ON THE FUNDAMENTAL EXISTENCE THEOREM OF KISHI

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1. Notation and terminology. Let Ω be a locally compact Hausdorff space and $G(x, y)$ be a strictly positive lower semicontinuous function on the product space $\Omega \times \Omega$ of Ω. Such a function $G(x, y)$ is called a ***kernel*** on Ω. The ***adjoint kernel*** $\check{G}(x, y)$ of $G(x, y)$ is defined by $\check{G}(x, y) = G(y, x)$. Whenever we say a measure on Ω, we mean a positive regular Borel measure on Ω. The ***potential*** $G_\mu(x)$ and the ***adjoint potential*** $\check{G}_\mu(x)$ of a measure μ relative to the kernel $G(x, y)$ is defined by

$$G_\mu(x) = \int G(x, y)\, d\mu(y) \quad \text{and} \quad \check{G}_\mu(x) = \int \check{G}(x, y)\, d\mu(y)$$

respectively. These are also strictly positive lower semicontinuous functions on Ω provided $\mu \neq 0$.

We say that a kernel $G(x, y)$ on Ω satisfies the ***continuity principle*** when, for any measure μ with compact support S_μ, the finite continuity of the restriction of $G_\mu(x)$ to S_μ implies the global finite continuity of $G_\mu(x)$ on Ω.

A property is said *to hold G-p.p.p.* on a subset X in Ω, when the property holds on X except a set E which does not contain any compact support S_ν of a measure $\nu \neq 0$ with finite G-energy $\int G_\nu(x)\, d\nu(x)$. Notice that $\int G_\nu(x) d\nu(x) = \int \check{G}_\nu(x)\, d\nu(x)$. Hence the notion G-p.p.p. is equivalent to that of $\check{G}$-p.p.p.

2. Result. M. Kishi [4] [5] proved the following important existence theorem in the potential theory with non-symmetric kernel:

Assume that the adjoint kernel $\check{G}(x, y)$ of $G(x, y)$ satisfies the continuity principle. Given a non-empty separable compact subset K of Ω and a strictly positive finite upper semicontinuous function $u(x)$ on K. Then there exists a measure μ with support S_μ in K such that

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$$G_\mu(x) \geq u(x)$$

G-p.p.p. on K and

$$G_\mu(x) \leq u(x)$$

everywhere on S_μ.

In contrast with the symmetric case and also in the view point of the application, it is desirable to avoid the separability condition in Kishi's theorem on the given compact set K and the aim of this note is to do this. Namely, we shall prove

THEOREM: *Assume that the adjoint kernel* $\check{G}(x, y)$ *satisfies the continuity principle. Given an arbitrary non-empty compact subset* K *of* Ω *and a strictly positive finite upper semicontinuous function* $u(x)$ *on* K. *Then there exists a measure* μ *with compact support* S_μ *in* K *such that*

$$G_\mu(x) \geq u(x)$$

G-p.p.p. on K and

$$G_\mu(x) \leq u(x)$$

everywhere on S_μ.

To prove this theorem, we may assume without loss of generality that

$$u(x) = 1 \qquad \text{identically on } K.$$

In fact, consider the function $G'(x, y)$ on $K \times K$ defined by

$$G'(x, y) = G(x, y)/u(x).$$

Clearly $G'(x, y)$ is again a kernel on K. Moreover, the adjoint kernel $\check{G}'(x,y)$ satisfies the continuity principle on K along with $\check{G}(x, y)$. To see this, assume that $\check{G}'_\mu(x)$ is finitely continuous on the support S_μ $(\subset K)$ of a measure μ. Then

$$\check{G}'_\mu(x) = \int (G(y, x)/u(y))\, d\mu(y) \geq (\inf_{K\times K} G(z, z')) \int (1/u(y))\, d\mu(y).$$

Taking x in S_μ and noticing that $\inf_{K\times K} G(z, z') > 0$, we see that the function $1/u(y)$ is μ-integrable on K. Hence the set function

$$\nu(X) = \int_X (1/u(y))\, d\mu(y)$$

is again a positive regular Borel measure on K and $S_\nu = S_\mu$ and $\check{G}'_\mu(x) = \check{G}_\nu(x)$ on K and so $\check{G}_\nu(x)$ is finitely continuous on S_ν. Thus by the continuity principle assumed for $\check{G}(x, y)$, $\check{G}_\nu(x)$ is finitely continuous on Ω and a fortiori on K. Hence $\check{G}'_\mu(x)$ is finitely continuous on K.

If we get the theorem concerning this new kernel $G'(x, y)$ and the constant function 1, then we get the theorem concerning $G(x, y)$ and $u(x)$. Hence hereafter *we always assume that* $u(x) = 1$ *identically on* K.

3. Fundamental lemma of Kishi. The method of our proof is an uncountable version of the interesting method of Kishi [4], [5]. The starting point and also the key of Kishi's method is the following lemma of algebraic character which we shall give an alternating proof using the finite points version of Gauss-Frostman-Ninomiya's variation:

LEMMA 1 (Kishi [5], Theorem I. 3). *Given strictly positive numbers* a_{ki} $(k, i = 1, 2, \dots, n)$. *Then there exists a system* $t_1, t_2, \dots, t_n$ *of non-negative numbers such that*

$$\sum_{i=1}^{n} a_{ki} t_i \geq 1$$

for all $k = 1, 2, \dots, n$ *and*

$$\sum_{i=1}^{n} a_{ji} t_i \leq 1$$

for all j *such that* $t_j \neq 0$.

For the proof, we shall use the following theorem (see Kakutani [3] or Nikaidô [7]):

Kakutani's fixed point theorem: *Let* X *be a compact convex set in the euclidean n-space* R^n *and* f *be a closed "point to set" mapping of* X *in* X *such that the set* $f(x)$ *is a non-empty convex set in* X *for any point* x *in* X. *Then there exists a point* x *in* X *such that* x *belongs to* $f(x)$.

In our case, we take X as follows:

$$X = (x = (x_1, \dots, x_n) \in R^n;\ \sum_{i=1}^{n} x_i = 1,\ x_i \geq 0\ (i = 1, 2, \dots, n)).$$

Then the set X is a compact convex set in R^n. We also denote by $\tilde{X}$ the totality of non-empty convex subset of X. We consider the bilinear form

$$[x, y] = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i y_k,$$

where $x=(x_1, \dots, x_n)$ and $y=(y_1, \dots, y_n)$ are in R^n. Clearly $[x, y]$ is continuous on $R^n \times R^n$ and $[x, x]>0$. We set, for each x in X,

$$f(x)=(y \in X;\ \inf_{z \in X}[z, x]=[y, x]).$$

Clearly $f(x)$ belongs to $\tilde{X}$. Hence f defines a mapping of X into $\tilde{X}$. Moreover f is a closed mapping, i.e. for any sequences (x^ν) in X and (y^ν) in X such that $y^\nu \in f(x^\nu)$ and $\lim_\nu x^\nu = x$ and $\lim_\nu y^\nu = y$, we get $y \in f(x)$. In fact, for any z in X, we have

$$[z, x^\nu] \geq [y^\nu, x^\nu].$$

Hence by making $\nu \nearrow \infty$,

$$[z, x] \geq [y, x]$$

for any z in X, which shows that $\inf_{z \in X}[z, x]=[y, x]$ and so by the definition of $f(x)$, we obtain $y \in f(x)$.

Thus we may use Kakutani's fixed point theorem for our X and f. So there exists a point x in X such that $x \in f(x)$, or equivalently

$$[z, x] \geq [x, x] \qquad \text{for any } z \text{ in } X.$$

Now let

$$t=(t_1, \dots, t_n)=(x_1/[x, x], \dots, x_n/[x, x]).$$

Then

$$[z, t] \geq 1 \qquad \text{for any } z \text{ in } X$$

and

$$[x, t]=1.$$

Let z^k be the point in X whose k-th coordinate is 1 $(k=1, 2, \dots, n)$. Then we get, for all $k=1, 2, \dots, n$,

$$\sum_{i=1}^n a_{ki} t_i = [z^k, t] \geq 1.$$

These relations with

$$\sum_{k=1}^n [z^k, t] x_k = [x, t] = 1$$

give

$$\sum_{i=1}^n a_{ji} t_i = 1 \qquad \text{for all } j \text{ with } t_j \neq 0.$$

If this is not the case, there exists a j with $t_j \neq 0$ such that $[z^j, t]=\sum_{i=1}^n a_{ji} t_i > 1$. Then $x_j=[x, x] t_j \neq 0$ and

$$1=[x,\ t]=\sum_{k=1,\,k\neq j}^{n}[z^k,\ t]x_k+[z^j,\ t]x_j$$
$$>\sum_{k=1,\,k\neq j}^{n}x_k+x_j=\sum_{k=1}^{n}x_k=1,$$

which is clearly a contradiction. Q.E.D.

4. The case where $G(x, y)$ is finitely continuous. Although we cannot always choose a countable dense subset of K in general, we can always construct a sort of countable dense subset of K relative to the kernel $G(x, y)$. For the aim, let

$$d(y,\ y')=\sup_{z\in K}|G(y,\ z)-G(y',\ z)|.$$

This $d(y, y')$ defines a pseudo-metric on K. Concerning this, we have

LEMMA 2. *For any x in K and for any positive number ε, there exists a neighborhood U of x in K such that $d(x, x')<\varepsilon$ for any x' in U.*

Proof. By the finite continuity of $G(x, y)$ on K, for each point z in K, there exist open neighborhoods U_z of x and V_z of z in K such that

$$|G(x,\ z)-G(x',\ z')|<\varepsilon/2$$

for any (x', z') in $U_z\times V_z$. Since K is compact, $(V_z)_{z\in K}$ contains a finite subcovering $(V_{z_i})_{i=1}^{n}$ of K. Let $U=\bigcap_{i=1}^{n}U_{z_i}$. Suppose that x' and z are arbitrary points in U and K respectively. Then there exists a neighborhood V_{z_i} such that z belongs to V_{z_i}. Since (x', z) belongs to $U_{z_i}\times V_{z_i}$,

$$|G(x,\ z)-G(x',\ z)|\leq|G(x,\ z)-G(x,\ z_i)|+|G(x,\ z_i)-G(x',\ z)|<\varepsilon.$$

Thus $d(x, x')<\varepsilon$ for any x' in U. Q.E.D.

LEMMA 3. *There exist a sequence $(x_n)_{n=1}^{\infty}$ of points in K and a strictly increasing sequence $(\nu(m))_{m=1}^{\infty}$ of positive integers and a family $(U_m(x_k)\,;\ 1\leq k\leq\nu(m))$ of open sets in K such that x_k belongs to $U_m(x_k)$ and $\bigcup_{k=1}^{\nu(m)}U_m(x_k)=K$ and $U_m(x_k)\supset U_{m+1}(x_k)$ $(k=1, 2, \ldots, \nu(m))$ and*

$$d(x,\ x_k)<1/2m\qquad \textit{for any } x \textit{ in } U_m(x_k)\ \ (1\leq k\leq\nu(m)).$$

Proof. We construct such a system by induction on m. Firstly, for each point z in K, Lemma 2 assures the existence of an open neighborhood $V_1(z)$ of z in K such that $d(x, z)<1/2$ for each point x in $V_1(z)$. By the compactness of K, we can choose a finite subcovering $(V_1(z_i))_{i=1}^{n_1}$ of K of the covering $(V_1(z))_{z\in K}$. We set $\nu(1)=n_1$ and $x_k=z_k$ and $U_1(x_k)=V_1(z_k)$ $(k=1, 2, \ldots, \nu(1))$.

Assume that the system is constructed for $m=1, 2, \ldots, s-1$. By Lemma 1, for each point z in K, there exists an open neighborhood $V_s(z)$ of z in K such that $d(x, z)<1/2s$ for any x in $V_s(z)$. By the compactness of K, we can choose a finite subcovering $(V_s(z_i))_{i=1}^{n_s}$ of K of the covering $(V_s(z))_{z\in K}$. We set $\nu(s)=\nu(s-1)+n_s$ and $x_{\nu(s-1)+k}=z_k$ $(k=1, 2, \ldots, n_s)$ and we also put $U_s(x_k)=V_s(x_k)\cap U_{s-1}(x_k)$ $(k=1, 2, \ldots, \nu(s-1))$ and $V_s(x_k)$ $(k=\nu(s-1)+1, \nu(s-1)+2 \ldots, \nu(s-1)+n_s)$. This completes the induction. Q.E.D.

Let $M(K)$ be the totality of all positive regular Borel measures on K. The *vague topology* in $M(K)$ is the topology defined by the total family of semi-norms $\mu\to|\int f(x)\,d\mu(x)|$, where f runs over all elements in the Banach space of all finitely continuous functions on K. A subset F of $M(K)$ is said to be bounded if $\sup(\mu(K);\ \mu\in F)<\infty$. Combining Tychonoff-Kakutani theorem (see Loomis [6], Theorem 9 B, p. 22) with the representation theorem of Riesz-Markoff-Kakutani (see Halmos [2], Theorem D, p. 247), we see that

LEMMA 4. *The vague-closure of a bounded set in $M(K)$ is vaguely compact.*

Using these lemmas, we shall prove the theorem under the assumption that *the kernel $G(x, y)$ is finitely continuous on $K\times K$*. Let $K_m=(x_1, x_2, \ldots, x_{\nu(m)})$, where $(x_k)_{k=1}^{\infty}$ is as in Lemma 3. By Lemma 1, there exists a positive measure μ_m with support S_{μ_m} in K_m such that

$$\int G(x, y)\,d\mu_m(y)\geq 1$$

on K_m and

$$\int G(x, y)\,d\mu_m(y)\leq 1$$

on S_{μ_m}. Let $\alpha=\min_{K\times K}G(x, y)$. Then $\alpha>0$ and for a point x in S_{μ_m}, $\alpha\int d\mu_m(x)\leq G_{\mu_m}(x)\leq 1$. Hence $\mu_m(K)\leq 1/\alpha$ $(m=1, 2, \ldots)$. Let $F_n=(\mu_m;\ m\geq n)$ and $\overline{F}_n$ be the vague-closure of F_n in $M(K)$. Since $(F_n)_{n\geq 1}$ has the finite intersection property, by Lemma 4,

$$\cap_{n=1}^{\infty}\overline{F}_n\neq\emptyset.$$

Fix a μ in $\cap_{n=1}^{\infty}\overline{F}_n$. Let $\Lambda=(\lambda)$ be the totality of open neighborhoods λ of μ in $M(K)$ and $A=((\lambda, m);\ \lambda\in\Lambda,\ m=1, 2, \ldots)$. For each $a=(\lambda, m)$ and $a'=(\lambda', m')$ in A, we define $a\geq a'$ if $\lambda\subset\lambda'$ and $m\geq m'$. By this ordering, A becomes a directed set. For each $a=(\lambda, m)$ in A, we choose a measure μ_a such that $\mu_a\in\lambda\cap F_m$. If $\mu_a=\mu_n$, then we fix such a μ_n in $\lambda\cap F_m$ for μ_a and set $[a]=n$. Since $[a]\geq m$, we see that

$$\lim_a \mu_a = \mu \text{ (vaguely)} \quad \text{and} \quad \lim_a [a] = \infty.$$

Let x be an arbitrary point in K. Since $\bigcup_{k=1}^{\nu([a])} U_{[a]}(x_k) = K$, x belongs to a $U_{[a]}(x_k)$ $(1 \leq k \leq \nu([a]))$. Then by the definition of $U_{[a]}(x_k)$, $d(x, x_k) < 1/2[a]$ and so

$$G(x, y) + 1/2[a] \geq G(x_k, y)$$

for any y in K. Hence by using the fact that $x_k \in K_{[a]}$,

$$G_{\mu_a}(x) + \mu_a(K)/2[a] \geq G_{\mu_a}(x_k) \geq 1.$$

Hence

$$G_\mu(x) = \lim_a (G_{\mu_a}(x) + \mu_a(K)/2[a]) \geq 1.$$

Next, suppose that x be an arbitrary point in S_μ. For any fixed positive integer m, we can find a $U_m(x_k)$ such that $x \in U_m(x_k)$ $(1 \leq k \leq \nu(m))$. We assert that there exists an a_0 in A such that $[a_0] \geq m$ and

$$S_{\mu_a} \cap U_m(x_k) \neq \emptyset$$

for any a in A with $a \geq a_0$. In fact, let f be a finitely continuous function on K such that $f(x) = 1$ and $f = 0$ outside $U_m(x_k)$. Then

$$0 < \int f(y)\, d\mu(y) = \lim_a \int f(y)\, d\mu_a(y).$$

From this our assertion follows. Hence we can find a point x_a in $S_{\mu_a} \cap U_m(x_k)$. Then since $d(x, x_k) < 1/2m$ and $d(x_a, x_k) < 1/2m$,

$$G(x, y) \leq G(x_k, y) + 1/2m \leq (G(x_a, y) + 1/2m) + 1/2m$$

for any y in K. Hence $G_{\mu_a}(x) \leq G_{\mu_a}(x_a) + \mu_a(K)/m \leq 1 + 1/\alpha m$ and so

$$G_\mu(x) = \lim_a G_{\mu_a}(x) \leq 1 + 1/\alpha m.$$

Thus by making $m \nearrow \infty$, we get $G_\mu(x) \leq 1$. Q.E.D.

5. General case. We shall prove the theorem without any additional assumption. Let $\mathfrak{G}$ be the totality of finitely continuous functions $g(x, y)$ on $K \times K$ dominated by $G(x, y)$ and dominating the constant $\alpha = \min_{K \times K} G(x, y) > 0$. The set $\mathfrak{G}$ becomes a directed set with the usual function ordering and by considering $\mathfrak{G}$ as a directed sequence,

$$\lim_{\mathfrak{G} \ni g} g(x, y) = G(x, y)$$

on $K \times K$. By the preceding proof in §4, there exists a measure μ_g with support S_{μ_g} in K such that

$$g_{\mu_g}(x) \geq 1$$

on K and

$$g_{\mu_g}(x) \leq 1$$

on S_{μ_g}. Let x be in S_{μ_g}, then $\alpha\int d\mu_g(y) \leq G_{\mu_g}(x) \leq 1$ and so $\mu_g(K) < 1/\alpha$. Let $F_g = (\mu_{g'})_{\mathfrak{G} \ni g' \geq g}$ and $\overline{F}_g$ be the vague-closure of F_g in $M(K)$. Since $(F_g)_{g \in \mathfrak{G}}$ has the finite intersection property, by Lemma 4, there exists a measure μ in the set $\bigcap_{\mathfrak{G} \ni g} \overline{F}_g$. Let $\Lambda = (\lambda)$ be the totality of vague-neighborhoods λ of μ and $A = ((\lambda, g);\ \lambda \in \Lambda,\ g \in \mathfrak{G})$. For any $a = (\lambda, g)$ and $a' = (\lambda', g')$ in A, we define $a \geq a'$ if $\lambda \subset \lambda'$ and $g \geq g'$. By this ordering, A is a directed set. For each $a = (\lambda, g)$ in A, we choose a measure μ_a such that $\mu_a \in \lambda \cap F_g$ and if $\mu_a = \mu_{g'}$, then we fix such a $\mu_{g'}$ in $\lambda \cap F_g$ for μ_a and set $[a] = g'$. Since $g' \geq g$, it is clear that

$$\lim_a \mu_a = \mu \text{ (vaguely) and } \lim_a [a](x, y) = G(x, y) \text{ on } K \times K.$$

We first show that $G_\mu(x) \geq 1$ G-p.p.p. on K. Contrary to the assertion, assume that there exists a measure $\nu \neq 0$ with support S_ν in K such that $\int G_\nu(x)\, d\nu(x) = \int \check{G}_\nu(x)\, d\nu(x) < \infty$ and $G_\mu(x) < 1$ on S_ν. By Lusin's theorem (see Halmos [2], p.p. 242–243), since $\int G_\mu(x)\, d\nu(x) < \int d\nu(x) < \infty$, there exists a compact subset K_1 of S_ν with positive ν-measure such that $G_\mu(x)$ and $\check{G}_\nu(x)$ are finitely continuous on K_1. Hence $G_\mu(x) - 1$ is continuous on K_1 and so

$$p = \sup_{x \in K_1} (G\mu(x) - 1) < 0.$$

Let ν_1 be defined by $\nu_1(X) = \nu(X \cap K_1)$. Then ν_1 belongs to $M(K)$ and the lower semicontinuous function $\check{G}_{\nu_1}(x)$ is equal to the finitely upper semicontinuous function $\check{G}_\nu(x) - \check{G}_{\nu - \nu_1}(x)$ on S_{ν_1}. Hence $\check{G}_{\nu_1}(x)$ is finitely continuous on S_{ν_1}. So by the continuity principle assumed for the kernel $\check{G}(x, y)$, $\check{G}_{\nu_1}(x)$ is finitely continuous on Ω and so on K. Thus

$$\begin{aligned}\int G_\mu(x)\, d\nu_1(x) = \int \check{G}_{\nu_1}(x)\, d\mu(x) &= \lim_a \int \check{G}_{\nu_1}(x)\, d\mu_a(x)\\ &= \lim_a \int G_{\mu_a}(x)\, d\nu_1(x)\\ &\geq \limsup_a \int [a]_{\mu_a}(x)\, d\nu_1(x) \geq \int d\nu_1(x),\end{aligned}$$

or

$$\int (G_\mu(x)-1)\,d\nu_1(x)\geq 0.$$

On the other hand, $\int (G_\mu(x)-1)\,d\nu_1(x)\leq \int p\,d\nu_1(x)=p\nu_1(K)<0$, which is a contradiction.

Finally we show that $G_\mu(x)\leq 1$ on S_μ. For the aim, let x be an arbitrary point in S_μ and $T=(\tau)$ be the totality of neighborhoods τ of x. We set $B=((\tau, \lambda, g);\ \tau\in T, \lambda\in\Lambda, g\in\mathfrak{G})$ and for any $b=(\tau, \lambda, g)$ and $b'=(\tau', \lambda', g')$ in B, we define $b\geq b'$ if $\tau\subset\tau'$, $\lambda\subset\lambda'$ and $g\geq g'$. Then B is a directed set. As in § 4, we can find an a_0 in A such that

$$S_{\mu_a}\cap\tau\neq\emptyset \qquad \text{for any } a \text{ in } A \text{ with } a\geq a_0.$$

Hence for any $b=(\tau, a)=(\tau, \lambda, g)$ in B, we can find a point x_b in $S_{\mu_a}\cap\tau$ for some a' with $a'\geq a$. We set $[b]=a'$ and $\langle b\rangle=\tau$. Then we see that

$$\lim_b \mu_{[b]}=\mu \text{ (vaguely)} \quad \text{and} \quad \lim_b[[b]](x, y)=G(x, y) \text{ on } K\times K$$
$$\text{and} \quad \lim_b x_b=x.$$

Hence for any fixed g' in $\mathfrak{G}$, we get

$$1\geq \limsup_b [[b]]_{\mu_{[b]}}(x_b)\geq \limsup_b g'_{\mu_{[b]}}(x_b).$$

Let ε be an arbitrary positive number. Since $g'(x, y)$ is continuous on $K\times K$, we can find, for any y in K, neighborhoods τ_y in T and V_y of y such that $|g'(x', y')-g'(x, y)|<\varepsilon$ for any (x', y') in $\tau_y\times V_y$. Since K is compact, there exists a finite subcovering $(V_{y_i})_{i=1}^s$ of K of $(V_y)_{y\in K}$. Let $\tau_0=\bigcap_{i=1}^s\tau_{y_i}$, which is in T. Then for any (x', y) in $\tau_0\times K$, $|g'(x', y)-g'(x, y)|<\varepsilon$. Clearly there exists a b_0 in B such that for any $b\geq b_0$, $\langle b\rangle\subset\tau_0$ or $x_b\in\tau_0$. Hence for any $b\geq b_0$,

$$|g'_{\mu_b}(x_b)-g'_\mu(x)|\leq|g'_{\mu_b}(x_b)-g'_{\mu_b}(x)|+|g'_{\mu_b}(x)-g'_\mu(x)|$$
$$\leq\varepsilon\mu_b(K)+|g'_{\mu_b}(x)-g'_\mu(x)|.$$

Thus we have $\limsup_b g'_{\mu_b}(x_b)\geq g'_\mu(x)-\varepsilon\mu(K)$ and as ε is arbitrary,

$$\limsup_b g'_{\mu_b}(x_b)\geq g'_\mu(x).$$

Thus we obtain $1\geq g'_\mu(x)$. By a Fatou type theorem (see Brelot [1], p. 7);

$$1\geq \sup_{g'\in\mathfrak{G}} g'_\mu(x)=\sup_{g'\in\mathfrak{G}}\int g'(x, y)\,d\mu(y)=\int \sup_{g'\in\mathfrak{G}} g'(x, y)\,d\mu(y)$$
$$=\int G(x, y)\,d\mu(y)=G_\mu(x). \qquad \text{Q.E.D.}$$

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