ON THE EXPONENTIAL MAPS AND THE TRIANGULAR 2-CHOMOLOGY OF GRADED LIE RINGS OF LENGTH THREE

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1. Let H be a group. We mean by an N-series in H a decreasing series of subgroups $H_1 = H, H_2, \dots, H_{n+1} = \{e\}$ such that the commutator $xyx^{-1}y^{-1}$ of two elements x and y respectively in H_i and H_j belongs to H_{i+j}, where $H_s = \{e\}$ for $s \geq n+1$. We call n the length of the N-series (H_i). We mean by a graded Lie ring of length n a Lie ring $\mathfrak{L}$ which is a direct sum $A_1 + \cdots + A_n$ of additive subgroups $A_1, \dots, A_n$ such that $[A_i, A_j] \subset A_{i+j}$, where $A_s = \{0\}$ for $s \geq n+1$. For each N-series (H_i) of length n the graded Lie ring $\mathfrak{L}[(H_i)]$ is associated with (H_i) as follows[1]:

1) $\mathfrak{L}[(H_i)]$ is the direct sum of the additively written factor groups $A_i = H_i/H_{i+1}$ $(i = 1, 2, \dots, n)$, and this direct sum gives the addition in $\mathfrak{L}[(H_i)]$.

2) The Lie product $[a, b]$ of $a \in A_i$ and $b \in A_j$ is the group commutator $xyx^{-1}y^{-1}$ modulo H_{i+j+1} of the representatives x and y respectively of a and b in H_1. We shall call $\mathfrak{L}[(H_i)]$ the graded Lie ring associated with the N-series (H_i).

In the present note we shall introduce triangular 2-cocycles of a graded Lie ring $\mathfrak{L}$ of length three and shall show that for each triangular 2-cocycle γ of $\mathfrak{L}$ we can define the Exponential Map Exp_γ of $\mathfrak{L}$ (onto a group) that is a bijective map of $\mathfrak{L}$ such that 1) $H_1 = \mathrm{Exp}_\gamma(A_1 + A_2 + A_3)$, $H_2 = \mathrm{Exp}_\gamma(A_2 + A_3)$, $H_3 = \mathrm{Exp}_\gamma(A_3)$, $H_4 = \{e\}$ form an N-series and 2) $\mathfrak{L}$ is regarded as the graded Lie ring $\mathfrak{L}[(H_i)]$ associated with (H_i). Two triangular 2-cocycles γ and γ' are called to be equivalent if the corresponding N-series (H_i) and (H_i') are isomorphic. We shall call the equivalent classes of triangular 2-cocycles the triangular cohomology classes of $\mathfrak{L}$. We shall also show that for any N-series (H_i) of length three there exists a triangular 2-cocycle β of $\mathfrak{L}[(H_i)]$ such that the N-series corresponding to the pair $(\mathfrak{L}[H_i)], \beta)$ is isomorphic to (H_i). So

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1) See [1] 18. 4 p. 329.

we can conclude that the set of N-series of length three corresponds bijectively to the set of pairs consisting of a graded Lie ring $\mathfrak{L}$ of length three and a triangular 2-cohomology class of $\mathfrak{L}$. This is a generalization of theory of central extensions of abelian groups by abelian groups.

2. Triangular 2-cocycles. Let $\mathfrak{L} = A_1 + A_2 + A_3$ be a graded Lie ring of length three. We regard A_j as an A_i-module on which A_i operates simply, and denote by $C^2(A_i, A_j)$ the additive group of 2-cochains of A_i with coefficients in A_j. We mean by a triangular 2-cochain a triangular matrix

$$\gamma = \begin{pmatrix} \gamma_{21} & 0 \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$

with components γ_{ji} in $C^2(A_i, A_j)$, $(i<j)$. We denote by $C^2(L)$ the set of triangular 2-cochains of L. For $a = a_1 + a_2 + a_3$, $b = b_1 + b_2 + b_3$ $(a_i, b_i \in A_i;\ i = 1, 2, 3)$ and $\gamma \in C^2(\mathfrak{L})$ we mean by $\gamma(a, b)$ the triangular matrix

$$\begin{pmatrix} \gamma_{21}(a_1, b_1), & 0 \\ \gamma_{31}(a_1, b_1), & \gamma_{32}(a_2, b_2) \end{pmatrix}$$

We shall now define triangular 2-cocycles of $\mathfrak{L}$:

Definition. A triangular 2-cocycle of $\mathfrak{L}$ is a triangular 2-cochain γ of $\mathfrak{L}$ satisfying

(1) $$\partial\gamma_{21} = 0,\ \partial\gamma_{32} = 0^{2)},$$

(2) $$\partial\gamma_{31}(a_1. b_1, c_1) + [a_1, \gamma_{21}(b_1, c_1)] + \gamma_{32}(\gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1 + c_1)) - \gamma_{32}(\gamma_{21}(a_1, b_1), \gamma_{21}(a_1 + b_1, c_1)) = 0, \quad (a_1, b_1, c_1 \in A_1),$$

(3) $$\gamma(0, a) = \gamma(a, 0) = 0,\ (a \in \mathfrak{L}),$$

(4) $$\gamma_{21}(a_1, b_1) - \gamma_{21}(b_1, a_1) = [a_1, b_1], \quad (a_1, b_1 \in A_1),$$

(5) $$\gamma_{32}(a_2, b_2) = \gamma_{32}(b_2, a_2),\ (a_2, b_2 \in A_2).$$

LEMMA. *Let γ be a triangular 2-cocycle of $\mathfrak{L}$. Then*

(6) $$\gamma_{31}(a_1, -a_1) - \gamma_{31}(-a_1, a_1) = [a_1, \gamma_{21}(-a_1, a_1)], \quad (a_1 \in A_1),$$

Proof. From (1), (2), (3) it follows

[2] We mean by ∂ the ordinary coboundary operator.

$$\begin{aligned}0 &= \partial\gamma_{31}(a_1, -a_1, a_1) + [a_1, \gamma_{21}(-a_1, a_1)] \\ &\quad + \gamma_{32}(\gamma_{21}(-a_1, a_1), \gamma_{21}(a_1, 0)) - \gamma_{32}(\gamma_{21}(a_1, -a_1), \gamma_{21}(0, a_1)) \\ &= \gamma_{31}(-a_1, a_1) - \gamma_{31}(a_1, -a_1) \\ &\quad + [a_1, \gamma_{21}(-a_1, a_1)], \ (a_1 \in A_1).\end{aligned}$$

This proves (6).

3. *The Exponential Map associated to a triangular 2-cocycle.* Let γ be a triangular 2-cocycle of $\mathfrak{L}$. For $a_i, b_i \in A_i$ $(i = 1, 2, 3)$, put

$$\begin{aligned}(7) \qquad \eta_\gamma(a_1 + a_2 + a_3, b_1 + b_2 + b_3) &= \gamma_{21}(a_1, b_1) + \gamma_{31}(a_1, b_1) + \gamma_{32}(a_2, b_2) \\ &\quad + [a_1, b_2] + \gamma_{32}(a_2 + b_2, \gamma_{21}(a_1, b_1))\end{aligned}$$

and

$$\begin{aligned}(8) \qquad \rho_\gamma(a_1 + a_2 + a_3) &= c_1 + c_2 + c_3, \\ c_1 &= -a_1, \ c_2 = -a_2 - \gamma_{21}(a_1, c_1), \\ c_3 &= -a_3 - \gamma_{31}(a_1, c_1) - \gamma_{32}(a_2, c_2) - [a_1, c_2] - \gamma_{32}(a_2 + c_2, \gamma_{21}(a_1, c_1)).\end{aligned}$$

We shall first show the following properties of η_γ and ρ_γ.

PROPOSITION 1. *Let γ be a triangular 2-cocycle of $\mathfrak{L}$. Then*

$$(9) \qquad \eta_\gamma(a, 0) = \eta_\gamma(0, a) = 0, \ (a \in \mathfrak{L}),$$

$$(10) \qquad \eta_\gamma(b, c) - \eta_\gamma(a + b + \eta_\gamma(a, b), c) + \eta_\gamma(a, b + c + \eta_\gamma(b, c)) - \eta_\gamma(a, b) = 0, \quad (a, b, c \in \mathfrak{L}),$$

$$(11) \qquad a + \rho_\gamma(a) + \eta_\gamma(\rho_\gamma(a), a) = a + \rho_\gamma(a) + \eta_\gamma(a, \rho_\gamma(a)) = 0, \ (a \in \mathfrak{L}).$$

Proof. By virtue of (3), (7) it follows $\eta_\gamma(0, a) = \eta_\gamma(a, 0) = 0$, $(a \in \mathfrak{L})$. By virtue of (1), (2), (3), (6), (7) for $a = a_1 + a_2 + a_3$, $b = b_1 + b_2 + b_3$, $c = c_1 + c_2 + c_3$, $(a_i, b_i, c_i \in A_i;\ i = 1, 2, 3)$, we have

$$\begin{aligned}&\eta_\gamma(b, c) - \eta_\gamma(a + b + \eta_\gamma(a, b), c) + \eta_\gamma(a, b + c + \eta_\gamma(b, c)) - \eta_\gamma(a, b) \\ &= \partial\gamma_{21}(a_1, b_1, c_1) + \partial\gamma_{31}(a_1, b_1, c_1) + \gamma_{32}(b_2, c_2) - \gamma_{32}(a_2 + b_2 + \gamma_{21}(a_1, b_1), c_2) \\ &\quad + \gamma_{32}(a_2, b_2 + c_2 + \gamma_{21}(b_1, c_1)) - \gamma_{32}(a_2, b_2) + [b_1, c_2] - [a_1 + b_1, c_2] \\ &\qquad + [a_1, b_2 + c_2 + \gamma_{21}(b_1, c_1)] - [a_1, b_2] \\ &\quad + \gamma_{32}((b_2 + c_2, \gamma_{21}(b_1, c_1)) - \gamma_{32}(a_2 + b_2 + c_2 + \gamma_{21}(a_1, b_1), \gamma_{21}(a_1 + b_1, c_1)) \\ &\quad + \gamma_{32}(a_2 + b_2 + c_2 + \gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1 + c_1)) - \gamma_{32}(a_2 + b_2, \gamma_{21}(a_1, b_1)) \\ &= \partial\gamma_{31}(a_1, b_1, c_1) + [a_1, \gamma_{21}(b_1, c_1)] \\ &\quad + \gamma_{32}(a_2 + b_2 + c_2, \gamma_{21}(b_1, c_1)) - \gamma_{32}(a_2 + b_2 + c_2, \gamma_{21}(a_1, b_1)) \\ &\quad + \gamma_{32}(a_2 + b_2 + c_2 + \gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1 + c_1)) - \gamma_{32}(a_2 + b_2 + c_2 + \gamma_{21}(a_1, b_1), \\ &\qquad \gamma_{21}(a_1 + b_1, c_1))\end{aligned}$$

$$
\begin{aligned}
&= \partial\gamma_{31}(a_1, b_1, c_1) + [a_1, \gamma_{21}(b_1, c_1)] \\
&\quad + \gamma_{32}(a_2+b_2+c_2, \gamma_{21}(a_1, b_1+c_1) + \gamma_{21}(b_1+c_1)) + \gamma_{32}(\gamma_{21}(a_1, b_1+c_1), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma_{21}(b_1, c_1)) \\
&\quad - \gamma_{32}(a_2+b_2+a_2, \gamma_{21}(a_1+b_1, c_1) + \gamma_{21}(a_1, b_1)) - \gamma_{32}(\gamma_{21}(a_1+b_1, c_1), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \gamma_{21}(a_1, b_1)) \\
&= \partial\gamma_{31}(a_1, b_1, c_1) + [a_1, \gamma_{21}(b_1, c_1)] \\
&\quad + \gamma_{32}(\gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1+c_1)) - \gamma_{32}(\gamma_{21}(a_1, b_1), \gamma_{21}(a_1+b_1, c_1)) = 0.
\end{aligned}
$$

This proves (10). From the definition (8) of ρ_γ, putting $\rho_\gamma(a) = c_1+c_2+c_3$ ($c_i \in A_i$: $i=1, 2, 3$), we have

$$
\begin{aligned}
& a + \rho_\gamma(a) + \eta_\gamma(a, \rho_\gamma(a)) = (a_1+a_2+a_3) + (c_1+c_2+c_3) + \gamma_{21}(a_1, c_1) \\
&\quad + \gamma_{31}(a_1, c_1) + \gamma_{32}(a_2, c_2) + [a_1, c_2] + \gamma_{32}(a_2+c_2, \gamma_{21}(a_1, c_1)) = 0, \\
& a + \rho_\gamma(a) + \eta_\gamma(\rho_\gamma(a), a) = a + \rho_\gamma(a) + \eta_\gamma(a, \rho_\gamma(a)) + \eta_\gamma(\rho_\gamma(a), a) - \eta_\gamma(a, \rho_\gamma(a)) \\
&\quad = \eta_\gamma(\rho_\gamma(a), a) - \eta_\gamma(a, \rho_\gamma(a)) = \gamma_{21}(c_1, a_1) - \gamma_{21}(a_1, c_1) + \gamma_{32}(c_2, a_2) - \gamma_{32}(a_2, c_2) \\
&\quad + \gamma_{31}(c_1, a_1) - \gamma_{31}(a_1, c_1) + [c_1, a_1] - [a_1, c_2] + \gamma_{32}(a_2+c_2, \gamma_{21}(c_1, a_1)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - \gamma_{32}(a_2+c_2, \gamma_{21}(a_1, c_1)).
\end{aligned}
$$

Since $c_1 = -a_1$ and $c_2 = -a_2 - \gamma_{21}(a_1, -a_1)$, by virtue of (5), (6) we have

$$\gamma_{21}(a_1, c_1) = \gamma_{21}(c_1, a_1),\ \gamma_{32}(c_2, a_2) = \gamma_{32}(a_2, c_2),$$

and

$$
\begin{aligned}
& \gamma_{31}(c_1, a_1) - \gamma_{31}(a_1, c_1) + [c_1, a_2] - [a_1, c_2] \\
&\quad = \gamma_{31}(-a_1, a_1) - \gamma_{31}(a_1, -a_1) + [-a_1, a_2] - [a_1, -a_2 - \gamma_{21}(a_1, -a_1)] \\
&\quad = \gamma_{31}(-a_1, a_1) - \gamma_{31}(a_1, -a_1) + [a_1, \gamma_{21}(a_1, -a_1)] = 0.
\end{aligned}
$$

Hence $a + \rho_\gamma(a) + \eta_\gamma(\rho_\gamma(a), a) = 0$. This completes the proof of Proposition 1.

For each triangular 2-cocycle γ of $\mathfrak{L}$ we shall define the set of symbols $\{\mathrm{Exp}_\gamma(a) \mid a \in \mathfrak{L}\}$ with the following law of composition:

$$\mathrm{Exp}_\gamma(a)\,\mathrm{Exp}_\gamma(b) = \mathrm{Exp}_\gamma(a+b+\eta_\gamma(a, b)), \qquad (a, b \in \mathfrak{L}). \tag{12}$$

We call the map Exp_γ the Exponential Map of $\mathfrak{L}$ with a base γ, and call γ the basic triangular 2-cocycle of the Exponential Map Exp_γ.

PROPOSITION 2. $\mathrm{Exp}_\gamma(\mathfrak{L}) = \{\mathrm{Exp}_\gamma(a) \mid a \in \mathfrak{L}\}$ *is a group and* Exp_γ *is a bijective map of* $\mathfrak{L}$ *onto* $\mathrm{Exp}_\gamma(\mathfrak{L})$.

Proof. By virtue of (9) we have $\mathrm{Exp}_\gamma(a)\mathrm{Exp}_\gamma(0) = \mathrm{Exp}_\gamma(0)\mathrm{Exp}_\gamma(a) = \mathrm{Exp}_\gamma(a)$, and thus $\mathrm{Exp}_\gamma(0)$ is the unit element. From (11) it tollows $\mathrm{Exp}_\gamma(a)\cdot$

$\mathrm{Exp}_\gamma(\rho_\gamma(a)) = \mathrm{Exp}_\gamma(\rho_\gamma(a))\,\mathrm{Exp}_\gamma(a) = \mathrm{Exp}_\gamma(0)$. This shows that $\mathrm{Exp}_\gamma(\rho_\gamma(a))$ is the inverse element of $\mathrm{Exp}_\gamma(a)$. From (10) it follows

$$\begin{aligned}(\mathrm{Exp}_\gamma(a)\,\mathrm{Exp}_\gamma(b))\,\mathrm{Exp}_\gamma(c) &= \mathrm{Exp}_\gamma(a+b+\eta_\gamma(a, b))\,\mathrm{Exp}_\gamma(c)\\ &= \mathrm{Exp}_\gamma(a+b+c+\eta_\gamma(a, b)+\eta_\gamma(a+b+\eta_\gamma(a, b), c)\\ &= \mathrm{Exp}_\gamma(a+b+c+\eta_\gamma(b, c)+\eta_\gamma(a, b+c+\eta_\gamma(b, c))\\ &= \mathrm{Exp}_\gamma(a)\,(\mathrm{Exp}_\gamma(b+c+\eta_\gamma(b, c)) = \mathrm{Exp}_\gamma(a)\,(\mathrm{Exp}_\gamma(b)\,\mathrm{Exp}_\gamma(c)).\end{aligned}$$

This shows that $\mathrm{Exp}_\Gamma(\mathfrak{L})$ satisfies the associative law. Therefore $\mathrm{Exp}_\gamma(\mathfrak{L})$ is a group. Since $\mathrm{Exp}_\gamma(a)$ are symbols, we have $\mathrm{Exp}_\gamma(a) = \mathrm{Exp}_\gamma(b)$ if and only if $a = b$. This completes the proof of Proposition 2.

We mean by the Logarithmic Map Log_γ the inverse map of the Exponential Map Exp_γ. Namely Log_γ is a bijective map of $\mathrm{Exp}_\gamma(\mathfrak{L})$ onto $\mathfrak{L}$ such that

$$\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a)) = a, \qquad (a\in\mathfrak{L}). \tag{13}$$

PROPOSITION 3. *For any a_i, $b_i\in A_i$ $(i=1, 2, 3)$ we have*

$$\begin{aligned}\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a_1+a_2+a_3)\,\mathrm{Exp}_\gamma(b_1+b_2+b_3)) &= (a_1+b_1)+(a_2+b_2+\gamma_{21}(a_1, b_1))\\ &\quad +(a_3+b_3+\gamma_{31}(a_1, b_1)+\gamma_{32}(a_2, b_2)+[a_1, b_2]+\gamma_{32}(a_2+b_2, \gamma_{21}(a_1, b_1))),\end{aligned} \tag{14}$$

$$\begin{aligned}\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a_1+a_2+a_3)^{-1}) &= \mathrm{Log}_\gamma(\rho_\gamma(a_1+a_2+a_3))\\ &= -a_1-(a_2+\gamma_{21}(a_1, -a_1)-(a_3+\gamma_{31}(a_1, -a_1)\\ &\qquad +\gamma_{32}(a_2, -a_2-\gamma_{21}(a_1, -a_1))\\ &\quad +[a_1, -a_2-\gamma_{21}(a_1, a_1)]+\gamma_{32}(-\gamma_{21}(a_1, a_1), \gamma_{21}(a_1, -a_1))).\end{aligned} \tag{15}$$

Proof. Since $\mathrm{Exp}_\gamma(a_1+a_2+a_3)\,\mathrm{Exp}_\gamma(b_1+b_2+b_3) = \mathrm{Exp}_\gamma(a_1+b_1+a_2+b_2+a_3+b_3+\eta_\gamma(a_1+a_2+a_3, b_1+b_2+b_3)$, (14) follows from (7). (15) is an immediate consequence from (8).

For the sake of simplicity we denote by B_1, B_2, B_3, B_3, B_4 the ideals of $\mathfrak{L} = A_1+A_2+A_3$, A_2+A_3, A_3, $\{0\}$, respectively.

PROPOSITION 4. *If $a\in B_i$ and $b\in B_j$, we have*

$$\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a)\,\mathrm{Exp}_\gamma(b))\in B_{\min(i,j)}, \tag{16}$$

$$\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a)^{-1})\in B_i, \tag{17}$$

$$\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a)\,\mathrm{Exp}_\gamma(b)\,\mathrm{Exp}_\gamma(a)^{-1}\mathrm{Exp}_\gamma(b)^{-1}) \equiv [a, b] \bmod B_{i+j+1}, \tag{18}$$

where $B_s = \{0\}$ for $s\geq 4$.

Proof. (16) and (17) are immediate consequences of (14) and (15), respectively. By virtue of (14), if $a\in\mathfrak{L}$ and $b_3\in A_3$, we have $\mathrm{Exp}_\gamma(a)\,\mathrm{Exp}_\gamma(b_3)$

$=\mathrm{Exp}_\gamma(a+b_3)=\mathrm{Exp}_\gamma(b_3)\,\mathrm{Exp}_\gamma(a)$. This shows that $\mathrm{Exp}_\gamma(A_3)$ is a subgroup contained in the center of $\mathrm{Exp}_\gamma(\mathfrak{L})$. Using this fact and (1), (3), (4), (14), (15) we have for $a=a_1+a_2+a_3$, $b=b_1+b_2+b_3$ $(a_i,\ b_i\in A_i)$

$$\begin{aligned}&\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a)\,\mathrm{Exp}_\gamma(b)\,\mathrm{Exp}_\gamma(a)^{-1}\mathrm{Exp}_\gamma(b)^{-1})\\ &\equiv \gamma_{21}(a_1,\ b_1)-\gamma_{21}(a_1,\ -a_1)\gamma_{21}(b_1,\ -b_1)+\gamma_{21}(-a_1,\ -b_1)+\gamma_{21}(a_1+b_1,\ -a_1-b_1)\\ &\equiv \gamma_{21}(a_1,\ b_1)-\gamma_{21}(b_1,\ a_1)\equiv[a_1,\ b_1] \bmod A_3.\end{aligned}$$

This proves (16) for $i=j=1$. Since $\mathrm{Exp}_\gamma(A_3)$ is contained in the center, (16) is also true for $i=3$ or $j=3$. So it is sufficient to prove (16) for $a=a_2$ and $b=b_1+b_2$ $(a_2,\ b_2\in A_2,\ b_1\in A_1)$. Using (14), (15), (6) we have

$$\begin{aligned}&\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a_2)\,\mathrm{Exp}_\gamma(b_1+b_2)\,\mathrm{Exp}_\gamma(a_2)^{-1}\mathrm{Exp}_\gamma(b_1+b_2)^{-1})=\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(a_2)\,\mathrm{Exp}_\gamma\\ &\qquad\qquad (b_1+b_2)(\mathrm{Exp}_\gamma(b_1+b_2)\,\mathrm{Exp}_\gamma(a_2))^{-1})\\ &=\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(b_1+a_2+b_2+\gamma_{32}(a_2,\ b_2))\mathrm{Exp}_\gamma(b_1+a_2+b_2+\gamma_{32}(b_2,\ a_2)+[b_1,\ a_2])^{-1})\\ &=\mathrm{Log}_\gamma(\mathrm{Exp}_\gamma(b_1+a_2+b_2+\gamma_{32}(a_2,\ b_2))\,\mathrm{Exp}_\gamma(b_1+a_2+b_2+\gamma_{32}(a_2,\ b_2))^{-1}\\ &\qquad\qquad \mathrm{Exp}_\gamma([b,\ a])^{-1})\\ &=\ -[b_1,\ a_2]=[a_2,\ b_1].\end{aligned}$$

This completes the prove of (16).

We shall now sum up the results in this paragraph in the following theorem.

THEOREM 1. *Let γ be a triangular 2-cocycle of a graded Lie ring of length three $\mathfrak{L}=A_1+A_2+A_3$. Then $H_1=\mathrm{Exp}_\gamma(A_1+A_2+A_3)$, $H_2=\mathrm{Exp}_\gamma(A_2+A_3)$, $H_3=\mathrm{Exp}_\gamma(A_3)$, $H_4=\{e\}$ form an N-series such that $\mathfrak{L}$ is regarded as the graded Lie ring associated with (H_i).*

Proof. (16), (17) and (18) in Proposition 4 show that H_1, H_2, H_3, H_4 form an N-series. Identifying $\mathrm{Exp}_\gamma(a_i)$ mod H_{i+1} with a_i mod B_{i+1}, we get the identification of $\mathfrak{L}[(H_i)]$ with $\mathfrak{L}$.

4. In this paragraph we shall prove the following theorem:

THEOREM 2. *Let $H_1=H$, H_2, H_3, $H_4=\{e\}$ be an N-series. Then there exists a triangular 2-cocycle γ of the graded Lie ring $\mathfrak{L}[(H_i)]$ associated with (H_i) such that the N-series associated with the pair $(\mathfrak{L}[(H_i)],\ \gamma)$ (in the sense of Theorem 1) is isomorphic to the N-series (H_i).*

In the proof of Theorem 2, we shall also show the structural meaning of the triangular 2-cocycle γ.

The proof of Theorem 2: We denote by $\mathfrak{L}[(H_i)] = A_1 + A_2 + A_3$ the graded Lie ring associated with (H_i). We shall identify H_i/H_{i+1} with A_i and shall use the both notations (the additive one and the multiplicative one) freely for the sake of simplicity. Let us choose a family of representatives $\{v_{ji}(\xi) \in H_1/H_{j+1} | \xi \in H_1/H_{i+1};\ 1 \leq i \leq j \leq 3\}$ such that

$$(19)\qquad v_{i,i}(\xi) = \xi,$$

$$(20)\qquad v_{k,j}(v_{j,i}(\xi)) = v_{k,i}(\xi), \qquad (\xi \in H_1/H_{i+1};\ 1 \leq i \leq j \leq k \leq 3),$$

$$(21)\qquad v_{k,j}(\eta v_{ji}(\xi)) = v_{k,j}(\eta) v_{k,i}(\xi), \ (\xi \in H_1/H_{i+1};\ \eta \in H_l/H_{i+1};\ 1 \leq i < l \leq j \leq k \leq 3),$$

$$(22)\qquad v_{j,i}(e) = \text{the unit element } e \text{ of } H_1/H_{j+1}.$$

Put

$$(23)\qquad c_{i-1}(\xi, \xi') = v_{i,i-1}(\xi) v_{i,i-1}(\xi') v_{i,i-1}(\xi\xi')^{-1}, \qquad (\xi, \xi' \in H_1/H_i;\ i = 2, 3),$$

$$(24)\qquad \gamma_{21}(a_1, b_1) = c_1(a_1, b_1),\ \gamma_{32}(a_2, b_2) = c_2(a_2, b_2), \qquad (a_i, b_i \in A_i;\ i = 1, 2)$$

and

$$(25)\qquad \gamma_{31}(a_1, b_1) = c_2(v_{2,1}(a_1), v_{2,1}(b_1)), \qquad (a_1, b_1 \in A_1).$$

We shall prove that the triangular 2-cochain

$$\gamma = \begin{pmatrix} \gamma_{21} & 0 \\ \gamma_{31} & \gamma_{32} \end{pmatrix}$$

is a triangular 2-cocycle of $\mathfrak{L}[(H_i)]$. Since γ_{21} (resp. γ_{32}) are the 2-cocycles associated with the extension H_1/H_3 (resp. H_2/H_4) of A_2 (resp. A_3) by A_1 (resp. A_2), we have $\partial\gamma_{21} = 0$ (resp. $\partial\gamma_{32} = 0$). Since $v_{j,i}(e) =$ the unit element e of H_1/H_{j+1}, we have $\gamma_{ji}(0, a_j) = \gamma_{ji}(a_i, 0) = 0$ for $a_i \in A_i$. Namely $\gamma(0, a) = \gamma(a, 0) = 0$ for $a \in \mathfrak{L}$. Since H_3 is contained in the center of H, from the definition of γ_{31} we have for $a_1, b_1, c_1 \in A_1$

$$\begin{aligned}
(v_{31}(a_1) v_{31}(b_1)) v_{31}(c_1) &= v_{32}(v_{21}(a_1)) v_{32}(v_{21}(b_1)) v_{31}(c_1) \\
&= c_3(v_{21}(a_1), v_{21}(b_1)) v_{32}(v_{21}(a_1) v_{21}(b_1)) v_{31}(c_1) \\
&= \gamma_{31}(a_1, b_1) v_{32}(c_{21}(a_1, b_1) v_{31}(a+b) v_{31}(c_1)) \\
&= \gamma_{31}(a_1, b_1) v_{32}(c_{21}(a_1, b_1)) \gamma_{31}(a_1 + b_1, c_1) v_{32}(c_{21}(a_1 + b_1, c_1)) \\
&\quad\ v_{31}(a_1 + b_1 + c_1) \\
&= \gamma_{31}(a_1, b_1) \gamma_{31}(a_1 + b_1, c_1) \gamma_{32}(\gamma_{21}(a_1, b_1), \gamma_{21}(a_1 + b_1, c_1)) \\
&\quad\ v_{32}(\gamma_{21}(a_1, b_1) + \gamma_{21}(a_1 + b_1, c_1)) v_{31}(a_1 + b_1 + c_1)
\end{aligned}$$

and

$$v_{31}(a_1)(v_{31}(b_1)v_{31}(c_1)) = v_{31}(a_1)\gamma_{31}(b_1, c_1)v_{32}(c_2(b_1, c_1))v_{31}(b_1+c_1)$$
$$= \gamma_{31}(b_1, c_1)v_{31}(a_1)v_{32}(c_2(b_1, c_1))v_{31}(a_1)^{-1}v_{31}(c_2(b_1, c_1))^{-1}$$
$$v_{32}(c_2(b_1, c_1))v_{31}(a_1)v_{31}(b_1+c_1)$$
$$= \gamma_{31}(b_1, c_1)[a_1, \gamma_{21}(b_1, c_1)]v_{32}(c_2(b_1, c_1))\gamma_{31}(a_1, b_1+c_1)$$
$$v_{32}(c_2(a_1, b_1+c_1)v_{31}(a_1+b_1+c_1)$$
$$= \gamma_{31}(b_1, c_1)\gamma_{31}(a_1, b_1+c_1)[a_1, \gamma_{21}(b_1, c_1)]\gamma_{32}(\gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1+c_1))$$
$$v_{32}(\gamma_{21}(b_1, c_1)+\gamma_{21}(a_1, b_1+c_1))v_{31}(a_1+b_1+c_1).$$

Hence by the associative law and the equality $\partial\gamma_{21}=0$ we have

$$\partial\gamma_{31}(a_1, b_1, c_1)+[a_1, \gamma_{21}(b_1, c_1)]+\gamma_{32}(\gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1+c_1))$$
$$-\gamma_{32}(\gamma_{21}(a_1, b_1), \gamma_{21}(a_1+b_1, c_1))=0$$

in the additive notation. This shows that γ is a triangular 2-cocycle of $\mathfrak{L}[(H_i)]$. We denote by σ the map of H_1 onto $\mathrm{Exp}_\gamma(\mathfrak{L}[(H_i)]$ defined by

$$\sigma(v_{33}(a_3)v_{32}(a_2)v_{31}(a_1)) = \mathrm{Exp}_\gamma(a_1+a_2+a_3), \qquad (a_i\in A_i).$$

We shall shows that σ is an isomorphism. From the definition of γ and $\{v_{ji}(\xi)\}$ it follows

$$(v_{33}(a_3)v_{32}(a_2)v_{31}(a_1))(v_{33}(b_3)v_{32}(b_2)v_{31}(b_1))$$
$$= v_{33}(a_3)v_{33}(b_3)v_{32}(a_2)v_{32}(b_2)v_{32}(b_2)^{-1}v_{31}(a_1)v_{32}(b_2)v_{31}(a_1)^{-1}v_{31}(a_1)v_{31}(b_1)$$
$$= v_{33}(a_3+b_3+[-b_2, a_1])c_2(a_2, b_2)v_{32}(a_2+b_2)v_{32}(v_{21}(a_1))v_{32}(v_{21}(b_1))$$
$$= v_{33}(a_3+b_3+[a_1, b_2]+\gamma_{32}(a_2, b_2))v_{32}(a_2+b_2)c_2(v_{21}(a_1), v_{21}(b_1))$$
$$v_{32}(v_{21}(a_1)v_{21}(b_1))$$
$$= v_{33}(a_3+b_3+[a_1, b_2]+\gamma_{32}(a_2, b_2)\gamma_{31}(a_1, b_1))v_{32}(a_2+b_2)v_{32}(c_1(a_1, b_1))$$
$$v_{21}(a_1+b_1))$$
$$= v_{33}(a_3+b_3+\gamma_{31}(a_1, b_1)+\gamma_{32}(a_2, b_2)+[a_1, b_2]+\gamma_{32}(a_2+b_2, \gamma_{21}(a_1, b_1)))$$
$$v_{32}(a_2+b_2+\gamma_{21}(a_1, b_1))v_{31}(a_1+b_1).$$

Hence by virtue of (14) in Proposition 3 we have

$$(v_{33}(a_3)v_{32}(a_2)v_{31}(a_1)v_{33}(b_3)v_{32}(b_2)v_{31}(b_1))$$
$$= \mathrm{Exp}_\gamma(a_1+b_1+a_2+b_2+\gamma_{21}(a_1, b_1)+a_3+b_3+\gamma_{31}(a_1, b_1)+\gamma_{32}(a_3, b_2)$$
$$+\gamma_{32}(a_2+b_2, \gamma_{21}(a_1, b_1)))$$
$$= \mathrm{Exp}_\gamma(a_1+a_2+a_3)\mathrm{Exp}_\gamma(b_1+b_2+b_3) = \sigma(v_{33}(a_3)v_{32}(a_2)v_{31}(a_1))$$
$$\sigma(v_{33}(b_3)v_{32}(b_2)v_{31}(b_1)).$$

This proves that σ is a homomorphism. Since obviousely σ is bijective, σ is an isomorphism. This completes the proof of Theorem 2.

Two triangular 2-cocycles γ and γ' of $\mathfrak{L} = A_1 + A_2 + A_3$ are called to be equivalent if the N-series associated with the pairs $(\mathfrak{L}, \gamma)$ and $(\mathfrak{L}, \gamma')$ (in the means of Theorem 1) are isomorphic. We call the equivalent classes of triangular 2-cocycles the triangular 2-cohomology classes of $\mathfrak{L}$. Then by virtue of Theorems 1 and 2 we can conclude that the set of pairs consisting of a graded Lie ring $\mathfrak{L}$ of length three and a triangular 2-cohomology class of $\mathfrak{L}$ corresponds bijiectively to the set of N-series of length three by means of the Exponential Maps.

REFERENCE

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Supplement: Let $\mathfrak{L} = A_1 + A_2 + A_3$ be a graded Lie ring of length three. Let γ_{21} be a 2-cocycle of A_1 with coefficients in A_2 and γ_{32} be a 2-cocycle of A_2 with coefficients in A_3 such that the extended group N_3 of by N_2 with respect to γ_{32} is abelian, i.e.

$$0 \to A_3 \to N_2 \to A_2 \to 0 \qquad (*)$$

is an exact sequence of abelian groups. We denote by $H_a^2(A_2, A_3)$ the group of all the abelian extensions of A_3 by A_2. We denote by $\delta^{\gamma_{32}}$ the coboundary operation of $H^2(A_1, A_2)$ into $H^3(A_1, A_3)$ with respect to the exact sequence $(*)$, then we have the following identity

$$\delta^{\gamma_{32}}(\gamma_{21})(a_1, b_1, c_1) = \gamma_{32}(\gamma_{21}(b_1, c_1), \gamma_{21}(a_1, b_1 + c_1)) - \gamma_{32}(\gamma_{21}(a_1, b_1), \gamma_{21}(a_1 + b_1, c_1)), \qquad (a_1, b_1, c_1 \in A_1).$$

On the other hand if we put $\gamma'_{32} = \gamma_{32} + \partial\beta$ with a 1-cochain β of A_2, we have

$$\delta^{\gamma_{32}'}(\gamma_{21}) = \delta^{\gamma_{32}}(\gamma_{21}) + \partial g, \; g(a_1, b_1) = \beta(\gamma_{21}(a_1, b_1)), \qquad (a_1, b_1 \in A_1).$$

These identities show that the mapping $(\gamma_{32}, \gamma_{21}) \to \delta^{\gamma_{32}}(\gamma_{21})$ induces the zero-map of $H_a^2(A_2, A_3) \times H^2(A_1, A_2)$ into $H^3(A_1, A_3)$. The map: $\gamma_{21} \to [a_1, \gamma_{21}(b_1, c_1)]$ induces a homomorphism χ of $H^2(A_1, A_2)$ into $H^3(A_1, A_3)$. We denote by K the kernel of χ, then we can parametrize by $K \times H_a^2(A_2, A_3) \times H^2(A_1, A_3)$ all N-series $\{N_1, N_2, N_3\}$ such that associated Lie ring of $\{N_i\}$ is canonically isomorphic to $\mathfrak{L} = A_1 + A_2 + A_3$.

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