ON MULTIPLY TRANSITIVE GROUPS I

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Dedicated to the memory of Professor TADASI NAKAYAMA

The purpose of this paper is to prove the following three theorems which were announced in [2].

THEOREM 1. *Let G be a quadruply transitive group on $\{1, 2, \ldots, n\}$ and H the subgroup of G consisting of all the elements leaving the two letters* 1 *and* 2 *invariant. If G is of even degree and H contains a normal subgroup Q which is regular on $\{3, 4, \ldots, n\}$, then G is one of the following groups: S_4, S_6 or A_6.*

THEOREM 2. *Let G be a quintuply transitive group on $\{1, 2, \ldots, n\}$ and H the subgroup of G consisting of all the elements leaving the three letters* 1, 2 *and* 3 *invariant. If H contains a normal subgroup Q which is regular on $\{4, 5, \ldots, n\}$, then G is one of the following groups: S_5, S_6, S_7, A_7 or M_{12}.*

The following theorem is an improvement of a theorem of Wielandt ([4], Satz 1).

THEOREM 3. *Let G be a k-fold transitive group of degree n. If the outer automorphism group of any simple subgroup of G is solvable, then $k \leq 6$ unless G is S_n* or *A_n.*

We use standard notations throughout. For a set X let $|X|$ denote the number of elements of X. For a subset X of a group G let $N_G(X)$ denote the normalizer of X in G, and the centralizer of X in G is denoted by $C_G(X)$.

1. Proof of Theorem 1

We first prove the following lemma which will be used in this and the next sections.

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LEMMA[1]. *Let V be a vector space over a field and ρ a nilpotent linear transformation of V. If $\rho^n = 0$ then*

$$\dim V \leq n \dim V_0,$$

where $V_0 = \{v \in V;\ \rho v = 0\}$.

Proof. We prove the lemma by the induction on n. For $n = 1$, the lemma is trivial. Let $W = \rho V$. Then $W \simeq V/V_0$. Since $\rho^{n-1}W = 0$ we have, by the hypothesis of induction,

$$\dim W \leq (n-1) \dim W_0,$$

where $W_0 = W \cap V_0$. Therefore we have

$$\begin{aligned}\dim V &= \dim W + \dim V_0 \\ &\leq (n-1) \dim W_0 + \dim V_0 \\ &\leq n \dim V_0.\end{aligned}$$

Proof of Theorem 1. Since Q is regular on $\{3, 4, \ldots, n\}$ and n is even, Q is of even order. Now Q is a regular normal subgroup of H which is doubly transitive on $\{3, 4, \ldots, n\}$, therefore Q is an elementary abelian subgroup of exponent 2 ([3], 11.3, (a)) and the unique minimal normal subgroup of H ([3], 11.4, 11.5).

Let $s \neq 1$ be an element of Q. We may assume

$$s = (1)(2)(3, 4)\cdots.$$

Since G is quadruply tranistive there is an element x in G such that

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 \cdots \\ 3 & 4 & 1 & 2 \cdots \end{pmatrix}.$$

Let $t = x^{-1}sx$. Then

$$t = (1, 2)(3)(4)\cdots$$

and t fixes only two letters 3 and 4. Since t is in $N_G(H)$ and Q is the unique minimal normal subgroup of H, $t^{-1}Qt = Q$ and t induces an automorphism τ of Q. Let Q_0 be the subgroup of Q consisting of all the elements left invariant by τ. From the regularity of Q, s is in Q_0. Let

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[1] The lemma of this general form is due to the suggestion by Professor N. Ito. The lemma was first stated in more special form.

$$r=(1)(2)(3,\ \alpha)\cdots$$

be an element in Q which is different from s, then $\alpha\neq 1, 2, 3, 4$. If $\alpha\to\alpha'$ under t then $\alpha'\neq\alpha$ and

$$r^{\tau}=t^{-1}rt=(1)(2)(3,\ \alpha')\cdots$$

is different from r. Thus we have $Q_0=\{1,\ s\}$ and $|Q_0|=2$. Applying Lemma for $\rho=\tau-1$, we have $|Q|\leq 4$, therefore $|Q|=n-2=2$ or 4, $n=4$ or 6. The quadruply transitive group of degree 4 or 6 is clearly S_4, A_6 or S_6.

2. Proof of Theorem 2

In the same way as Theorem 1 we have first the following proposition.

Proposition. *Let G be a quintuply transitive group on $\{1, 2, \ldots, n\}$ and H the subgroup of G consisting of all the elements leaving the three letters 1, 2 and 3 invariant. If n is divisible by 3 and H contains a normal subgroup Q which is regular on $\{4, 5, \ldots, n\}$, then G is S_6 or M_{12}.*

Proof. Since H is doubly transitive on $\{4, 5, \ldots, n\}$, where n is a multiple of 3, and Q is a regular normal subgroup of H, Q is an elementary abelian subgroup of exponent 3 and the unique minimal normal subgroup of H.

Let $s\neq 1$ be an element of Q. We may assume

$$s=(1)(2)(3)(4, 5, 6)\cdots.$$

Since G is quintuply transitive there is an element x in G such that

$$x=\begin{pmatrix}1 & 2 & 4 & 5 & 6\cdots\\ 4 & 5 & 1 & 2 & 3\cdots\end{pmatrix}.$$

Let $t=x^{-1}sx$. If $3\to\alpha$ under x then

$$t=(1, 2, 3)(4)(5)(\alpha)\cdots$$

and t fixes only three letters 4, 5, α. Since $t^{-1}Ht=H$, t induces an automorphism $\tau:\ x\to t^{-1}xt$ of Q, whose order is 3. Let Q_0 be the subgroup of Q consisting of all the elements left invariant by τ. Since Q is regular on $\{4, 5, \ldots, n\}$ and both s and $s^{\tau}=t^{-1}st$ take 4 to 5, we have $s=s^{\tau}$, $s\in Q_0$ and t fixes 6. Therefore $\alpha=6$. Let

$$r=(1)(2)(3)(4,\ \beta,\ \gamma)\cdots$$

be an element in Q which is different from s and s^2, then $\beta \neq 1, 2, 3, 4, 5, 6$. If $\beta \rightarrow \beta'$ under t, then $\beta \neq \beta'$ and

$$r^{\tau} = t^{-1}rt = (1)(2)(3)(4, \beta', \gamma') \cdots$$

is different from r. Thus we have $Q_0 = \{1, s, s^2\}$ and $|Q_0| = 3$. Applying Lemma for $\rho = \tau - 1$, we have $|Q| \leq |Q_0|^3 = 27$, since $(\tau - 1)^3 = 0$. Therefore $|Q| = n - 3 = 3$, 9 or 27, $n = 6$, 12 or 30. If $n = 6$, G must be S_6. It is known that a quadruply transitive group of degree 11 is S_{11}, A_{11} or M_{11} ([1], p. 77). Therefore if $n = 12$, G is one of the groups S_{12}, A_{12} or M_{12}. But among these groups only M_{12} satisfies the assumption. If $n = 30$, then $n = 2 \cdot 13 + 4$ and by a theorem of Miller ([1], Theorem 5.7.2) G must be S_{30} or A_{30}. But in both cases G does not satisfy the assumption.

Proof of Theorem 2. Since H is doubly transitive on $\{4, 5, \ldots, n\}$, Q is an elementary abelian subgroup. Let V be the subgroup consisting of all the elements leaving the five letters 1, 2, 3, 4 and 5 invariant, and let $\Delta = \{1, 2, 3, 4, 5, \cdots\}$ be the set of all letters left invariant by V. By a theorem of Witt [5] $N = N_G(V)$ is quintuply transitive on Δ. Let N^{Δ} be the restriction of N on Δ. Then the kernel of the natural homomorphism $\varphi: N \rightarrow N^{\Delta}$ is V and we have $N/V \simeq N^{\Delta}$. The permutation group N^{Δ} on Δ is a quintuply transitive group such that only the identity leaves five letters invariant. By a theorem of Jordan ([1], p. 72) N^{Δ} is one of the following groups: S_5, S_6, A_7 or M_{12}. Therefore $|\Delta| = 5$, 6, 7 or 12.

Let $H_0 = H \cap N$. Then $H_0^{\Delta} = \varphi(H_0)$ is the subgroup of N^{Δ} consisting of all the elements leaving the three letters 1, 2 and 3 invariant. Let $Q_0 = Q \cap N$. Since Q is regular on $\{4, 5, \ldots, n\}$, there is an element s in Q such that

$$s = (1)(2)(3)(4, 5, \ldots) \cdots$$

and then, by the regularity of Q, $s \in C_G(V)$, $s \in Q_0$. Thus $Q_0 \neq 1$. Q_0 is isomorphic to $Q_0^{\Delta} = \varphi(Q_0)$ and Q_0^{Δ} is a normal subgroup of a doubly transitive group H_0^{Δ} on $\Delta - \{1, 2, 3\}$. Therefore Q_0^{Δ} is transitive on $\Delta - \{1, 2, 3\}$ and hence regular on it. Thus we have $|Q_0^{\Delta}| = |Q_0| = |\Delta| - 3 = 2$, 3, 4 or 9. Since Q_0 is a subgroup of the elementary abelian group Q, the exponent of Q must be 2 or 3. If the exponent is 2, by Theorem 1, G is a transitive extension of S_4, S_6 or A_6, therefore G must be one of the groups S_5, S_7 or A_7. If the exponent is 3, by Proposition, G is S_6 or M_{12}.

3. Proof of Theorem 3

Let X be a 7-fold transitive group on $\{1, 2, \ldots, n\}$, which is different from S_n and A_n, G the subgroup of X consisting of all the elements leaving the two letters 1 and 2 invariant, and let H be the subgroup consisting of all the elements leaving the five letters 1, 2, 3, 4 and 5 invariant. The group G is quintuply transitive on $\{3, 4, \ldots, n\}$. By Hilfssatz (2) in [4], H contains a normal subgroup which is regular on $\{6, 7, \ldots, n\}$. Therefore, by Theorem 2, G is one of the following groups: S_5, S_6, S_7, A_7 or M_{12}. Since M_{12} has no transitive extension, G is a symmetric or alternating group and hence X is S_n or A_n. This is a contradiction.

References

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