NIL SEMI-GROUPS OF RINGS WITH A POLYNOMIAL IDENTITY

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In the memory of Professor TADASI NAKAYAMA

The basic properties of associative rings R satisfying a polynomial identity $p[x_1, \ldots, x_n] = 0$ were obtained under the assumptions that the ring was an algebra [e.g., [4] Ch. X], or with rather strong restrictions on the ring of operators ([1]). But it is desirable to have these properties for arbitrary rings, and the present paper is the first of an attempt in this direction. The problem is almost trivial for prime or semi-prime rings but quite difficult in arbitrary rings. The known proofs for algebras have to be modified and in some cases new proofs have to be obtained as the existing proofs fail to exploit the known structure. In the present paper we extend the results of [1] on the nil subalgebras of a ring with an identity for arbitrary multiplicative nil semi-groups of the ring and for arbitrary rings.

Finally, we extend our results to rings with a pivotal monomial and as a consequence we show that the nil multiplicative semigroups of a simple ring of bounded index are nilpotent.

1. Notations. Let Ω be a set of linear mappings of a ring R into a ring T, i.e., given a mapping $\Omega \times R \to T$, denoted by $w.r.$ and satisfying

$$(1.1) \qquad \begin{matrix} w(rs) = (wr)s = r(ws) \\ w(r+s) = wr + ws \end{matrix}, \quad w \in \Omega; \; r, s \in R.$$

Let $x_1, x_2, \ldots,$ be an infinite set of indeterminates. Let $\tilde{\Omega}[x]$ be the free ring generated by the $\{x_i\}$ and the symbols of Ω, and among the elements of $\tilde{\Omega}[x]$, we restrict ourselves to the set $\Omega[x]$ of all polynomials $p[x] = \sum w_{(i)} x_{i_1} \cdots x_{i_n}$ which are finite sums of different monomials $x_{i_1} \cdots x_{i_n}$ preceded by an element $w_{(i)}$ of the set Ω.

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In most applications (e.g., algebras over a field $F=\Omega$) Ω is a commutative ring, then $\Omega[x]$ is the free ring generated by Ω and $\{x_i\}$ as long as only additive structure of $\Omega[x]$ is considered.

For $p[x]=\sum w_{(i)}x_{i_1}\cdots x_{i_n}$ we define degree, linearity, multilinearity in the usual way, and we set $\Omega(p)=\{w_{(i)}\}$ the set of all coefficients of $p[x]$. Thus Ker $\Omega(p)=\{r \mid r\in R,\ wr=0$ for all $w\in\Omega(p)\}$. We shall also decompose $p[x]=p_0[x]+p_1[x]+\cdots+p_l[x]$ in homogeneous components $p_j[x]$ of degree j and note that also $p_j[x]\in\Omega[x]$.

If $p[x]\in\Omega[x]$ then for every substitution $x_i=r_i\in R$, the element $p[r_1, r_2,\ldots]$ is a well defined element in K, and if $p[r]=0$ for all substitutions we say that $p=0$ is a polynomial identity of R.

In the linearization process of a polynomial identity one starts with a monomial $\pi(x)=\pi(x_1,\ldots,x_r)=m_1x_1m_2x_1\cdots m_kx_1m_{k+1}$, where m_i is either 1 or a monomial not containing x_1; then one replaces x_1 by x_1+x_{s+1} and write

$$\pi(x_1+x_{r+1})=\pi_1+\pi_2+\cdots+\pi_{2^k} \tag{1.2}$$

where the sum ranges over all 2^k monomials obtained from π by the distribntive law. $\pi_1=\pi(x_1,x_2,\ldots,x_r)$ and $\pi_{2^k}=\pi(x_{r+1},x_2,\ldots,x_r)$ and all other π_j are different monomials of degree $<k$ in x_1 and x_{r+1}.

This simple observation is applied to the following extension of the linearization process.

LEMMA 1. *Let R satisfy a polynomial identity $p[x_1,x_2,\ldots,x_d]=0$ of degree d, and let $p=p_0+p_1+\cdots+p_d$ be the decomposition of p in homogeneous component p_j of degree j. Let $\pi(x)=wx_{i_1}x_{i_2}\cdots x_{i_r}$ be a monomial of degree r appearing in $p[x]$, then R satisfies a polynomial identity:*

$$\bar{p}[x_1,\ldots,x_r]=wx_1\cdots x_r+\bar{p}_r[x_1,\ldots,x_r]+\bar{p}_{r+1}[x_1,\ldots,x_r]+\cdots+\bar{p}_d[x_1,\ldots,x_r] \tag{1.3}$$

where $\bar{p}_j\in\Omega[x]$ is homogeneous of degree j and the coefficients $\Omega(\bar{p}_j)\subseteq\Omega(p_j)$ for $r\le j\le d$, and $\bar{p}_r[x]=\sum w_{(i)}x_{i_1}\cdots x_{i_r}$ is homogeneous multilinear with monomials $\neq x_1x_2\cdots x_r$.

In particular for $r=d$, $\bar{p}[x]$ is multilinear and homogeneous.

Proof. Let ν_i be the degree of x_i in the monomial $\pi(x)$ and $\nu=\text{Max}\ \nu_i$, and let τ be the number of x_i of degree ν in $\pi(x)$. Consider the pairs (ν,τ)

ordered lexicographically and our proof will be by induction on these pairs (ν, τ):

Let $\pi(x)$ contain k different x_i then by setting $x_j = 0$ for all x_j not appearing in $\pi(x)$, and choosing $x_1, \ldots, x_r$ for the x_j appearing in $\pi(x)$, we clearly get from $p[x]$ a polynomial $\tilde{p}[x_1, \ldots, x_r] \in \Omega[x]$, which holds in R, containing the monomial $\pi(x)$ and for which $\Omega(\tilde{p}_i) \subseteq \Omega(p_j)$.

If $\nu = 1$ (then $\tau = r$), π is multilinear and we can assume that $\pi(x) = wx_1x_2 \cdots x_r$. Next we obtain a polynomial $\bar{p}[x] \in \Omega[x]$ satisfied in R of the same type as $\tilde{p}$ but whose monomial contain all the x_i, $i = 1, 2, \ldots, r$, in the following way: $\tilde{p}_1[x_1, \ldots, x_r] - \tilde{p}[0, x_2, \ldots, x_r] \in \Omega[x]$, and it is satisfied in R and with the same properties i.e., $\pi(x)$ is a monomial in it and the set of coefficients of the monomials of degree $j \subseteq \Omega(\tilde{p}_j) \subseteq \Omega(p_j)$, and all monomials of $\tilde{p}_1$ contain x_1. Repeat this process with $\tilde{p}_1$ to obtain a polynomial identity $\tilde{p}_2$ whose monomials will contain both x_1 and x_2; and so on$\cdots$. Finally the polynomial $\bar{p}[x_1, \ldots, x_r]$ is necessarily of the form (1.3) as all its monomials are of degree $\geq r$, and those of degree r must contain all $x_1, x_2, \ldots, x_r$; furthermore none of the monomials is repeated.

So let $\nu > 1$, and $x_1, \ldots, x_k$ be x's appearing in $\pi(x)$. Consider the polynomial identity of R, $q[x_1, \ldots, x_{k+1}] = 0$ given by:

$$q[x_1, \ldots, x_{k+1}] = \tilde{q}[x_1 + x_{k+1}, x_2, \ldots, x_k] - \tilde{q}[x_1, x_2, \ldots, x_k] - \tilde{q}[x_{k+1}, x_2, \ldots, x_k]$$

It follows readily from the remarks preceding (1.2) that $q[x] \in \Omega[x]$ and $\Omega(q_j) \subseteq \Omega(\tilde{p}_j) \subseteq \Omega(p_j)$. Furthermore, $q = 0$ in R is a consequence of the distributive law of (1.1); and finally it follows by (1.2) that $q[x]$ contains a monomial with the coefficient w as that of $\pi(x)$, but for which we have the pair $(\nu, \tau - 1)$ if $\tau > 1$ or $(\nu - 1, \lambda)$ for some λ if $\tau = 1$. In both cases we can apply our induction to obtain the required polynomial identity $\bar{p} = 0$ of (1.3).

2. Multiplicative nil semigroups of rings.

The theory of the Lower Radical (e.g., [4]) is well known to hold also for semi-groups with a zero. For further references we recall some of the definitions and results required later:

Let S denote a semi-group with a zero, which in our applications will always be a multiplicative subset of a ring R. A subset $M \subseteq S$ is an ideal if SM and

$MS \subseteq M$. As usual we construct for each ideal the quotient semi-group S/M, which is the set S with all elements of M are identified with the zero.

An ideal $I \subseteq S$ is said to be nilpotent modulo M of index k if $I^k = \{a_1 a_2 \cdots a_k \mid a_i \in I\} \subseteq M$. Then the union of a finite number of ideals nilpotent mod M is also an ideal and it is nilpotent mod M. Denote by $N_1(S/M)$ the union of all nilpotent ideals mod M, which is an ideal in S containing M, but need not be nilpotent.

We define for every ordinal λ:

$N_0(S/M) = M$

$N_\lambda(S/M) = N_1(S/N_{\lambda-1})$ if λ is not a limit ordinal

$N_\lambda(S/M) = \bigcup_{\rho<\lambda} N_\rho(S/M)$ for limit ordinals λ.

The basic properties of the Lower Radical is the following:

LEMMA 2. *i) There exists an ordinal σ such that $N_\sigma = N_\tau$ for all $\tau \geq \sigma$. The ideal N_σ is the minimal ideal Q in S containing M such that $N_1(S/Q) = Q$ (i.e., S does not contain ideals $\neq Q$ which are nilpotent mod Q)*

ii) Each ideal N_λ is locally nilpotent (mod M); that is, every finite set in N_λ generates a nilpotent (mod M) semi-group.

Proof. The proofs are well known for the case of rings (e.g. [4]) and it is even simpler for semigroups. As we shall need here only the fact that $N_1(S/N_\rho) = N_\rho$ and (ii) we reproduce their proofs. The first is evident by chosing σ to be the first ordinal for which $N_\rho = N_{\rho+1}$, and to prove (ii) let $s_1, \ldots, s_t \in N_\lambda$, and the proof is carried by induction on λ. If λ is a limit ordinal, then the finiteness of t puts our set in an N_ρ with $\sigma < \lambda$ where induction can be used.

If $\lambda = \rho + 1$, then $s_1, \ldots, s_t$ belongs to a union of a finite set of nilpotent ideals mod N_ρ which is nilpotent—hence, $s_{(i)} = s_{i_1} s_{i_2} \cdots s_{i_k} \in N_\rho$ for some fixed k and all products of k elements of the s_j's. This set $\{s_{(i)}\}$ is finite and therefore it follows by induction that $\{s_{(i)}\}$ is nilpotent mod M which clearly implies the nilpotency of $\{s_1, \ldots, s_t\}$.

We need also the following property of nil semi-groups:

LEMMA 3. *Let M be an ideal in S, and let S be nil mod M. If S does not contain nilpotent ideals mod M, then there is an infinite set $a_1, a_2, \ldots, a_n, \ldots$ of elements in S with the property that $a_1 a_2 a_3 \cdots a_n \notin M$ for every n, but $a_i a_j \in$*

M if $i \geq j$.

Proof. As S is nil mod M, choose in S an element $a_1 \notin M$ but such that $a_1^2 \in M$. Suppose $a_1, \ldots, a_{n-1}$ have been chosen such that $a_1 a_2 \cdots a_{n-1} \notin M$ but $a_i a_j \in M$ when $j \leq i$, then choose a_n as follows:

Let $b = a_1 a_2 \ldots a_{n-1} \in M$, and S does not have nilpotent ideal—the ideal $\{b\}$ generated by b is not nilpotent. This implies that $bSb \nsubseteq M$, as otherwise the ideal $\{b\}^3 \subseteq M$. So let $bxb \notin M$ for some $x \in S$ as S is nil we can choose x such that $b(xb)^2 \in M$. Repeating our process with bxb replacing b, we choose $y \in S$ with $bxbybxb \notin M$, and finally we set $a_n = xbybxb \notin M$. Now $a_1 a_2 \cdots a_{n-1} a_n = ba_n = bxbybxb \notin M$, and $a_i a_j \in M$ for $j \leq i \leq n-1$. Finally $a_n a_j = xbybx\,(a_1 \cdots a_{n-1})\,a_j \in M$ since $a_{n-1} a_j \in M$, and $a_n^2 = xbyb(xb)^2 ybxb \in M$ as $b(xb)^2 \in M$. This completes the proof of the lemma.

The following lemma takes into account the addition of the ring R which contains the semi-group S:

LEMMA 4. *Let* $\Omega_0 = (w_1, \ldots, w_n)$ *be a finite set of operators of* Ω, *and* $M = S \cap Ker\, \Omega_0 = \{s \mid s \in S,\ \ w_i s = 0 \ \ for\ \ every\ \ w_i \in \Omega_0\}$. *For* $s \in S$, *if* $\Omega_0 s \subseteq \sum \Omega_0 N_\lambda(R/M)$ *then* $s \in N_\lambda(R/M)$.

Proof. We use induction on λ. For $\lambda = 0$, our condition requires that $\Omega_0 s \subseteq \Omega_0 N_0 = \Omega_0 M = 0$, hence $s \in MN_0(S/M)$.

Let $\lambda > 0$, we get $w_i s = \sum w_j a_{ij}$, with $a_{ij} \in N_\lambda(R/M)$ and the number of the a_{ij} is finite, hence our proof is immediate if λ is a limit ordinal. If $\lambda = \rho + 1$, then the finite set of elements a_{ij} generates a nilpotent ideal modulo $N_\rho(S/M)$ and say it is of index n, then
$x_0 a^{(1)} x_1 a^{(2)} x_2 \cdots a^{(n)} x_n \in N_\rho(S/M)$ for all $x_i \in S$ and all $a^{(i)}$ of the set $\{a_{ij}\}$. Hence, by (1.1)

$$w_i x_0 s x_1 s x_2 \cdots x_{n-1} s x_n = (x_0 s \cdots s x_{n-1})(w_i s) x_n = (x_0 s \cdots x_{n-1} s x_{n-1}) \cdot \sum_{j_n} w_{j_n} a_{ij_n} x_n$$
$$= (x_0 s \cdots s x_{n-2}) \sum_{j_n} (w_{j_n} s) x_{n-1} a_{ij_n} x_n = (x_0 s \cdots s x_{n-2}) \sum_{j_{n-1},\, j_n} w_{j_{n-1}} a_{j_n j_{n-1}} x_{n-1} a_{i_n} x_n$$
$$= \cdots = \sum w_{j_1} x_0 a_{j_2 j_1} x_1 a_{j_3 j_2} \cdots x_{n-1} a_{ij_n} x_n \subseteq \sum \Omega_0 N_\rho(S/M).$$

Thus, $\Omega_0(x_0 s x_1 s \cdots s x_n) \subseteq \sum \Omega_0 N_\rho(S/M)$ for arbitrary $x_i \in S$. It follows, therefore, by induction that $x_0 s x_1 s \cdots s x_n \in N_\rho(S/M)$ and consequently the ideal $\{s\}$ generated by s in S is nilpotent modulo N_ρ, and hence $s \in N_{\rho+1}(S/M)$. q.e.d.

3. Our main result is now:

THEOREM 5. *Let R be a ring with a polynomial identity $p[x]=0$ of degree d and coefficients $\Omega(p)\subseteq\Omega$, and let $M=Ker\ \Omega(p)$, and S be a multiplicative subset of R. Then:*

i) If S is nilpotent mod $M\cap S$ of index n then $S^{[d/2]}$ generates a nilpotent ideal modulo M in the ring R, of index $\leq(d+1)^{2n}$.

ii) If S is nil mod $M\cap S$ then $S^{[d/2]}\subseteq N_1(R/M)$.

Remark. Clearly in our case M is a two sided ideal in the ring R and hence $N_1(R/M)$ is the ***sum*** of all nilpotent ideals in the ring.

Proof. Let R^* denote the ring obtained by adjoining a unit to R. Let R^*TR^* denote the two sided ideal in R generated by T, where T is any subset of R.

For every integer m set $\mu(m)$ the index of nilpotency of the ideal $R^*S^mR^*$ modulo M. Thus $\mu(n)=1$ in case (i) of our theorem since $S^n\subseteq M$.

Let m be any integer $>\left[\frac{d}{2}\right]$, where [ ] denotes the largest integer $\leq\frac{d}{2}$. The proof begins similarly to the proof of this theorem for algebras given in [1]:

Consider the sets $T_{2j-1}=S^{m-j}R^*S^{j-1}$, $T_{2j}=S^{m-j}R^*S^j$ for $j=1,2,\dots,m$. Note that if $a_i\in T_i$ then $a_ia_k\in R^*S^mR^*$ if $k<i$, and $a_i^2\in R^*S^{m-1}R^*$. By choosing a_i arbitrary in T_i, the products $a_1a_2\cdots a_r$ (for any r) will range on a set of generators of the additive sets $T_1T_2\cdots T_r=(S^{m-1}R^*)^rS^j$ where $2j=r$ or $2j+1=r$.

For any $w\in\Omega[p]$, a coefficient of the polynomial $p[x]$, we apply Lemma 1 and obtain the polynomial $\bar{p}[x_1,\dots,x_r]$ which we write in the form:

$$wx_1x_2\cdots x_r=-\bar{p}_r[x_1,\dots,x_r]-\bar{p}_{r+1}[x_1,\dots,x_r]-\cdots-\bar{p}_d[x_1,\dots,x_r]. \tag{3.1}$$

Letting $x_i=a_i\in T$, the last relation shows in view of the preceding remarks that

$$w(S^{m-1}R^*)^rS^j\subseteq\Omega(p_r)(R^*S^mR^*)+\sum_{i>r}\Omega(p_i)(R^*S^{m-1}R^*) \tag{3.2}$$

Indeed, $\Omega(\bar{p}_r)\subseteq\Omega(p_r)$ and the monomials $x_{i_1}\cdots x_{i_r}$ in $\bar{p}_r$ differ from $x_1x_2\cdots x_r$, hence it must contain a product x_ix_j with $j\leq i$ so that the substitution $x_i=a_i$ yields an element in $\Omega(p_r)(R^*S^mR^*)$; and as for other monomials of $\bar{p}_k$, being of degree $k>r$, they necessarily contain a product x_ix_j with $j\leq i$ so they yield elements in $\Omega(\bar{p}_k)(R^*S^{m-1}R^*)\subseteq\Omega(p_k)(R^*S^{m-1}R^*)$, which completes the proof of (3.2).

The validity of (3.2), for any $w \in \Omega(p_r)$ yields by multiplying on the left by R^* and on the right by $S^{r-j}R^*$:

(3.3) $$\Omega(p_r)(R^*S^{m-1}R^*)^{r+1} \subseteq \Omega(p_r)(R^*S^mR^*) + \sum_{i>r} \Omega(p_i)(R^*S^{m-1}R^*).$$

Note that for $r=d$, the second summand does not appear. Now multiply both sides of (3.3) by $(R^*S^{m-1}R^*)^{r+1}$ and apply (3.3) for $i=r+1$ in the terms under sum, then noting that $(R^*S^mR^*)(R^*S^{m-1}R^*)^r \subseteq R^*S^mR$ we get:

$$\Omega(p_r)(R^*S^{m-1}R^*)^{2(r+1)} \subseteq \Omega(p_r)(R^*S^mR^*) + \Omega(p_{r+1})(R^*S^mR) + \sum_{i>r+1} \Omega(p_i)(R^*S^{m-1}R^*).$$

Repeating this process we finally get:

$$\Omega(p_r)(R^*S^{m-1}R^*)^{t(r)} \subseteq \sum_{i \geq r} \Omega(p_i)(R^*S^mR^*)$$

where $t(r) = (r+1) + [(r+1)+(r+2)+ \cdots + d] \leq 1 + \dfrac{d(d+1)}{2} = \delta < (d+1)^2$. This being true for $r=0, 1, \ldots, d$ yields

(3.4) $$\Omega(p)(R^*S^{m-1}R^*)^{\delta} \subseteq \Omega(p)(R^*S^mR^*).$$

Next multiplying both sides of (3.3) by $(R^*S^{m-1}R^*)^{\delta}$ and apply (3.4) to the right side we get $\Omega(p)(R^*S^{m-1}R^*)^{2\delta} \subseteq \Omega(p)(R^*S^{m-1}R^*)^{\delta}(R^*S^mR^*) \subseteq \Omega(p)(R^*S^mR^*)^2$. Continuing and multiplying again by $(R^*S^{m-1}R^*)^{\delta}$ and so on we finally obtain:

$$\Omega(p)(R^*S^{m-1}R^*)^{\mu\delta} \subseteq \Omega(p)(R^*S^mR^*)^{\mu}.$$

From the preceding definition of $\mu = \mu(m)$ it follows that $\Omega(p)(R^*S^mR^*)^{\mu} = 0$. Hence $\Omega(p)(R^*S^{m-1}R^*)^{\mu\delta} = 0$ which shows that for $m > \left[\dfrac{d}{2}\right]$, $\mu(m-1) \leq \mu(m)\delta$. Now $\mu(n) = 1$, so $\mu\left(\left[\dfrac{d}{2}\right]\right) \leq \mu\left(\left[\dfrac{d}{2}\right]+1\right)\delta \leq \cdots \leq \mu(n)\delta^{n-[d/2]} = \delta^{n-[d/2]} < (d+1)^{2n}$, which means that $S^{[d/2]}$ generates a nilpotent ideal mod M of index $\leq (d+1)^{2n}$, and (i) is proved.

To prove (ii), let $M_0 = S \cap M = \{s \mid s \in S, \Omega(p)s = 0\}$. Consider the Lower Radical $N_\sigma(S/M_0)$ of Lemma 2. If $N_\sigma \neq S$, then S has no nilpotent ideals mod $N_\sigma(S/M_0)$ and hence we obtain by Lemma 3 a set of elements $a_1, a_2, \ldots, a_r$ in S such that $a_1a_2 \cdots a_r \notin N_\sigma$ but every other product $a_{i_1} \cdots a_{i_k}$ for which some $i_{v+1} \leq i_v$ will belong to N_σ. This will always be the case if $k > r$, or if it is the product of these r elements but in a different order.

Substituting these a_i's in the polynomial (3.1), for every $w \in \Omega(p)$ we get

$w(a_1\cdots a_r)\subseteq\sum\Omega(p)N_\sigma(S/M_0)$ and therefore, also $\Omega(p)(a_1\cdots a_d)\subseteq\sum\Omega(p)N_\sigma$ by multiplying on the right by $a_{r+1}\cdots a_d$. It follows now by lemma 4 that $a_1a_2\cdots a_d\in N_\sigma$ which is a contradiction. Hence $N_\sigma(S/M_0)=S$.

In particular, this yields that S is locally nilpotent modulo M_0, which means that for arbitrary $s_1, s_2, \ldots, s_{[d/2]}\in S$, the multiplicative set $S_0=\{s_1, \ldots, s_{[d/2]}\}$ generated by the s_i is nilpotent. It follows, therefore, by part (i) of our theorem that $S_0^{[d/2]}$ generates a nilpotent ideal in R. Thus $s_1s_2\cdots s_{[d/2]}\in S_0^{[d/2]}\subseteq N_1(R/M)$, and this being true for all $s_i\in S$ yield that $S^{[d/2]}\subseteq N_1(R/M)$ as required.

4. We extend our result now to rings with a (two-sided) pivotal monomial. This notion has been introduced in [3] and followed in [2], and in the present paper we try to define it in its most general form where the method of the preceding section can be applied.

Let π be a function from the set of integers $(1, 2, \ldots, d)$ into the positive integers. We make correspond to π a monomial $\pi(x)=x_{\pi(1)}\cdots x_{\pi(d)}$ of degree d. Let $C_\pi=\{\sigma\}$ be the set of all functions σ defined on $(1, 2, \ldots, q)$ for arbitrary q and satisfying one of the following:

1) $q>d$

2) The non ordered sets $(\pi(1), \ldots, \pi(d))\neq(\sigma(1), \sigma(2), \ldots, \sigma(d))$

3) If $q\leq d$ and the sets $(\pi(i))$ and $(\sigma(j))$ are the same, then for some $1\leq i\leq q$ we have $[\sigma(i+1)-\sigma(i)][\pi(i+1)-\pi(i)]<0$, i.e., the order (in magnitude) of the pairs $(\sigma(i), \sigma(i+1))$ and $(\pi(i+1), \pi(i))$ are different.

We now define $\pi(x)$ to be a *pivotal* (two-sided) *monomial* of degree d if for every substitution $x_i=a_i$:

$$R\pi(a)R\subseteq\sum_{\sigma\in C_\pi}R\sigma(a)R.$$

This definition is a slight generalization of the notion of strong pivotal monomial of [3] and [2].

Our aim is to show

THEOREM 6. *If R has a pivotal monomial of degree d and S is a nil multiplicative semi-group in R then:*

i) Is S is nilpotent then S^d generates a nilpotent ideal in R.

ii) If S is nil then $S^d\subseteq N_1(R)$ which is the union of all nilpotent ideals in R (In particular, S is locally nilpotent).

Proof. First one follows the proof of Theorem 2 of [3] to show that we can assume that $\pi(x)=\varepsilon(x)=x_1x_2\cdots x_d$. This is justified, since in the linearization process of (1.2) used in the proof of the quoted theorem, we have that for each π_i in (1.2)--$\pi_j\in C_{\pi_i}$ if $i\neq j$ and $\sigma_j\in C_{\pi_i}$ for all j if $\sigma\in C_\pi$, and therefore the arguments of that proof hold and so will be the first result that we may take $\pi=\varepsilon$.

Theorem 6 has an interesting corollary which is well known for matrix rings over division rings:

COROLLARY 7. *Let R be a simple ring whose nilpotent elements are of index bounded by n, then the nil multiplicative semi-groups of R are nilpotent of index $\leq n$.*

Proof. The ring R of our theorem has $\pi(x)=x^n$ as a pivotal monomial. Indeed, for every $a\in R$: if $a^{n+1}\neq 0$, then $Ra^nR=R=Ra^{n+1}R\subseteq\sum_{\sigma\in C_\pi}R\sigma(a)R$, since R is a simple ring, and if $a^{n+1}=0$ then a is nil and hence also $a^n=0$, which shows that $0=Ra^nR=Ra^{n+1}R\subseteq\sum_{\sigma\in C_\pi}R\sigma(a)R$, which shows that in any case x^n is a pivotal monomial.

The rest of the proof follows now from the fact that if R is simple and not nilpotent then $N_1(R)=0$, and our result is a consequence of Theorem 6. If R is nilpotent, then $R^2=0$ and our corollary holds trivially.

A final remark is that one can define also pivotal monomials with operators Ω by requiring that $\Omega_0(R\pi(x)R)\subseteq\sum_{\sigma\in C_\pi}\Omega_0(R\sigma(x)R)$ for finite sets Ω_0. In this case the preceding results will be valid mod $M=\text{Ker }\Omega_0$.

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