

ON CARTAN PSEUDO GROUPS

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Let M be a domain in an Euclidean space and let Γ be a pseudo group of transformations* of M . We say that Γ is a Cartan pseudo group [1, 2] if the following conditions are satisfied:

1) There exists a domain M' and a projection $\rho : M \rightarrow M'$, such that the orbits of Γ are the fibers of the projection ρ . We assume moreover that there is a system of coordinates (x) of M' and a system of coordinates (x, y) of M such that the fibers of ρ are defined by $(x) = \text{constants}$,

2) There are forms $\omega^i, \bar{\omega}^\lambda, i = 1 \cdots m, \lambda = 1 \cdots n$, defined in D such that

a) $(\omega^i, \bar{\omega}^\lambda)$ is a basis of the space of linear forms at every point of M ,

$$(1) \quad b) d\omega^i = \frac{1}{2} c_{jk}^i \omega^j \wedge \omega^k + a_{j\lambda}^i \omega^j \wedge \bar{\omega}^\lambda$$

where $c_{jk}^i, a_{j\lambda}^i$ are functions on M which depend on (x) only,

c) $\omega^r = dx^r$ for $1 \leq r \leq \text{dimension } M'$,

d) The matrices $a_\lambda = \|a_{j\lambda}^i\|$ are linearly independent at every point of M ,

e) Let π_1 and π_2 be respectively the projections of $M \times M$ into the first and second factors. The closed differential system $\mathcal{L}$ on $M \times M$, with independent variables $x \circ \pi_1, y \circ \pi_1$ generated by

$$\begin{aligned} x^r \circ \pi_1 - x^r \circ \pi_2 &= 0, \quad 1 \leq r \leq \text{dimension } M', \\ \pi_1^* \omega^i - \pi_2^* \omega^i &= 0 \quad 1 \leq i \leq m \end{aligned}$$

is in involution at every integral point,

3) A local transformation f of M is in Γ if and only if f preserves the forms ω^i , i.e. $f^* \omega^i = \omega^i, i = 1, \dots, m$.

In this note we prove that every differential form on M which is invariant under all transformations of a Cartan pseudo group Γ is a linear combination of the forms ω^i the coefficient being functions of x only.

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* All maps and differential forms considered in this note are assumed to be analytic.

Put $\sigma^i = \pi_1^* \omega^i - \pi_2^* \omega^i$, $\tau^\lambda = \pi_1^* \tilde{\omega}^\lambda - \pi_2^* \tilde{\omega}^\lambda$. The set $(\pi_1^* \omega^i, \pi_1^* \tilde{\omega}^\lambda, \sigma^i, \tau^\lambda)$ is a basis of differential forms on every point of $M \times M$. Let E be a contact element of $M \times M$ at the integral point $(u_1, u_2) \in M \times M$ such that the forms $\pi_1^* \omega^i$ and $\pi_1^* \tilde{\omega}^\lambda$ are linearly independent on E and denote by σ^i/E the restriction of σ^i to E . Put $x = \rho(u_1) = \rho(u_2)$ and write

$$\begin{aligned}\sigma^i/E &= p_j^i(\pi_1^* \omega^j)/E + p_\lambda^i(\pi_1^* \tilde{\omega}^\lambda)/E \\ \tau^\lambda/E &= q_j^\lambda(\pi_1^* \omega^j)/E + q_\mu^\lambda(\pi_1^* \tilde{\omega}^\mu)/E.\end{aligned}$$

E is an integral element of Σ if and only if

$$(2) \quad \begin{aligned}p_j^i &= p_\lambda^i = q_\mu^\lambda = 0 \\ a_{j\lambda}^i(x) q_k^\lambda - a_{k\lambda}^i(x) q_j^\lambda &= 0\end{aligned}$$

for every choice of the indices i, j, k .

Let V and V' be vector spaces of dimensions v and v' over the real field and let L be a vector space of linear maps of V into V' of dimension n . Take basis of V and V' and let $a_\lambda = \|a_{j\lambda}^i\|$, $i = 1, \dots, v'$, $j = 1, \dots, v$, $\lambda = 1, \dots, n$ be a basis of L . The space $\mathcal{B}(L)$ of all linear maps $b : V \rightarrow L$, $b = \|b_j^\lambda\|$ such that

$$a_{j\lambda}^i b_k^\lambda - a_{k\lambda}^i b_j^\lambda = 0$$

for every choice of i, j, k , is called the derived space of L .

Let s_1 be the maximum rank of the matrix $A_1 = \|a_{j\lambda}^i t_1^j\|$ when the vector $(t_1^1, t_1^2, \dots, t_1^{v'})$ varies in $R^{v'}$. Put $A_2 = \|a_{j\lambda}^i t_2^j\|$ and let s_2 be the maximum rank of the matrix $\begin{vmatrix} A_1 \\ A_2 \end{vmatrix}$ when the vectors $(t_1^1, \dots, t_1^{v'}), (t_2^1, \dots, t_2^{v'})$, vary independently in $R^{v'}$. Define an integer s_i in a similar way for each i , $1 \leq i \leq v-1$. The integers s_i are called the characters of L . If δ is the dimension of $\mathcal{B}(L)$ it can be proved [3, page 4] that

$$(3) \quad \delta \leq n \cdot v - (s_1 + \dots + s_{v-1}).$$

The space L is called involutive when the equality holds in (3).

Let $L(x)$ be the space of endomorphisms of R^m generated by the matrices $a_\lambda(x) = \|a_{j\lambda}^i(x)\|$ and denote by $s_1(x), \dots, s_{m-1}(x)$, $\delta(x)$ the characters and the dimension of the derived space of $L(x)$; Σ is involutions at every integral point if and only if $\delta(x)$ is constant and $L(x)$ is involutive for every x . When Σ is involutive the characters $s_i(x)$ are independent of x .

Let now f be a transformation of Γ . Applying f^* to equation (1) we have

$$d\omega^i = \frac{1}{2} c_{jk}^i \omega^j \wedge \omega^k + a_{jk}^i \omega^j \wedge f^* \tilde{\omega}^k$$

hence,

$$a_{j\lambda}^i (f^* \tilde{\omega}^\lambda - \tilde{\omega}^\lambda) \wedge \omega^j = 0$$

and the linear form $a_{j\lambda}^i (f^* \tilde{\omega}^\lambda - \tilde{\omega}^\lambda)$ is a linear combination of the form ω^j . Since the matrices a_λ are linearly independent we have

$$(4) \quad f^* \tilde{\omega}^\lambda = \tilde{\omega}^\lambda + h_j^\lambda \omega^j.$$

Substituting (4) in (1) we have

$$a_{j\lambda}^i h_k^\lambda - a_{j\lambda}^i h_j^\lambda = 0.$$

Conversely let u_1, u_2 be two points of M such that $\rho(u_1) = \rho(u_2) = x$ and let h_j^λ be an element of the derived space of $L(x)$. Let E be the integral contact element of Σ at the point $(u_1, u_2) \in M \times M$ whose coordinates are $p_j^i = p_\lambda^i = q_\mu^\lambda = 0$, $q_j^\lambda = h_j^\lambda$. Let f be the transformation of M defined by an integral manifold of Σ whose tangent space at the point (u_1, u_2) is E . Then $f(u_1) = u_2$ and, at the point u_1

$$f^* \tilde{\omega}^\lambda = \tilde{\omega}^\lambda + h_i^\lambda \omega^i$$

for $1 \leq \lambda \leq n$.

Assume now that the differential form ω is invariant under Γ and write

$$\omega = \alpha_i \omega^i + \beta_\lambda \tilde{\omega}^\lambda.$$

Given $u_1, u_2 \in M$ with $\rho(u_1) = \rho(u_2) = x$ there exists $f \in \Gamma$ such that $f(u_1) = u_2$ and, at the point u_1 , $f^* \tilde{\omega}^\lambda = \tilde{\omega}^\lambda$. It follows that α_i, β_λ depend only on x . Assume that there exists x such that not all coefficients $\beta_\lambda(x)$ are zero. Then we can take ω to be one of the forms $\tilde{\omega}^\lambda$. Hence, there exists a system of forms $(\omega^i, \tilde{\omega}^\lambda)$ which satisfy conditions 2) and 3) and such that $\tilde{\omega}^\lambda$ is an invariant form of Γ . Then, if h_i^λ is an element of $\mathcal{D}(L(x))$ we have necessarily $h_i^\lambda = 0$ for every i . Let L_0 be the subspace $L(x)$ generated by $a_1(x), \dots, a_{n-1}(x)$. Any element of $\mathcal{D}(L(x))$ has values in L_0 hence, the dimension of $\mathcal{D}(L_0)$ is $\delta(x)$. Let $s'_1 \cdots s'_{m-1}$ be the characters of (L_0) . By (3)

$$\delta(x) \leq m(n-1) - [s'_1 + \cdots + s'_{m-1}].$$

By the definition of the characters $s_i \leq s'_i + 1$. Therefore

$$\delta(x) < m \cdot n - [s_1 + \dots + s_{m-1}]$$

and $L(x)$ is not involutive. Hence all coefficients β are zero and ω is a linear combination of the ω^i with coefficients depending on x only.

The following example shows that the result is not true if we drop the condition that Σ is in involution even if the coefficients a_{jk}^i, c_{jk}^i are constant and Γ is transitive. Let Γ be the pseudo group operating on R^n obtained by localization of the group of rigid motions. Γ is a Lie pseudo group of order 1. Let $\mathcal{F}$ be the space of orthonormal frames of R^n and denote by $\tilde{\Gamma}$ the prolongation of Γ to $\mathcal{F}$. In $\mathcal{F}$ we have differential forms $\omega^i, \omega_j^i (\omega_j^i + \omega_i^j = 0)$, canonically defined which satisfy equations (1) with constant coefficients. A transformation f of $\mathcal{F}$ is in $\tilde{\Gamma}$ if and only if f preserves the forms ω^i . On the other hand all the forms ω^i, ω_j^i are invariant by the elements of $\tilde{\Gamma}$.

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