho_{\hat{s}}(H) = H_{\hat{s}}$. Thus if we prove the finiteness of $(J; H)$, then the first part of the theorem follows from prop. 3. So we shall prove the finiteness. Put $H_1 = \overline{J_{\hat{s}} D}$. Then H_1 is G-invariant and admissible subgroup of J. Moreover if $\mathfrak{P}$ is contained in $\hat{s}$, then $\iota_{\mathfrak{P}} K_{\mathfrak{P}}$ is contained in H_1. Let A_1 be the abelian extension over K corresponding to H_1 by the class field theory. Then A_1 is normal over k and we have $S(A_1/k) \supset S(K/k)$ by prop. 2. Hence $A_1 \subset K$ by prop. 1. We have necessarily $A_1 = K$, hence H_1

$=\overline{J_{\hat{s}}D}=J$. Thus we see algebraically $J/H=\overline{J_{\hat{s}}D}/H\subset J_{\hat{s}}H/H\cong J_{\hat{s}}/J_{\hat{s}}H=J_{\hat{s}}/J_{\hat{s}}(H)$ $=J_{\hat{s}}/H_{\hat{s}}$. Since $(J_{\hat{s}}; H_{\hat{s}})$ is finite, $(J; H)$ is finite. Thus the finiteness and so the one to one correspondence in the theorem is proved.

Now let again H be a G-invariant admissible subgroup of J of finite index. Since the restriction $\pi_{\hat{s}'}$ is an open map of J to $J_{\hat{s}'}$, $\pi_{\hat{s}'}(H)$ is closed in $J_{\hat{s}'}$. Moreover $\pi_{\hat{s}'}(H)$ is obviousely G-invariant and contains $\pi_{\hat{s}'}(D)$. Put $J_{\hat{s}}\times\pi_{\hat{s}'}(H)$ $=H_1$. Then $(J; H_1)$ is finite and H_1 is a G-invariant admissible subgroup of J. Let A_1 be the class field over K corresponding to H_1. Then A_1 is a normal extension over k and $S(A_1/K)\supset\hat{s}=\hat{S}(K/k)$ by prop. 2, hence $S(A_1/k)\supset S(K/k)$. This implies $A_1\subset K$ by prop. 1. Then we have necessarily $A_1=K$, which implies $J_{\hat{s}'}=\pi_{\hat{s}'}(H_1)$. Since $\pi_{\hat{s}'}(H_1)=\pi_{\hat{s}'}(H)$, we have $J_{\hat{s}'}=\pi_{\hat{s}'}(H)$. Hence $J/H\cong J_{\hat{s}}/H_{\hat{s}}$ by (3).

Thus the theorem is completely proved.

The following theorem is an immediate consequence of the class field theory and theorem 1.

THEOREM 2. *Let K be a normal extension of k of finite degree and put $G=G(K/k)$, $\hat{s}=\hat{S}(K/k)$. Then there is a one to one correspondence between the set of all G-invariant $\hat{s}$-admissible subgroups $H_{\hat{s}}$ of $J_{\hat{s}}$ of finite index and the set of all abelian extensions M of K of finite degree which is normal over k. If M corresponds to $H_{\hat{s}}$, then $G(M/K)\cong J_{\hat{s}}/H_{\hat{s}}$. Furthermore a prime $\mathfrak{P}$ of $\hat{s}$ decomposes completely in M if and only if $\iota_{\mathfrak{P}}K_{\mathfrak{P}}\subset H_{\hat{s}}$.*

§ 3. Conditions of the s-admissibility

Let F be any algebraic number field, J be the idèle group of F, $\mathfrak{C}=J/F^{\times}$ be its idèle class group, and $\mathfrak{D}$ be the connected component of the unity in $\mathfrak{C}$. Let further s be a set of infinitely many primes of F and J_s be the restricted idèle group of F. We have called, in the previous section, a subgroup of H_s of J_s s-admissible if $H_s=\rho_s(\overline{H_sD})$ where D is the complete inverse image of $\mathfrak{D}$ by the natural homomorphism of J to $\mathfrak{C}$. Now we prove

THEOREM 3. *Let H_s be a closed subgroup of J_s of finite index. Then H_s is s-admissible if and only if there exists a neighborhood U' of the unity in $J_{s'}$ such that H_s contains $\pi_s(D\cap(J_s\times U'))$.*

Proof. Suppose that H_s is s-admissible, and put $H=\overline{H_sD}$. Then H is admissible in J and $\rho_S(H) = H_S$. Let U be a neighborhood of the unity in J whith $H \supset U$. Such a neighborhood always exists because of the finiteness of $(J; H)$. Set $U = U_S \times U_{S'}$ where $U_S \subset J_S$ and $U_{S'} \subset J_{S'}$. Then $U_{S'}$ is a neighborhood of the unity in $J_{S'}$ and we see $H_S = \rho_S(H) \supset \pi_S(H \cap (J_S \times U_{S'}) \supset \pi_S(D \cap (J_S \times U_{S'}))$ by lemma 4.

Suppose conversely that there exists a neighborhood U' of the unity in $J_{S'}$ such that $H_S \supset \pi_S(D \cap (J_S \times U'))$. Let $\mathfrak{x}$ be any element of $\rho_S(\overline{H_S D})$, then $\mathfrak{x} \times 1 \in \overline{H_S D}$. Now let $U(\mathfrak{x})$ be any neighborhood of $\mathfrak{x}$ in J_S and put $U = U(\mathfrak{x}) \times U'$. Then since U is a neighborhood of $\mathfrak{x} \times 1$ in J, $U \cap (H_S D)$ is not empty. Let $(\mathfrak{h} \times 1)(\mathfrak{a} \times \mathfrak{b})$ be in this intersection, where $\mathfrak{h} \in H_S$, $\mathfrak{a} \times \mathfrak{b} \in D$, $\mathfrak{a} \in J_S$, $\mathfrak{b} \in J_{S'}$. Then $(\mathfrak{h} \times 1)(\mathfrak{a} \times \mathfrak{b}) = \mathfrak{h}\mathfrak{a} \times \mathfrak{b} \in U$, $\mathfrak{b} \in U'$ and $\mathfrak{h}\mathfrak{a} \in U(\mathfrak{x})$. Hence $\mathfrak{a} \in \pi_S(D \cap (J_S \times U')) \subset H_S$ and $\mathfrak{h}\mathfrak{a} \in H_S$. Therefore $U(\mathfrak{x}) \cap H_S$ is not empty, which implies $\mathfrak{x} \in \overline{H_S} = H_S$, since H_S is closed in J_S. Thus we have $\rho_S(\overline{H_S D}) \subset H_S$, hence $\rho_S(\overline{H_S D}) = H_S$. This proves that H_S is S-admissible, and the theorem is proved.

Now the structure of D is known by Artin [1][3]. Let $\mathbf{Z}$ be the additive group of rational integers and $\overline{\mathbf{Z}}$ be the completion of $\mathbf{Z}$ under the topology where the ideals of $\mathbf{Z}$ form a fundamental system of neighborhoods of 0. For each unit ε of F, we denote by $\bar{\varepsilon}$ the idèle which has components ε at all finite primes and components 1 at all infinite primes. Then for $x \in \overline{\mathbf{Z}}$, $\bar{\varepsilon}^x$ can be defined as the extension of the exponentiation ε^m by an ordinary integer m. Let r_1, r_2 be as usual the numbers of real primes and complex primes of F respectively and put $r = r_1 + r_2 - 1$. Then we have by Artin [1] immediately the following

Proposition 4. *Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ be any system of independent totally positive units of F. Let further E be the group of all idèles $\bar{\varepsilon}^{x_1} \cdots \bar{\varepsilon}_r^{x_r}$ where $x_1 \ldots, x_r \in \overline{\mathbf{Z}}$, and J_∞ be the group of those idèles which have components* 1 *at all finite primes. Then we can choose a system of representatives of* D mod. $F^\times$ in EJ_∞.

Notations being as above, let $x \in \overline{\mathbf{Z}}$ and ε be one of the ε_i. Let further U' be a neighborhood of the unity in $J_{S'}$ containing $\pi_{S'}(J_\infty)$. Then there exists an integer $m \in \mathbf{Z}$ such that $\bar{\varepsilon}^x/\bar{\varepsilon}^m \in J_S \times U'$. Indeed let $\mathfrak{p}$ be a finite prime. If $x' \equiv 0$ mod $\mathfrak{p}^\kappa$ for a sufficiently large κ, then the $\mathfrak{p}$-component of the idèle $\bar{\varepsilon}^{x'}$ is in the $\mathfrak{p}$-component of $J_S \times U'$. Since the $\mathfrak{p}$-component of $J_S \times U'$ containes $U_\mathfrak{p}$ for almost

[3] See also E. Artin and J. Tate [2], Ch. 9.

all $\mathfrak{p}$, we can set $\kappa = 0$ for such almost all $\mathfrak{p}$. This implies immediately the existence of the above integer m, being $x \equiv m$ mod. $\mathfrak{p}^{\kappa}$ for all finite $\mathfrak{p}$. Hence by prop. 4 we have

$$(D \cap (J_S \times U')) \cdot F = D. \tag{4}$$

LEMMA 5. *Let G be a topological group which is locally compact and satisfies the second axiom of countability. Let N be a discrete normal subgroup and A be a closed subgroup of G. Suppose further AN is closed in G. Then we have* $AN/N \cong A/AN$.

Proof. Let f be the canonical map of G to G/N. Then f is an open map, and we have $f(A) = f(AN) = AN/N$. Since N is discrete, G/N is locally isomorphic to G. Hence G/N is also locally compact and satisfies the second axiom of countability. Both A and AN are locally compact, because they are closed in G. Furthermore AN/N is also locally compact owing to the continuity of f. It is known[4] that if g is an onto-homomorphic mapping between two locally compact groups satisfying the second axiom of countability, then f is open. Therefore the restriction of f to A, whose image is AN/N, is also open. Then since the kernel of f is AN, we have $f(A) = A/AN$. Thus the lemma is proved.

PROPOSITION 5. *Let U' be a neighborhood of the unity in $J_{S'}$ containing $\pi_{S'}(J_\infty)$, and H_S be a closed subgroup of J_S of finite index. If $H_S \supset \pi_S(F^\times \cap (J_S \times U'))$, then $H_S \supset \pi_S(D \cap \times U'))$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} J & \xrightarrow{\quad j' \quad} & \mathfrak{C}' = J/(F^\times \cap (J_S \times U')) \\ \Big\downarrow \pi_S & & \Big\downarrow \pi'_S \\ J_S & \xrightarrow{\quad j'_S \quad} & \mathfrak{C}'_S = J_S/\pi_S(F^\times \cap (J_S \times U')). \end{array}$$

where j' and j'_S are the canonical homomorphisms respectively. First we shall prove that $j'_S\pi_S(D \cap (J_S \times U'))$ is connected in $\mathfrak{C}'_S$. Since J, $F^\times$ and $(D \cap (J_S \times U'))$ satisfy the conditions of G, N and A in lemma 5 respectively, we have alge-

[4] See L. S. Pontrjagin [12], Ch. 3, § 19, theorem 13. In this book, the topology is T_1 and every subgroup is closed. But the proof of the theorem 13 in this book hold in general case.

braically and topologioally $D/F^{\times}(D \cap (J_s \times U'))F^{\times}/F^{\times} \cong D \cap (J_s \times U') = j'(D \cap (J_s \times U'))$. Hence $j'(D \cap (J_s \times U'))$ is connected in $\mathfrak{C}'_s$, because $D/F^{\times} = \mathfrak{D}$ is the connected component of the unity in $\mathfrak{C}$. Since j', j'_s, π_s are all continuous maps, π'_s is also continuous and we have $\pi'_s j'(D \cap (J_s \times U')) = j'_s \pi_s(D \cap (J_s \times U'))$. Hence $j'_s \pi_s(D \cap (J_s \times U'))$ is connected in $\mathfrak{C}'_s$. Now since H_s is closed in J_s and contains $\pi_s(F^{\times} \cap (J_s \times U'))$, $j'_s(H_s)$ is also closed in $\mathfrak{C}'_s$. On the other hand, $j'_s(H_s)$ is of finite index in $\mathfrak{C}'_s$, since H_s is so in J_s. Hence $j'_s(H_s)$ is open in $\mathfrak{C}'_s$. Therefore $j'_s(H_s)$ must contain $j'_s \pi_s(D \cap (J_s \times U'))$ which is connected in $\mathfrak{C}'_s$ as proved above. This implies $H_s \supset \pi_s(D \cap (J_s \times U'))$, since $H_s \supset \pi_s(F^{\times} \cap (J_s \times U'))$ by the assumption. Thus the proposition is proved.

Let again K be a normal extension of k of finite degree, and let $J = J_K$. Put $s = \hat{S}(K/k)$, $G = G(K/k)$. Then we have

THEOREM 4. *Let $H_{\hat{s}}$ be a G-invariant closed subgroup of $J_{\hat{s}}$ of finite index. Then, in order that $H_{\hat{s}}$ is $\hat{s}$-admissible, it is necessary and sufficient that there exists a noighborhood U' of the unity in $J_{\hat{s}}$ such that $U' \supset \pi_{\hat{s}'}(J_\infty)$ and $H_{\hat{s}} \supset \pi_{\hat{s}}(K^{\times} \cap (J_{\hat{s}} \times U'))$.*

Proof. Suppose that $H_{\hat{s}}$ be s-admissible and put $H = \overline{H_s D}$. Then $\rho_{\hat{s}}(H) = H_{\hat{s}}$. If $K = k$, then $\hat{s}'$ is empty and the theorem is true by the class field theory. Thus we assume $K \neq k$. As the proof of theorem 3, let U be a neighborhood of the unity in J such that $H \supset U$ and set $U = U_{\hat{s}} \times U_{\hat{s}'}$ where $U_{\hat{s}} \subset J_{\hat{s}}$, $U_{\hat{s}'} \subset J_{\hat{s}'}$. Then we can assume that $U_{\hat{s}'}$ contains $\pi_{\hat{s}'}(J_\infty)$, because every infinite prime of $\hat{s}'$ is ramified in K/k, and consequently complex. Hence it is not ramified in any extension over K. Therefore, if $\mathfrak{p}$ is an infinite prime of $\hat{s}'$, then $U_{\mathfrak{p}}$ is contained in H. This implies $U'_\infty \subset H$. Now we see, as the proof of the theorem 3, that $H_{\hat{s}} = \rho_{\hat{s}}(H) \supset \pi_{\hat{s}}(H \cap (J_s \times U_{\hat{s}'})) \supset \pi_{\hat{s}}(K \cap (J_{\hat{s}} \times U_{\hat{s}'})$, which proves the necessity of the theorem. The sufficiency follows immediately from theorem 3 and prop. 5.

§ 4. k-meta-abelian extensions

Let K/k be, as in the previous sections, an extension of degree n of an algebraic number field k, and put $s = S(K/k)$, $\hat{s} = \hat{S}(K/k)$, $G = G(K/k)$. For every prime $\mathfrak{p}$ of k we fix a prime $\mathfrak{P}$ of K which divides $\mathfrak{p}$, and set $(\mathfrak{p}, \sigma) = \mathfrak{P}^{\sigma}$ for $\sigma \in G$. Then $\hat{s}$ consists of all $(\mathfrak{p}, \sigma)$ such that $\mathfrak{p} \in s$ and $\sigma \in G$. Denote by $\hat{s}(\sigma)$ the set of all $(\mathfrak{p}, \sigma)$ where $\mathfrak{p} \in s$. Then $\hat{s}$ is the union of those $\hat{s}(\sigma)$ where $\sigma \in G$, and we have naturally

$$J_{\hat{s}} \cong \prod_{\sigma \in G} J_{\hat{s}}(\sigma) \quad \text{(direct)}.$$

Hereafter we shall fix this isomorphism.

If $\mathfrak{p} \in \hat{s}$, any element α of $K_{\mathfrak{P}\sigma}$ is expressed by $\alpha = \sum a_i \pi^i$ where $a_i \in k$ and π is any fixed prime element of $k_{\mathfrak{p}}$. Thus the element of $K_{\mathfrak{P}\sigma}$ is naturally embedded in $k_{\mathfrak{p}}$. We denote this embedding by $g_{\mathfrak{p}\sigma}$ or $g_{(\mathfrak{p},\sigma)}$, and we have $g_{\mathfrak{P}\sigma}(\alpha^{\sigma}) = g_{\mathfrak{P}}(\alpha)$ for $\alpha \in K_{\mathfrak{P}}$, $\sigma \in G$. Denote by g_{σ} the mapping of $J_{\hat{s}(\sigma)}$ to J_s, which maps every $(\mathfrak{p},\sigma)$-component by $g_{\mathfrak{P}\sigma} = g_{(\mathfrak{p},\sigma)}$. Set $g = \prod_{\sigma \in G} g_{\sigma}$. Then g_{σ} and g imply algebraically and topologically

$$J_{\hat{s}(\sigma)} \cong J_s \tag{5}$$

and

$$J_{\hat{s}} \cong J_s^n, \tag{6}$$

where J_s^n is the direct product of n-fold of J_s.

Let $\sigma_1, \ldots, \sigma_n$ be all elements of G and P_{σ} be the permutation $(\sigma_1, \ldots, \sigma_n) \rightarrow (\sigma_1\sigma, \ldots, \sigma_n\sigma)$ for every element $\sigma \in G$. For $(a_1, \ldots, a_n) \in J_s^n$ set

$$(a_1, \ldots, a_n)^{\sigma} = P_{\sigma}(a_1, \ldots, a_n).$$

Then J_s^n has G as the operator domain, and the isomorphism (σ) becomes a G-isomorphism.

Now denote by A the maximal abelian extension of k. Let M be a meta-abelian extension of k of finite degree, that is, a normal extension of k of finite degree with an abelian group as commutator group of the Galois group. Then $\mathrm{A}M/\mathrm{A}$ is a Kummer extension. Let $\mathrm{A}M = \mathrm{A}(\sqrt[m_1]{a_1}, \ldots, \sqrt[m_t]{a_t})$[5]. Then if every a_i is an element of k, we called in [13] M a k-meta-abelian field over k, and proved that M contains necessarily the all m_i-th roots of unity $(i = 1, \ldots, t)$. Therefore if M is a k-meta-abelian field over k, there exists an abelian extension K of k which containes the m_i-th roots of unity and $K(\sqrt[m_1]{a_1}, \ldots, \sqrt[m_t]{a_t}) \supset M \supset K$ where $a_1, \ldots, a_t \in k$.

Put, as before, $s = S(K/k)$ and $\hat{s} = \hat{S}(K/k)$. For $\mathfrak{a} \in J_K$ and $a \in k$ let $(\mathfrak{a}, a)_m$ be the norm residue symbol for Kummer extension $K(\sqrt[m]{a})/K$[6]. Denote by $(\mathfrak{a}_{\mathfrak{p}}, a)_{\mathfrak{p},m}$ the norm residue symbol for the local Kummer extension $K_{\mathfrak{P}}(\sqrt[m]{a})/K_{\mathfrak{P}}$.

5) We assume $\sqrt[n_i]{a_i} \notin \underline{A}M$ for any integer $n_i > m_i$.

6) c. f. E. Artin and J. Tate [2], Ch. 12.

Then we have

$$(\mathfrak{a}, a)_m = \prod_{\mathfrak{P} \in S(K)} (\mathfrak{a}_{\mathfrak{P}}, a)_{\mathfrak{P}, m}.$$

Let $H_m(a)$ be the idele group of $\mathfrak{a} \in J_K$ such that $(\mathfrak{a}, a)_m = 1$. Then $H_m(a)$ is the admissible subgroup of J_K corresponding to $K(\sqrt[m]{a})$ by the classfield theory. We see immediately that $\rho_{\hat{S}}(H_m(a))$ consists of all $\mathfrak{a} \in J_{\hat{S}}$ such that $(\mathfrak{a}, a)_m = 1$, where $(\mathfrak{a}, a)_m = \prod_{\mathfrak{P} \in s} (\mathfrak{a}_{\mathfrak{p}}, a)_{\mathfrak{P}, m}$. The isomorphism g implies

$$(\mathfrak{a}, a)_m = \prod_{\substack{\mathfrak{P} \in s \\ \sigma \in G}} (\mathfrak{a}_{(\mathfrak{p}, \sigma)}, a)_m \quad \text{for} \quad \mathfrak{a} \in J_s^n.$$

Now we shall call a subgroup H_1 of J_s^n *k-admissible* if there exist elements $a_1, \ldots, a_t$ of k and natural numbers $m_1, \ldots, m_t$ such that H_1 contains the intersection of all $H_{m_1}(a_1), \ldots, H_{m_t}(a_t)$, provided that s contains all primes of k such that $N\mathfrak{p} - 1$ is divisible by all $m_1, \ldots, m_t$. Then we have by theorem 1 the following theorem, which essentially includes the main part of Kuroda [11], Fröhlich [5] and Furuta [7], [8].

THEOREM 5. *Let M be a k-meta-abelian extension over k of finite degree. Then there exist an admissible closed subgroup H of the (ordinary) idèle group J of k of finite index, say n, and a k-admissible closed subgroup H_1 of the restricted idèle group J_s^n of k of finite index which satisfies the following condition (♯), where s is the set of all primes $\mathfrak{p}$ of k such that $\iota_{\mathfrak{p}} k_{\mathfrak{p}} \subset H$.*

(♯) *If K is the class field over k corresponding to H, we have $G(M/K) = J_s^n/H_1$ and a prime $\mathfrak{p}$ of k decomposes completely in M/k if and only if $\mathfrak{p} \in s$ and $\iota_{\mathfrak{p}} k_{\mathfrak{p}} \subset H_1$.*

Conversely let H be an admissible closed subgroup of the (ordinary) idèle group J of k of finite index n and H_1 be a k-admissible closed subgroup of the restricted idèle group J_s^n of k of finite index, where s is the set of all primes $\mathfrak{p}$ of k such that $\iota_{\mathfrak{p}} k_{\mathfrak{p}} \subset H$. Then there exists a k-meta-abelian field M over k which satisfies the condition (♯).

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