RADON-NIKODYM DENSITIES BETWEEN HARMONIC MEASURES ON THE IDEAL BOUNDARY OF AN OPEN RIEMANN SURFACE

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Dedicated to the memory of Professor TADASI NAKAYAMA

1. Resolutive compactification and harmonic measures. Let R be an open Riemann surface. A compact Hausdorff space R^* containing R as its dense subspace is called a *compactification* of R and the compact set $\Delta = R^* - R$ is called an *ideal boundary* of R. Hereafter we always assume that R does not belong to the class O_G. Given a real-valued function f on Δ, we denote by $\overline{\varphi}_f^{R,R^*}$ (resp. $\underline{\varphi}_f^{R,R^*}$) the totality of lower bounded superharmonic (resp. upper bounded subharmonic) functions s on R satisfying

$$\liminf_{R \ni p \to p^*} s(p) \geq f(p^*) \qquad (\text{resp. } \limsup_{R \ni p \to p^*} s(p) \leq f(p^*))$$

for any point p^* in Δ. If these two families are not empty, then

$$\overline{H}_f^{R,R^*}(p) = \inf\,(s(p)\,;\ s \in \overline{\varphi}_f^{R,R^*}) \text{ and } \underline{H}_f^{R,R^*}(p) = \sup\,(s(p)\,;\ s \in \underline{\varphi}_f^{R,R^*})$$

are harmonic functions on R and $\overline{H}_f^{R,R^*} \geq \underline{H}_f^{R,R^*}$ on R. If these two functions coincide with each other on R, then we denote by H_f^{R,R^*} this common function and call f *resolutive* with respect to R^* (or Δ). We denote by $C(\Delta)$ the totality of bounded real valued continuous functions on Δ. If any function in $C(\Delta)$ is resolutive with respect to Δ, then following Constantinescu and Cornea [1] we say that R^* is a *resolutive compactification* of R. Important examples of resolutive compactifications are Wiener's, Martin's Royden's, Kuramochi's and Kerékjártó-Stoilow's compactifications (see [1]). Hereafter we always consider the resolutive compactification R^* of R.

Fix a point p in R. It is easy to see that $f \to H_f^{R,R^*}(p)$ is a positive linear functional on $C(\Delta)$ and so by Riesz-Markoff-Kakutani's theorem, there exists a positive regular Borel measure μ_p on Δ such that

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$$H_f^{R, R^*}(p) = \int_\Delta f(p^*)\, d\mu_p(p^*).$$

The measure μ_p is called the *harmonic measure* on Δ with the reference point p. We shall investigate the interdependence between each members of the family $(\mu_p;\ p \in R)$ of harmonic measures.

2. Harnack's function. Let k be the *Harnack's function* on $R\times R$, i.e. the function k defined by

$$k(p, p') = \inf(c>0;\ c^{-1}u(p) \leq u(p') \leq cu(p) \text{ for any } u \in HP(R)).$$

Then $1 \leq k(p, p') < \infty$ for any p and p' in R and $\lim_{p\to p'} k(p, p') = 1$. In fact, let U be a relatively compact simply connected domain in R containing p and p', and ϕ a $1:1$ conformal mapping of U onto $(z;\ |z|<1)$ with $\phi(p') = 0$. Then by putting $\phi(p) = re^{it}$

$$u(p) = (1/2\pi)\int_0^{2\pi} ((1-r^2)/(1-2r\cos(\theta - t) + r^2))\, u(\phi^{-1}(e^{i\theta}))d\theta$$

for any u in $HP(R)$ and so

$$((1-r)/(1+r))\, u(p) \leq u(p') \leq ((1+r)/(1-r))\, u(p).$$

Thus $1 \leq k(p, p') \leq (1+r)/(1-r) < \infty$ and if $p \to p'$, then $r \to 0$ and so $\lim_{p\to p} k(p, p') = 1$. Moreover it is easy to see that $k(p, p) = 1$, $k(p, p') = k(p', p)$ and $k(p, p'') \leq k(p, p')\, k(p', p'')$ for any p, p' and p'' in R.

3. Harmonic kernel. Let p and q belong to R. By the definition of $k(p, q)$, we see that

$$k(p, q)^{-1} d\mu_q \leq d\mu_p \leq k(p, q) d\mu_q. \tag{1}$$

Thus *measures* $\mu_p(p \in R)$ *are absolutely continuous with respect to each other* and so the μ_p-integrability and the μ_p-nullity do not depend on the special choice of p in R. We denote by $(d\mu_q/d\mu_p)(p^*)$ the *Radon-Nikodym density* of μ_q with respect to μ_p.

We fix a point o in R. Then we can easily see that the function $p \to \int_\Delta f(p^*)d\mu_p(p^*)$ is harmonic on R if f is μ_o-integrable on Δ. The main assertion in this note is the following

THEOREM. *There exists a function* $P_o(p, p^*)$ *on* $R\times\Delta$ *such that*

(a) $P_o(p, p^*) = (d\mu_p/d\mu_o)(p^*)$ (μ_o-*almost everywhere*) *on* Δ *as the function* *of* p^* *for any fixed* p *in* R;

(b) $P_o(p, p^*)$ *is harmonic on* R *as the function of* p *for any fixed* p^* *in* Δ;

(c) $k(o, p)^{-1} \leq P_o(p, p^*) \leq k(o, p)$ *for any* (p, p^*) *in* $R \times \Delta$;

(d) $P_o(p, p^*)$ *is Borel measurable on* $R \times \Delta$ *as the function of* (p, p^*).

Needless to say, such a function $P_o(p, p^*)$ is not unique in the proper sense, but unique in the following sense: if $\tilde{P}_o(p, p^*)$ is another function on $R \times \Delta$ satisfying the above four conditions, then $P_o(p, p^*) = \tilde{P}_o(p, p^*)$ $\tilde{\mu}_o$-almost everywhere on $R \times \Delta$. Here $\tilde{\mu}_o$ is the product measure $\tilde{\mu} \times \mu_o$, where $\tilde{\mu}$ is a measure on R which is equivalent to the Lebesgue measure in each parameter neighborhood of R. Such a $P_o(p, p^*)$ may be called a *harmonic kernel* (or Poisson type kernel) on $R \times \Delta$ with the reference point o. For any Borel function f, μ_o-integrable on Δ, we then have

$$H_f^{R, R^*}(p) = \int_\Delta P_o(p, p^*) f(p^*) \, d\mu_o(p^*).$$

The harmonicity of the function $p \to P_o(p, p^*)$ increases the usefulness of the above integral representation.

4. Proof of Theorem. Let $\tilde{P}(p, p^*)$ be an arbitrary but fixed function on $R \times \Delta$ such that $\tilde{P}(p, p^*) = (d\mu_p / d\mu_o)(p^*)$ (μ_o-almost everywhere) on Δ as the function of p^* for any fixed p in R. We may assume that $\tilde{P}(o, p^*) \equiv 1$ on Δ. Since R is separable, there exists a countable dense subset D of R with $o \in D$.

For any p and q in D, by (1), we see that

$$k(p, q)^{-1} (d\mu_q / d\mu_o)(p^*) \leq (d\mu_p / d\mu_o)(p^*) \leq k(p, q)(d\mu_q / d\mu_o)(p^*)$$

μ_o-almost everywhere on Δ as the function of p^*. Hence there exists a Borel set $E(p, q)$ in Δ such that

$$\mu_o(E(p, q)) = 0$$

and

$$k(p, q)^{-1} \tilde{P}(q, p^*) \leq \tilde{P}(p, p^*) \leq k(p, q) \tilde{P}(q, p^*)$$

for any p^* in $\Delta - E(p, q)$. Let

$$E = \cup_{p, q \in D} E(p, q).$$

Since D is countable, $\mu_o(E) = 0$. Hence

$$k(p, q)^{-1} \tilde{P}(q, p^*) \leq \tilde{P}(p, p^*) \leq k(p, q) \tilde{P}(q, p^*)$$

for any p and q in D and p^* in $\Delta - E$. In particular, since $\tilde{P}(o, p^*) = 1$ on Δ,

(2) $$k(p, o)^{-1} \leq \tilde{P}(p, p^*) \leq k(p, o)$$

for any p in D and p^* in $\Delta - E$. Thus

(3) $$|\tilde{P}(p, p^*) - \tilde{P}(q, p^*)| \leq k(p, o) \max(k(p, q) - 1, 1 - k(p, q)^{-1})$$

for any p and q in D and p^* in $\Delta - E$. We saw in Section 2 that

$$1 \leq k(p, q) \leq k(p, p_0) k(p, q_0), \qquad 1 \leq k(p, o) \leq k(o, p_0) k(p, p_0)$$

and

$$\lim_{D \ni p \to p_0} k(p, p_0) = \lim_{D \ni q \to p_0} k(q, p_0) = 1$$

for any p_0 in R. From these and (3), it follows that

$$\lim_{D \ni p, q \to p_0} |\tilde{P}(p, p^*) - \tilde{P}(q, p^*)| = 0,$$

or equivalently that

$$\lim_{D \ni p \to p_0} \tilde{P}(p, p^*)$$

exists for any p_0 in R and if p_0 belongs to D, then

$$\lim_{D \ni p \to p_0} \tilde{P}(p, p^*) = \tilde{P}(p_0, p^*).$$

Hence if we set

$$P(p, p^*) = \lim_{D \ni p' \to p} \tilde{P}(p', p^*)$$

in $R \times (\Delta - E)$, then the function $p \to P(p, p^*)$ is continuous on R for fixed p^* in $\Delta - E$. For arbitrary point p in R, take a sequence $(p_n)_{n=1}^{\infty}$ of points in D with $p_n \to p$. Then for any function f in $C(\Delta)$, by using (2), the definition of $P(p, p^*)$ and Lebesgue's convergence theorem,

$$\begin{aligned} \int_{\Delta} f(p^*)\, d\mu_p(p^*) &= H_f^{R, R^*}(p) \\ &= \lim_{n \to \infty} H_f^{R, R^*}(p_n) \\ &= \lim_{n \to \infty} \int_{\Delta - E} \tilde{P}(p_n, p^*) f(p^*) d\mu_o(p^*) \\ &= \int_{\Delta - E} \lim_{n \to \infty} \tilde{P}(p_n, p^*) f(p^*)\, d\mu_o(p^*) \\ &= \int_{\Delta - E} P(p, p^*) f(p^*)\, d\mu_o(p^*). \end{aligned}$$

This shows that $d\mu_p(p^*) = P(p, p^*)\, d\mu_o(p^*)$. Hence $P(p, p^*) = (d\mu_p / d\mu_o)(p^*)$.

μ_o-almost everwhere.

Let ϕ be an analytic mapping of the open unit disc $(z;\ |z|<1)$ onto R. Now we prove that the function $p\to P(p,p^*)$ is harmonic on R for almost every fixed p^* in Δ. For the aim, we have only to show that the function $z\to P(\phi(z),p^*)$ is harmonic on $(z;\ |z|<1)$ for almost every fixed p^* in Δ, since $p\to P(p,\ p^*)$ is continuous on R for any fixed p^* in $\Delta-E$. Since $p^*\to P(\phi(z),\ p^*)$ is Borel measurable on Δ for any fixed z in $(z;\ |z|<1)$ and $z\to P(\phi(z),\ p^*)$ is continuous on R for any fixed p^* in $\Delta-E$, it is easy to see that the function $(z,\ p^*)\to P(\phi(z),\ p^*)$ is Borel measurable on $R\times\Delta$.

Let $(z_n)_{n=1}^\infty$ be a countable dense subset of $(z;\ |z|<1)$. Fix an arbitrary positive integer n and choose a countable dense subset $(r_m)_{m=1}^\infty$ of the open interval $(0,\ 1-|z_n|)$. Then for any f in $C(\Delta)$, since $\int_\Delta P(\phi(z),\ p^*)f(p^*)d\mu_o(p^*)$ is harmonic in z of $(z;\ |z|<1)$, by Fubini's theorem,

$$\int_\Delta P(\phi(z_n),\ p^*)f(p^*)\,d\mu_o(p^*)=\frac{1}{2\pi}\int_0^{2\pi}\Big[\int_\Delta P(\phi(z_n+r_me^{i\theta}),\ p^*)f(p^*)\,d\mu_o(p^*)\Big]d\theta$$
$$=\int_\Delta\Big[\frac{1}{2\pi}\int_0^{2\pi}P(\phi(z_n+r_me^{i\theta}),\ p^*)d\theta\Big]f(p^*)d\mu_o(p^*).$$

Hence there exists a set $F_{n,m}$ in Δ with $\mu_o(F_{n,m})=0$ such that for any p^* in $\Delta-F_{n,m}$ it holds that

$$P(\phi(z_n),\ p^*)=\frac{1}{2\pi}\int_0^{2\pi}P(\phi(z_n+r_me^{i\theta}),\ p^*)\,d\theta. \tag{4}$$

Let $F_n=E\cup(\cup_{m=1}^\infty F_{n,m})$. Then $\mu_o(F_n)=0$ and the identity (4) holds for any $m=1,2,\ldots$ and p^* in $\Delta-F_n$. By the continuity of $P(\phi(z),\ p^*)$ in z for any fixed p^* in $\Delta-E$, we conclude that

$$P(\phi(z_n),\ p^*)=\frac{1}{2\pi}\int_0^{2\pi}P(\phi(z_n+re^{i\theta}),\ p^*)\,d\theta \tag{5}$$

for any r in $0<r<1-|z_n|$ and p^* in $\Delta-F_n$. Finally let $F=\cup_{n=1}^\infty F_n$. Then $\mu_o(F)=0$ and (5) holds for any $n=1,2,\ldots$ and any r in $0<r<1-|z_n|$ and any p^* in $\Delta-F$. By the continuity of $P(\phi(z),\ p^*)$ in z for any fixed p^* in $\Delta-E$, we conclude that

$$P(\phi(z),\ p^*)=\frac{1}{2\pi}\int_0^{2\pi}P(\phi(z+re^{i\theta}),\ p^*)d\theta$$

for any z in the unit disc and r in $0<r<1-|z|$ and p^* in $\Delta-F$, which shows that $z \rightarrow P(\phi(z), p^*)$ is harmonic in $(z;\ |z|<1)$ for any fixed p^* in $\Delta - F$.

Thus the function $p \rightarrow P(p, p^*)$ is harmonic on R for any fixed p^* in $\Delta - F$ with $\mu_o(F) = 0$. Let

$$P_o(p, p^*) = \begin{cases} P(p, p^*), & \text{for } (p, p^*) \text{ in } R \times (\Delta - F); \\ 1, & \text{for } (p, p^*) \text{ in } R \times F. \end{cases}$$

Then for any fixed p in R, $P_o(p, p^*) = P(p, p^*) = (d\mu_p/d\mu_o)(p^*)$ (μ_o-almost everywhere) on Δ. Thus (a) is satisfied by $P_o(p, p^*)$ thus constructed. It is also clear that $P_o(p, p^*)$ satisfies (b). The condition (c) follows immediately from (b), the definition of $k(o, p)$ and the fact that $P_o(o, p^*) \equiv 1$ for any p^* in Δ (see (2)). The last condition (d) is an easy consequence of (a) and (b).

References

[1] C. Constantinescu-A. Cornea: Ideale Ränder Riemannscher Flächen, Springer-Verlag, 1963.

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