IMBEDDING A REGULAR RING IN A REGULAR RING WITH IDENTITY

NENOSUKE FUNAYAMA

Dedicated to the memory of Professor TADASI NAKAYAMA

In [1] L. Fuchs and I. Halperin have proved that a regular ring R is isomorphic to a two-sided ideal of a regular ring with identity. ([1] Theorem 1). Their methed is to imbed the regular ring R in the ring of all pairs (a, ρ) with $a \in R$ and ρ from a suitable commutative regular ring S with identity such that R is an algebra over S. Thus S may be seen as the ring of $R-R$ endomorphisms of the additive group of R. The following question is naturally raised: Is it true that the ring of all $R-R$ endomorphisms of a rugular ring is a commutative regular ring? The main purpose of this paper is to answer this question affirmatively. (Theorem 1). After established this theorem we can follow the method in [1] to solve the problem in the title.

1. Endorphisms of R^+.

Let R^+ be the additive group of a given ring R with R as left and right operator domains, and let $\tilde{R}$ be the ring of all endomorphisms of R^+, that is the ring of all $R-R$ endomorphisms of the additive group R. $\tilde{R}$ has the identity $\bar{1}$ which is the identity mapping of R^+. Also let us denote by $\bar{0}$, $\bar{n}$ and $\bar{c}$ respectively the zero endomorphism, $\bar{n}$: $a \to na$, where a is an element in R and n is an integer, $\bar{c}$: $a \to ac$, where c is an element in the center C of R.

LEMMA 1. *If R has the identity* 1, *then $\tilde{R}$ is isomorphic to the center C of R.*

Proof. Let ρ be an element of $\tilde{R}$. Then for any element a in R we have $a\rho = (a1)\rho = a(1\rho)$ and $a\rho = (1a)\rho = (1\rho)a$. Thus $c = 1\rho$ is in the center C of R and $a\rho = ac = ca$. Conversely let c be an element in C, then $\bar{c}$: $a \to ac$ is an endomorphism of R^+. $\rho \to 1\rho$ sets up a ring isomorphism between $\tilde{R}$ and C.

LEMMA 2. *If $R^2 = R$, then $\tilde{R}$ is commutative.*

Received March 22, 1965.

Proof. Let ρ, τ be a pair of elements in $\tilde{R}$. We will show that $a(\rho\tau) = a(\tau\rho)$ for any element a in R. As $R^2 = R$ it is sufficient to show that $(bc)(\rho\tau) = (bc)(\tau\rho)$ for any pair of elements b, c in R, and this is easily shown using the fact that ρ, τ are $R-R$ endomorphisms.

LEMMA 3. *If R is a regular ring, then $\tilde{R}$ is commutative.*

Proof is clear by Lemma 2.

For an element ρ in $\tilde{R}$ denote the kernel and the image of ρ by

$$R_\rho = \rho^{-1}(0) = \{a \in R \mid a\rho = 0\},$$
$$\overline{R}_\rho = \{a\rho \mid a \in R\}.$$

R_ρ and $\overline{R}_\rho$ are ideals in R. *If ρ is idempotent then $R = R_\rho \oplus \overline{R}_\rho$.*

The converse is not always true, that is $R = R_\rho \oplus \overline{R}_\rho$ does not imply that ρ is idempotent, and so, for the later use, we seek for the condition for ρ which implies $R = R_\rho \oplus \overline{R}_\rho$.

LEMMA 4. *$R = R_\rho \oplus \overline{R}_\rho$ if and only if the following conditions are satisfied:*

$$x\rho^2 = 0 \text{ implies } x\rho = 0. \tag{1}$$

For any $x \in R$ there exists an element $y \in R$ such that

$$x\rho = y\rho^2. \tag{2}$$

Moreover the y in (2) is uniquely determined in $\overline{R}_\rho$.

Proof. Condition (1) is equivalent to the condition $R_\rho \cap \overline{R}_\rho = (0)$ as is easily shown. Condition (2) is equivalent to the condition $R = R_\rho + \overline{R}_\rho$. Indeed if $R = R_\rho + \overline{R}_\rho$, then any $x \in R$ may be written as $x = x_1 + x_2\rho$, where $x_1\rho = 0$ and then $x\rho = x_2\rho^2$. Conversely if the condition (2) is satisfied, any $x \in R$ may be written as $x = (x - y\rho) + y\rho$, where y satisfies $x\rho = y\rho^2$. Then $(x - y\rho)\rho = x\rho - y\rho^2 = 0$, which proves that $R = R_\rho + \overline{R}_\rho$. The proof of the last part is as follows: First the y in (2) may be chosen from $\overline{R}_\rho$ as $x\rho = y\rho^2$ and $y\rho = y\rho^2$ imply that $x\rho = (z\rho)\rho^2$. Secondly the uniqueness of y: If $x\rho = y\rho^2 = z\rho^2$, where y and z are in $\overline{R}_\rho$, then $(y - z)\rho^2 = 0$, which implies $(y - z)\rho = 0$ by (1). As y and z are in $\overline{R}_\rho$ $y = y'\rho$, $z = z'\rho$ for some y', $z' \in R$. Then $(y' - z')\rho^2 = 0$, and so again by (1) $(y' - z')\rho = 0$, that is $y = z$.

LEMMA 5. *If $\rho \in \tilde{R}$ satisfies $R = R_\rho \oplus \overline{R}_\rho$, then for some $\sigma \in \tilde{R}$,*

$$\rho\sigma\rho = \rho \tag{3}$$

$$\rho\sigma = \sigma\rho \tag{4}$$

$$\sigma\rho\sigma = \sigma \tag{5}$$

Proof. In Lemma 4 it is shown that $R = R_\rho \oplus \overline{R}_\rho$ implies that, for any $x \in R$ there exists uniquely determined $y \in \overline{R}_\rho$ with $x\rho = y\rho^2$. Define σ as $x\sigma = y$. As is easily seen σ is an endomorphism of the additive group of R. For any elements x, r in R we have

$$(xr)\rho = (x\rho)r = (y\rho^2)r = (yr)\rho^2.$$

As $\overline{R}_\rho$ is an ideal of R we have $yr \in \overline{R}_\rho$, showing that $(xr)\sigma = (x\sigma)r$. Similarly $(rx)\sigma = r(x\sigma)$. Thus $\sigma \in \tilde{R}$.

As the proofs of (3), (4) and (5) are similar we show only (5). To prove (5) it is sufficient to show that $x(\sigma\rho\sigma) = x\sigma$ for any $x \in R$. Put $x\sigma = y$ and $x(\sigma\rho\sigma) = z$. Then, by the definition of σ, we have $x\rho = y\rho^2$, $y \in \overline{R}_\rho$, and $(y\rho)\sigma = z$, that is $y\rho^2 = z\rho^2$, where y and z are in $\overline{R}_\rho$. Then $(y-z)\rho^2 = 0$, which implies $y = z$ as y and z are in $\overline{R}_\rho$. Thus we have $x\sigma = x(\sigma\rho\sigma)$.

THEOREM 1. *The ring $\tilde{R}$, ring of all endomorphisms of R^+, of a regular ring R is a commutative regular ring with identity.*

Proof. Commutativity was already shown in Lemma 3. To prove the regularity of R it is sufficient to prove $R = R_\rho \oplus \overline{R}_\rho$ for any $\rho \in \tilde{R}$, or equivalently, by Lemma 4, (1) and (2) in Lemma 4. Suppose that $x\rho \neq 0$. Then by the regularity of R there exists $y \in R$ such that $x\rho = (x\rho)y(x\rho)$. This implies $x\rho = (x\rho^2)yx$ and as $x\rho \neq 0$ we have that $x\rho^2 \neq 0$ showing (1). Also $x\rho = (x\rho)y(x\rho) = (xyx)\rho^2$ showing (2).

2. Imbedding a regular ring into a regular ring with identity.

Let R be an arbitrary ring.

Let S be a commutative subring of $\tilde{R}$, the ring of all $R-R$ endomorphisms of R^+, and let R^S be the set of all ordered pairs (a, ρ) where $a \in R$ and $\rho \in S$. In R^S define the equality, addition, and multiplication by

$$(a, \rho) = (b, \tau) \text{ if and only if } a = b \text{ and } \rho = \tau,$$
$$(a, \rho) + (b, \tau) = (a+b, \rho+\tau),$$
$$(a, \rho)(b, \tau) = (ab + b\rho + a\tau, \rho\tau).$$

Then R^S is a ring. Commutativity of S is used for the proof of associativity of R^S. If S has the identity then R^S has the identity $(0, \bar{1})$. The examples of

S are as follows: (a) $Z=\{\bar{n}:\ a\to na,\ n$ is an integer$\}$, (b) $\bar{C}=\{\bar{c}\mid\bar{c}:\ a\to ac$ $(=ca)$, c is in the center C of $R\}$, (c) $\bar{Z}+\bar{C}$, (d) $\tilde{R}$ when $\tilde{R}$ is commutative.

Remark 1. $R^{\bar{Z}}$ does not coincide with the classical imbedding $R^{\#}$. Indeed when R is of bounded order $R^{\bar{Z}}$ is of bounded order but $R^{\#}$ is not of bounded order.

R is imbedded in R^S as an ideal by the mapping $a\to(a, 0)$. Our idea is to give some properties to R^S selecting a suitable S. This idea is essentially included in [1], and the proof of the following theorem follows that in [1].

LEMMA 6. *If R and S are regular, then R^S is regular.*

Proof. Let (a, ρ) be any element in R^S. We will seek for (b, σ) such that $(a, \rho)(b, \sigma)(a, \rho)=(a, \rho)$, that is

$$\rho\sigma\rho=\rho,$$
$$aba+(ba)\rho+(ab)\rho+a^2\sigma+b\rho^2+a(\sigma\rho)+a(\rho\sigma)=a. \tag{6}$$

As S is regular there exists a σ such that $\rho\sigma\rho=\rho$. For the second equality: Let e be an idempotent in R such that $a=ae=ea$. (The existence such an e has been proved in [1] Lemma 2).

By the regularity of R there exists an element x such that

$$(a+e\rho)x(a+e\rho)=a+e\rho. \tag{7}$$

Put $y=exe$, then, as is easily calculated, y satisfies (7) replacing x by y. Put $b=y-e\sigma$, then b satisfies (6).

THEOREM 2. *$R^{\tilde{R}}$ is a regular ring with identity if R is regular. R is imbedded in $R^{\tilde{R}}$ as an ideal.*

Proof is clear from Theorem 1 and Lemma 6.

REFERENCE

[1] L. Fuchs and I. Halparin, On the embedding of a regular ring in a regular ring with identity, Fundamenta Mathematicae LIV (1964), pp. 287-290.

Yamagata University