ON THETA FUNCTIONS AND ABELIAN VARIETIES OVER VALUATION FIELDS OF RANK ONE

(II) THETA FUNCTIONS AND ABELIAN FUNCTIONS OF CHARACTERISTIC $p(>0)$

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To RICHARD BRAUER on his 60th Birthday

It may safely said that one of the most important problems in modern algebraic geometry is to elevate theory of abelian functions to the same level as theory of elliptic functions beyond the modern formulation of classical results. Being concerned in such a problem, we feel that one of the serious points is the lack of knowladge on the explicit expressions of abelian varieties and their law of compositions by means of their canonical systems of coordinates: Such expressions correspond to the cubic relation $\wp'^2 = 4\wp^3 - g_2\wp - g_3$ of Weierstrass' $\wp$-functions and their addition formulae in theory of elliptic functions.

In Part (I) we have introduced theta functions and abelian functions over fields of characteristic p with valuations of rank one,[1] and have shown that for each positive symmetric bimultiplicative function q valued in a valuation field of rank one there exists an abelian variety A_q such that A_q is embedded in a projective space by means of theta functions of some type with period (E, q).

In the present part (II) first we shall give the explicit addition formulae of the following abelian functions of characteristic $p(\geq 3)$

$$\{\Phi_{\mathfrak{g}_i}(u) = \vartheta_p[g_i, 0](q|u)/\vartheta_p[0, 0](q|u)\,|\,p\mathfrak{g}_i \in \mathfrak{M}\}$$

as an immediate consequence from the fact that $\{\vartheta_p[\mathfrak{g}_i, 0](q|u)\}$ form a base of theta functions of type $(p, 1)$ with period (E, q); the explicit addition formulae are comparatively simple, and they may be considered as the formulae

Received January 10, 1962.

1) We shall freely use the notations and results in Part (I), [2].

given by the reduction mod p of the explicit addition formulae of abelian functions of characteristic zero with a general module in some sense. As corollaries of the explicit addition formulae we have a system of equations satisfied by $\{\Phi_{\mathfrak{g}_i}(1)^{p-1}, \Phi_{\mathfrak{g}_j}(u)\}$; these equations are considered as a part of equations defining the locus of $(\vartheta_p[\mathfrak{g}_i, 0](q|u), \ldots, \vartheta_p[\mathfrak{g}_{p^r}, 0](q|u))$ over the field of coefficients, where r is the dimension of A_q. We, however, are not able to decide now whether the system generates all the relations of $\{\Phi_{\mathfrak{g}_i}(1)^{p-1}, \Phi_{\mathfrak{g}_j}(u)\}$ or not.

In the next section we shall give the explicit expressions of invariant differentials and invariant derivations of the abelian variety A_q of characteristic $p(>0)$ by mean of abelian functions $\{\Phi_{\mathfrak{g}_i}(u)\}$; these expressions are quite simple as we shall see in Theorem 3, 4 and 5.

In §3, following Bolza,[2] we shall introduce $\wp$-functions by means of the canonical base $\{D_1, \ldots, D_r\}$ of invariant derivations in §2 as follows:

$$\wp_{\mathfrak{g}_{i_1}\cdots\mathfrak{g}_{i_s}}(q|u) = -(D_{\mathfrak{g}_{i_1}}\cdots D_{\mathfrak{g}_{i_{s-1}}})\left(\frac{D'_{\mathfrak{g}_{i_s}}(\vartheta[0,0])(q|u)}{\vartheta[0,0](q|u)}\right),$$
$$(1\leq i_1, \ldots, i_s\leq p;\ s\geq 2).$$

We shall first show the following simple formula on a derivation D of a field of characteristic p:

$$D^{p-1}\left(\frac{D(y)}{y}\right) = \frac{D^p(y)}{y} - \left(\frac{D(y)}{y}\right)^p, \qquad y\neq 0.$$

Putting $y=\vartheta[0,0](q|u)$ and $D=D'_i$, $(1\leq i\leq r)$, in the above formula, we shall see that

$$\left(\frac{D'_i(\vartheta[0,0])(q|u)}{\vartheta[0,0](q|u)}\right)^p - \frac{D'_i(\vartheta[0,0])(q|u)}{\vartheta[0,0](q|u)} = \wp_{\underbrace{p^{-1}e_i\cdots p^{-1}e_i}_{p}}(q|u),$$
$$(1\leq i\leq r),$$

and $\{D'_i(\vartheta[0,0])(q|u)/\vartheta[0,0](q|u) | 1\leq i\leq r\}$ form a system of canonical Kummer-generators of separable unramified $(p, \ldots, p)$-covering of the abelian variety A_q with period (E, q). From this expression of $\wp_{\underbrace{p^{-1}e_i\cdots p^{-1}e_i}_{p}}(q|u)$ follows

[2] See [1], p. 327.

$$\mathfrak{r}_{\underbrace{p^{-1}\mathfrak{e}_i\ldots p^{-1}\mathfrak{e}_i}_{p+\nu}}(q\,|\,u) = \mathfrak{r}_{\underbrace{p^{-1}\mathfrak{e}_i\ldots p^{-1}\mathfrak{e}_i}_{\nu+1}}, \qquad (1\leq i\leq r;\ \nu=1,2,\ldots).$$

This shows that there exist only finite number of $\mathfrak{r}$-functions of characteristic $p(>0)$.

It seems to the author that the addition formulae in §1 will be a starting point of the study of moduli of abelian varieties of characteristic $p(\geq 3)$ and the study of details of $\mathfrak{r}$-functions of characteristic $p(>0)$ in §3 will also make rich the theory of abstract abelian varieties.

§1. The explicit addition formulae of abelian functions of characteristic $p\geq 3$

1.1. In the present section we assume that the characteristic p of the universal domain is not less than three. We shall now repeat some notations in (I):

$\mathfrak{M}_Q$: a vector space of dimension r over the field Q of rational numbers;

$\mathfrak{M}$: the module of all the vectors in $\mathfrak{M}_Q$ with integral coordinates;

Ω: an algebraically closed field complete with respect to a valuation v of rank one (valued in the additive group of real numbers). We assume that a mapping $(\xi, m/n)\to\xi^{m/n}$ of $\Omega^\times\times Q$ onto $\Omega^\times$ is given as follows: i) If m is an integer, ξ^m is the ordinary m-th power, ii) $(\xi^{m/n})^{m'/n'}=\xi^{mm'/nn'}$, iii) $(\xi\eta)^{m/n}=\xi^{m/n}\eta^{m/n}$, iv) $1^{1/p}=1$, v) if $(n,p)=1$, $1^{1/n}$ is a primitive n-th root of unity.

q: a function on $\mathfrak{M}_Q\times\mathfrak{M}_Q$ valued in $\Omega^\times$ such that i) $q(\mathfrak{m},\mathfrak{n})=q(\mathfrak{n},\mathfrak{m})$, ii) $q(\mathfrak{m}+\mathfrak{m}',\mathfrak{n})=q(\mathfrak{m},\mathfrak{n})q(\mathfrak{m}',\mathfrak{n})$, iii) $v(q(\mathfrak{m},\mathfrak{m}))>0$ for every $\mathfrak{m}\neq 0$ in $\mathfrak{M}_Q$;

$q(\mathfrak{n})$, $(\mathfrak{n}\in\mathfrak{M}_Q)$: the function on $\mathfrak{M}_Q$ defined by $q(\mathfrak{n})(\mathfrak{m})=q(\mathfrak{n},\mathfrak{m})^2$;

u: a variable function on $\mathfrak{M}_Q$ such that $u(\mathfrak{m}+\mathfrak{n})=u(\mathfrak{m})u(\mathfrak{n})$, i.e., for a base $(\mathfrak{m}_1,\ldots,\mathfrak{m}_r)$ of $\mathfrak{M}_Q$ $u(\mathfrak{m}_1),\ldots,u(\mathfrak{m}_r)$ are analytically independent variables;

K_q: the field of abelian functions with period (E, q) and with coefficients in Ω.

1.2. We choose a complete system $\{\mathfrak{g}_1=(0),\mathfrak{g}_2,\ldots,\mathfrak{g}_{p^r}\}$ of representatives of $p^{-1}\mathfrak{M}/\mathfrak{M}$ and denote by $\mathfrak{g}_i^{1/2}$ the element in $\{\mathfrak{g}_1,\ldots,\mathfrak{g}_{p^r}\}$ such that $2\mathfrak{g}_i^{1/2}-\mathfrak{g}_i\in\mathfrak{M}$. Since $p\neq 2$, there always exists such a vector $\mathfrak{g}_i^{1/2}$.

We put

$$(1)\qquad \vartheta[\mathfrak{g}_i, 0](q\,|\,u)=\sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i,\mathfrak{m}+\mathfrak{g}_i)u(\mathfrak{m}+\mathfrak{g}_i), \qquad (1\leq i\leq p^r).$$

Then $\vartheta[\mathfrak{g}_i, 0](q|u)$ $(1 \leq i \leq p^r)$ are converging series on u. It is obvious that $\vartheta[g_i, 0](q|u)$ does not depend of the choice of the representative $\mathfrak{g}_i$ of $p^{-1}\mathfrak{M}/\mathfrak{M}$, and $\vartheta[\mathfrak{g}_i, 0](q|u)^p$ is nothing but the theta series $\vartheta_p[\mathfrak{g}_i, 0](q|u) = \sum_{\mathfrak{m}\ni\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i, p(\mathfrak{m}+\mathfrak{g}_i))u(p(\mathfrak{m}+\mathfrak{g}_i))$ introduced in §2 (I). By virtue of Theorem 1 in §2 (I) $\{\vartheta_p[\mathfrak{g}_1, 0](q|u)|1 \leq i \leq p^r\}$ form a base of theta functions of type $(p, 1)$. Since 1 is the only one p-th root of unity we may put

$$\vartheta[\mathfrak{g}_i, 0](q|1) = \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i, \mathfrak{m}+\mathfrak{g}_i), \qquad (1 \leq i \leq p^r). \tag{2}$$

We denote briefly

$$\Phi_{\mathfrak{g}_i}(u) = \vartheta_p[\mathfrak{g}_i, 0](q|u)/\vartheta_p[0, 0](q|u), \qquad (1 \leq i \leq p^r). \tag{3}$$

We shall show some properties of $\vartheta[\mathfrak{g}_i, 0](q|u)$.

LEMMA 1.

$$\vartheta[\mathfrak{g}_i, 0](q|u^{-1}) = \vartheta[-\mathfrak{g}_i, 0](q|u), \tag{4}$$

$$\vartheta[\mathfrak{g}_i, 0](q|q(\mathfrak{n})u) = q(\mathfrak{n}, \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\vartheta[\mathfrak{g}_i, 0](q|u), \tag{5}$$
$$(\mathfrak{n} \in \mathfrak{M};\ 1 \leq i \leq p^r).$$

Proof. From (1) it follows

$$\begin{aligned}\vartheta[\mathfrak{g}_i, 0](q|u^{-1}) &= \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i, \mathfrak{m}+\mathfrak{g}_i)u(-\mathfrak{m}-\mathfrak{g}_i) \\ &= \sum_{\mathfrak{m}\in\mathfrak{M}} q(-\mathfrak{m}+\mathfrak{g}_i, -\mathfrak{m}+\mathfrak{g}_i)u(\mathfrak{m}-\mathfrak{g}_i) \\ &= \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}-\mathfrak{g}_i, \mathfrak{m}-\mathfrak{g}_i)u(\mathfrak{m}-\mathfrak{g}_t) \\ &= \vartheta[-\mathfrak{g}_i, 0](q|u),\end{aligned}$$

and for any $\mathfrak{n}$ in $\mathfrak{M}$

$$\begin{aligned}\vartheta[\mathfrak{g}_i, 0](q|q(\mathfrak{n})u) &= \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i, \mathfrak{m}+\mathfrak{g}_i)q(\mathfrak{n}, \mathfrak{m}+\mathfrak{g}_i)^2u(\mathfrak{m}+\mathfrak{g}_i) \\ &= q(\mathfrak{n}, \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{n}+\mathfrak{g}_i, \mathfrak{m}+\mathfrak{n}+\mathfrak{g}_i)u(\mathfrak{m}+\mathfrak{n}+\mathfrak{g}_i) \\ &= q(\mathfrak{n}, \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\vartheta[\mathfrak{g}_i, 0](q|u).\end{aligned}$$

LEMMA 2.

$$\vartheta[\mathfrak{g}_i, 0](q|q(\mathfrak{g}_j)u) = q(\mathfrak{g}_j, \mathfrak{g}_j)^{-1}u(\mathfrak{g}_j)^{-1}\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|u), \qquad (1 \leq i, j \leq p^r). \tag{6}$$

Proof. From (1) it follows

$$\begin{aligned}\vartheta[\mathfrak{g}_i, 0](q|q(g_i)u) &= \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i, \mathfrak{m}+\mathfrak{g}_i)q(\mathfrak{g}_j, \mathfrak{m}+\mathfrak{g}_i)^2u(\mathfrak{m}+\mathfrak{g}_i) \\ &= q(\mathfrak{g}_j, \mathfrak{g}_i)^{-1}u(\mathfrak{g}_j)^{-1}\sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}+\mathfrak{g}_i+g_j, \mathfrak{m}+\mathfrak{g}_i+\mathfrak{g}_j)u(\mathfrak{m}+\mathfrak{g}_i+\mathfrak{g}_j) \\ &= q(\mathfrak{g}_j, \mathfrak{g}_j)^{-1}u(\mathfrak{g}_j)^{-1}\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|u).\end{aligned}$$

LEMMA 3.

$$\vartheta[0, 0](q|1) \neq 0.$$

Proof. Since $v(q(\mathfrak{m}, \mathfrak{m}))>0$ for $\mathfrak{m}\neq 0$ and $\vartheta[0, 0](q|1) = \sum_{\mathfrak{m}\in\mathfrak{M}} q(\mathfrak{m}, \mathfrak{m})$, we have $v(\vartheta[0, 0](q|1)) = v(1) = 0$, and thus $\vartheta[0, 0](q|1)\neq 0$.

LEMMA 4. *The (p, p)-matrix $(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|1))$ of which (i, j)-element is $\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|1)$ is non-singular.*

Proof. Since $v(q(\mathfrak{m}+\mathfrak{g}_i, \mathfrak{m}+\mathfrak{g}_i)>0$ for every $\mathfrak{m}+\mathfrak{g}_i\neq 0$, it follows that $v(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|1)\neq 0$ if and only if $\mathfrak{g}_i+\mathfrak{g}_j\in\mathfrak{M}$. Hence det $(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|1))$ $= \pm\vartheta[0, 0](q|1)p+c$ with an element c such that $v(c)>0$. This shows that $v(\det(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|1))=0$, and thus det $(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j, 0](q|1))\neq 0$.

LEMMA 5. *$F(u, v) = \vartheta[0, 0](q|1)^2\vartheta[\mathfrak{g}_i, 0](q|uv)\vartheta[0, 0](q|uv^{-1})\vartheta[\mathfrak{g}_i^{1/2}, 0](q|u)^{p-2}\vartheta[\mathfrak{g}_i^{1/2}, 0](q|v)^{p-2}$ is a theta function of type $(p, 1)$ as a function of both u and v.*

Proof. Since $\mathfrak{g}_i+(p-2)\mathfrak{g}_i^{1/2} = p\mathfrak{g}_i^{1/2}+(\mathfrak{g}_i-2\,\mathfrak{g}_i^{1/2}\in\mathfrak{M}$, the function $F(u, v)$ in Lemma 5 is expressed as follows:

$\sum_{\mathfrak{m},\mathfrak{n}\in\mathfrak{M}} c_{\mathfrak{m},\mathfrak{n}}u(\mathfrak{m})v(\mathfrak{n})$. On the other hand, if for any $\mathfrak{n}$ in $\mathfrak{M}$ we put $q(\mathfrak{n})u$ instead of u in $F(u, v)$, by virture of Lemma 1 we have

$$\begin{aligned} F(q(\mathfrak{n})u, v) &= q(\mathfrak{n}, \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}v(\mathfrak{n})^{-1}\vartheta[0, 0](q|1)^2\vartheta[\mathfrak{g}_i, 0](q|uv) \\ &\quad q(\mathfrak{n}, \mathfrak{n})^{-1}v(\mathfrak{n})\vartheta[0, 0](q|uv^{-1})q(\mathfrak{n}, \mathfrak{n})^{-(p-2)}u(\mathfrak{n})^{-(p-2)} \\ &\quad \vartheta[\mathfrak{g}_i^{1/2}, 0](q|u)^{p-2}\vartheta[\mathfrak{g}_i^{1/2}, 0](q|v)^{p-2} \\ &= q(\mathfrak{n}, \mathfrak{n})^{-p}u(\mathfrak{n})^{-p}F(u, v). \end{aligned}$$

Since $\vartheta[0, 0](q|uv^{-1}) = \vartheta[0, 0](q|vu^{-1})$, we have $F(u, v) = F(v, u)$, and thus $F(u, q(\mathfrak{n})v) = F(q(\mathfrak{n})v, u) = q(\mathfrak{n}, \mathfrak{n})^{-p}v(\mathfrak{n})^{-p}F(u, v)$. This completes the proof of Lemma 5.

1.3. We shall first show the addition formulae of $\vartheta[\mathfrak{g}_i, 0](q|u)$.

THEOREM 1. *If $p\geq 3$, we have the following formulae:*

(7)
$$\begin{aligned} &\vartheta[0, 0](q|1)^2\vartheta[\mathfrak{g}_i, 0](q|uv)\vartheta[0, 0](q|uv^{-1}) \\ &\qquad \vartheta[\mathfrak{g}_i^{1/2}, 0](q|u)^{p-2}\vartheta[\mathfrak{g}_i^{1/2}, 0](q|v)^{p-2} \\ &= \sum_{i,l=1}^{p^r} c_{\mathfrak{g}_i-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\vartheta[\mathfrak{g}_j, 0](q|u)^p\vartheta[\mathfrak{g}_l, 0](q|v)^p, \qquad (1\leq i\leq p^r), \end{aligned}$$

where $\{c_{\mathfrak{g}_j,\mathfrak{g}_l}\}$ are the unique solutions of the following linear equations in

$\{\Lambda_{\mathfrak{g}_j, \mathfrak{g}_l}\}$:

$$\sum_{h,k=1}^{p^r} \vartheta[\mathfrak{g}_j+\mathfrak{g}_h, 0](q|1)\vartheta[\mathfrak{g}_l+\mathfrak{g}_k, 0](q|1)\Lambda_{\mathfrak{g}_h, \mathfrak{g}_k} = \vartheta[0, 0](q|1)^2\vartheta[\mathfrak{g}_j+\mathfrak{g}_l, 0](q|1)\vartheta[\mathfrak{g}_j-\mathfrak{g}_l, 0](q|1) \vartheta[\mathfrak{g}_j, 0](q|1)^{p-2}\vartheta[\mathfrak{g}_l, 0](q|1)^{p-2} \quad (1\le j, l\le p^r). \tag{8}$$

Moreover $\{c_{\mathfrak{g}_j, \mathfrak{g}_l}\}$ *satisfy*

$$c_{\mathfrak{g}_j, \mathfrak{g}_l} = c_{\mathfrak{g}_l, \mathfrak{g}_j}, \quad (1\le j, l\le p^r). \tag{9}$$

Proof. We put briefly

$$F_{\mathfrak{g}_i}(u, v) = \vartheta[0, 0](q|1)^2\vartheta[\mathfrak{g}_i, 0](q|uv)\vartheta[0, 0](q|uv^{-1}) \vartheta[\mathfrak{g}_i^{1/2}, 0](q|u)^{p-2}\vartheta[\mathfrak{g}_i^{1/2}, 0]^{p-2}(q|u), \quad (1\le i\le p^r).$$

It is sufficient to prove Theorem 1 for independent varible u and v. Since $\{\vartheta[\mathfrak{g}_i, 0](q|u)\,|\,1\le i\le p^r\}$ and $\{\vartheta[\mathfrak{g}_i, 0](q|v)\,|\,1\le i\le p^r\}$ form bases of theta functions of type $(p, 1)$ of u and v, respectively, by virtue of Lemma 5, we have

$$F_{\mathfrak{g}_1}(u, v) = \sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j, \mathfrak{g}_l}\vartheta[\mathfrak{g}_j, 0](q|u)^p\vartheta[\mathfrak{g}_l, 0](q|v)^p.$$

Putting $q(\mathfrak{g}_h)\,u$ and $q(\mathfrak{g}_k)v$ instead of u and v, by virtue of Lemma 2 we have

$$\begin{aligned}
&F_{\mathfrak{g}_1}(q(\mathfrak{g}_h)u, q(\mathfrak{g}_k)v)\\
&= \vartheta[0, 0](q|1)^2 q(\mathfrak{g}_h+\mathfrak{g}_k, \mathfrak{g}_h+\mathfrak{g}_k)^{-1}u(\mathfrak{g}_h+\mathfrak{g}_k)^{-1}v(\mathfrak{g}_h+\mathfrak{g}_k)^{-1}\\
&\quad \vartheta[\mathfrak{g}_h+\mathfrak{g}_k, 0](q|uv)q(\mathfrak{g}_h-\mathfrak{g}_k, \mathfrak{g}_h-\mathfrak{g}_k)^{-1}u(\mathfrak{g}_h-\mathfrak{g}_k)^{-1}v(\mathfrak{g}_h-\mathfrak{g}_k)^{-1}\\
&\quad \vartheta[\mathfrak{g}_h-\mathfrak{g}_k, 0](q|uv^{-1})\,q(\mathfrak{g}_h, \mathfrak{g}_h)^{-(p-2)}u(\mathfrak{g}_h)^{-(p-2)}\\
&\quad \vartheta[\mathfrak{g}_h, 0](q|u)^{p-2}q(\mathfrak{g}_k, \mathfrak{g}_k)^{-(p-2)}v(\mathfrak{g}_k)^{-(p-2)}\vartheta[\mathfrak{g}_k, 0](q|v)^{p-2}\\
&= q(\mathfrak{g}_h, \mathfrak{g}_h)^{-p}q(\mathfrak{g}_k, \mathfrak{g}_k)^{-p}u(\mathfrak{g}_h)^{-p}v(\mathfrak{g}_k)^{-p}\vartheta[0, 0](q|1)^2\\
&\quad \vartheta[\mathfrak{g}_h+\mathfrak{g}_k, 0](q|uv)\vartheta[\mathfrak{g}_h-\mathfrak{g}_k, 0](q|uv^{-1})\\
&\quad \vartheta[\mathfrak{g}_h, 0](q|u)^{p-2}\vartheta[\mathfrak{g}_k, 0](q|v)^{p-2},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j, \mathfrak{g}_l}\vartheta[\mathfrak{g}_j, 0](q|q(\mathfrak{g}_h)u)^p\vartheta[\mathfrak{g}_l, 0](q|q(\mathfrak{g}_k)v)^p\\
&= q(\mathfrak{g}_h, \mathfrak{g}_h)^{-p}q(\mathfrak{g}_k, \mathfrak{g}_k)^{-p}u(\mathfrak{g}_h)^{-p}v(\mathfrak{g}_k)^{-p}\\
&\quad \sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j, \mathfrak{g}_l}\vartheta[\mathfrak{g}_j+\mathfrak{g}_h, 0](q|u)^p\vartheta[\mathfrak{g}_l+\mathfrak{g}_k, 0](q|v)^p.
\end{aligned}$$

Hence

$$\vartheta[0,0](q|1)^2\vartheta[\mathfrak{g}_h+\mathfrak{g}_k,0](q|uv)\vartheta[\mathfrak{g}_h-\mathfrak{g}_k,0](q|uv^{-1})$$
$$\vartheta[\mathfrak{g}_h,0](q|u)^{p-2}\vartheta[\mathfrak{g}_k,0](q|v)^{p-2}$$
$$=\sum_{p^r}^{j,l=1} c_{\mathfrak{g}_j,\mathfrak{g}_l}\vartheta[\mathfrak{g}_j+\mathfrak{g}_h,0](q|u)^p\vartheta[\mathfrak{g}_l+\mathfrak{g}_k,0](q|v)^p.$$

Putting $\mathfrak{g}_h=\mathfrak{g}_k=\mathfrak{g}_i^{1/2}$, we have

$$F_{\mathfrak{g}_i}(u,v)=\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j,\mathfrak{g}_l}\vartheta[\mathfrak{g}_j+\mathfrak{g}_i^{1/2},0](q|u)^p\vartheta[\mathfrak{g}_l+\mathfrak{g}_i^{1/2},0](q|v)^p$$
$$=\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\vartheta[\mathfrak{g}_i,0](q|u)^p\vartheta[\mathfrak{g}_l,0](q|v)^p.$$

Putting $u=v=1$, we have

$$\vartheta[0,0](q|1)^2\vartheta[\mathfrak{g}_h+\mathfrak{g}_k,0](q|1)\vartheta[\mathfrak{g}_h-\mathfrak{g}_k,0](q|1)$$
$$\vartheta[\mathfrak{g}_h,0](q|1)^{p-2}\vartheta[\mathfrak{g}_k,0](q|1)^{p-2}$$
$$=\sum_{l=1}^{p^r} c_{\mathfrak{g}_j,\mathfrak{g}_l}\vartheta[\mathfrak{g}_j+\mathfrak{g}_h,0](q|1)^p\vartheta[\mathfrak{g}_l+\mathfrak{g}_k,0](q|1)^p,\qquad(1\le h,k\le p^r).$$

This is nothing but the system of equations (8). Since by virtue of Lemma 4 the (p^r,p^r)-matrix $(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j,0](q|1))$ is non-singular, the (p^{2r},p^{2r})-matrix of which $((i,j),(h,k))$-element is $\vartheta[\mathfrak{g}_i+\mathfrak{g}_h,0](q|1)\vartheta[\mathfrak{g}_j+\mathfrak{g}_k,0](q|1)$ is also non-singular, because the (p^{2r},p^{2r})-matrix is the tensor product

$$(\vartheta[\mathfrak{g}_i+\mathfrak{g}_j,0](q|1))\otimes(\vartheta[\mathfrak{g}_h+\mathfrak{g}_k,0](q|1)).$$

This shows that $\{c_{\mathfrak{g}_j,\mathfrak{g}_l}\}$ are the unique solutions ef (8). Since $F_{\mathfrak{g}_1}(u,v)=F_{\mathfrak{g}_1}(v,u)$, we get the formula (9).

As Corollaries of Theorem 1 we shall show some theta relations.

Putting $v=1$ in (7), we get

COROLLARY 1.

$$\vartheta[0,0](q|1)^2\vartheta[\mathfrak{g}_i^{1/2},0](q|1)^{p-2}\vartheta[\mathfrak{g}_i,0](q|u)$$
$$\vartheta[0,0](q|u)\vartheta[\mathfrak{g}_i^{1/2},0](q|u)^{p-2}$$
$$=\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\vartheta[\mathfrak{g}_j,0](q|1)^p\vartheta[\mathfrak{g}_l,0](q|u)^p,\qquad(1\le i\le p^r),$$

and

$$\vartheta[0,0](q|1)^p\vartheta[0,0](q|u)^p$$
$$=\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j,\mathfrak{g}_l}\vartheta[\mathfrak{g}_j,0](q|1)^p\vartheta[\mathfrak{g}_l,0](q|u)^p.$$

Putting $v=u^{-1}$ in (7), we have

COROLLARY 2.

$$\vartheta[0,0](q|1)^2\vartheta[\mathfrak{g}_i,0](q|1)\vartheta[0,0](q|u^2)\\ \vartheta[\mathfrak{g}_i^{1/2},0](q|u)^{p-2}\vartheta[-\mathfrak{g}_i^{1/2},0](q|u)^{p-2}\\ =\sum_{j,l=1}^{p^r}c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\vartheta[\mathfrak{g}_j,0](q|u)^p\vartheta[-\mathfrak{g}_l,0](q|u)^p. \tag{12}$$

Making the product (12) for $\mathfrak{g}_i$ and (12) for $\mathfrak{g}_1$, we have

COROLLARY 3.

$$\vartheta[\mathfrak{g}_i,0](q|1)\vartheta[\mathfrak{g}_i^{1/2},0](q|u)^{p-2}\vartheta[-\mathfrak{g}_i^{1/2},0](q|u)^{p-2}\\ \sum_{j,l=1}^{p^r}c_{\mathfrak{g}_j,\mathfrak{g}_l}\vartheta[\mathfrak{g}_j,0](q|u)^p\vartheta[-\mathfrak{g}_l,0](q|u)^p\\ =\vartheta[0,0](q|1)\vartheta[0,0](q|u)^{2(p-2)}\\ \sum_{j,l=1}^{p^r}c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\vartheta[\mathfrak{g}_j,0](q|u)^p\vartheta[-\mathfrak{g}_l,0](q|u)^p,\\ (1\le i\le p^r). \tag{13}$$

1.4. Let us now translate (7), . . . , (13) in the relations of the abelian functions $\{\Phi_{\mathfrak{g}_i}(u)\}$.

THEOREM 2. *If $p\ge 3$, we have the following addition formulae:*

$$\Phi_{\mathfrak{g}_i}(uv)=\frac{\left(\sum_{j,l=1}^{p^r}c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\Phi_{\mathfrak{g}_j}(u)\Phi_{\mathfrak{g}_l}(v)\right)^p}{\Phi_{\mathfrak{g}_i^{1/2}}(u)^{p-2}\Phi_{\mathfrak{g}_i^{1/2}}(v)^{p-2}\left(\sum_{j,l=1}^{p^r}c_{\mathfrak{g}_j,\mathfrak{g}_l}\Phi_{\mathfrak{g}_j}(u)\Phi_{\mathfrak{g}_l}(v)\right)^p}, \tag{14}$$

$$\Phi_{\mathfrak{g}_i}(u^{-1})=\Phi_{-\mathfrak{g}_i}(u),\qquad (1\le i\le p^r), \tag{15}$$

where the coefficient $\{c_{\mathfrak{g}_j,\mathfrak{g}_l}\}$ are the unique solution of the system of linear equations:

$$\sum_{h,k=1}^{p^r}A_{\mathfrak{g}_h,\mathfrak{g}_k}\Phi_{\mathfrak{g}_j+\mathfrak{g}_h}(1)\Phi_{\mathfrak{g}_l+\mathfrak{g}_k}(1)=\Phi_{\mathfrak{g}_j+\mathfrak{g}_l}(1)^{p-1}\Phi_{\mathfrak{g}_j-\mathfrak{g}_l}(1)^{p-1}\cdot\Phi_{\mathfrak{g}_j}(1)^{p-2/p}\cdot\Phi_{\mathfrak{g}_l}(1)^{p-2/p},\\ (1\le j,\ l\le p^r). \tag{16}$$

Proof. Since $\vartheta[0,0](q|1)\neq 0$ and $\vartheta[0,0](q|u)\neq 0$, we may divive (7) for $\mathfrak{g}_i$ by (7) for $\mathfrak{g}_1$. Making the p-th powers of both sides of the quotient of (7) for $\mathfrak{g}_i$ by (7) for $\mathfrak{g}_1$, we have (14). Dividing (8) by $\vartheta[0,0](q|1)^{2p}$, we get (16). (15) is a consequence from (4).

The formulae (14) and (16) are a system of explicit expression of the normal law of composition on the locus of $(\vartheta_p[\mathfrak{g}_1,0](q|u),\ldots,\vartheta_p[\mathfrak{g}_{p^r},0](q|u))$

over Ω.

Putting $v=u^{-1}$ in (14), we get

COROLLARY 1.

$$\Phi_{\mathfrak{g}_i}(1)\Big(\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j,\mathfrak{g}_l}\Phi_{\mathfrak{g}_j}(u)\Phi_{-\mathfrak{g}_l}(u)\Big)^p \Phi_{\mathfrak{g}_i^{1/2}}(u)^{p-2}\Phi_{-\mathfrak{g}_i^{1/2}}(u)^{p-2} = \Big(\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_l-\mathfrak{g}_i^{1/2}}\Phi_{\mathfrak{g}_j}(u)\Phi_{-\mathfrak{g}_l}(u)\Big)^p, \quad (1\leq i\leq p^r). \tag{17}$$

Putting $v=1$ in (31), we have

COROLLARY 2.

$$\Phi_{\mathfrak{g}_i}(u)\Phi_{\mathfrak{g}_i^{1/2}}(u)^{p-2}\Phi_{\mathfrak{g}_i^{1/2}}(1)^{p-2}\Big(\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j,\mathfrak{g}_l}\Phi_{\mathfrak{g}_j}(1)\Phi_{\mathfrak{g}_l}(u)\Big)^p = \Big(\sum_{j,l=1}^{p^r} c_{\mathfrak{g}_j-\mathfrak{g}_i^{1/2},\mathfrak{g}_j-\mathfrak{g}_i^{1/2}}\Phi_{\mathfrak{g}_j}(1)\Phi_{\mathfrak{g}_l}(u)\Big)^p, \quad (1\leq i\leq p^r). \tag{18}$$

(17) is the system of relations of $\{\Phi_{\mathfrak{g}_i}(u), \Phi_{\mathfrak{g}_j}(1)\}$ which is the explicit expression of the axiom: $x+(-x)=0$, and (18) is the system of relations of $\{\Phi_{\mathfrak{g}_i}(u), \Phi_{\mathfrak{g}_j}(1)\}$ which is the explicite expression of the axiom: $x+0=0$.

We shall now seek relations of $\{\Phi_{\mathfrak{g}_i}(u), \Phi_{\mathfrak{g}_j}(1)\}$ other than (17) and (18).

COROLLARY 3.

$$\Phi_{\mathfrak{g}_i^{1/2}+\mathfrak{g}_j}(1)^{p-2}\Phi_{\mathfrak{g}_i+\mathfrak{g}_j}(u)\Phi_{\mathfrak{g}_i^{1/2}}(u)^{p-2}\Big(\sum_{h,k=1}^{p^r} c_{\mathfrak{g}_h,\mathfrak{g}_k}\Phi_{\mathfrak{g}_h}(u)\Phi_{\mathfrak{g}_k+\mathfrak{g}_j}(1)\Big)^p = \Phi_{\mathfrak{g}_j}(1)^{p-2}\Phi_{\mathfrak{g}_j}(u)\Big(\sum_{h,k=1}^{p^r} c_{\mathfrak{g}_h-\mathfrak{g}_i^{1/2},\mathfrak{g}_k-\mathfrak{g}_i^{1/2}}\Phi_{\mathfrak{g}_h}(u)\Phi_{\mathfrak{g}_k+\mathfrak{g}_j}(1)\Big)^p. \quad (1\leq i,\ j\leq p^r). \tag{19}$$

Proof. Putting $v=q(\mathfrak{g}_j)$ in (7), we have

$$\Phi_{\mathfrak{g}_j}(uq(\mathfrak{g}_j))\Phi_{\mathfrak{g}_i^{1/2}}(u)^{p-2}\Phi_{\mathfrak{g}_i^{1/2}}(q(\mathfrak{g}_j))^{p-2}\Big(\sum_{h,k=1}^{p^r} c_{\mathfrak{g}_h,\mathfrak{g}_k}\Phi_{\mathfrak{g}_h}(u)\Phi_{\mathfrak{g}_k}(q(g_j))\Big)^p = \Big(\sum_{h,k=1}^{p^r} c_{\mathfrak{g}_h-,\mathfrak{g}_k-\mathfrak{g}_i^{1/2}\mathfrak{g}_i^{1/2}}\Phi_{\mathfrak{g}_h}(u)\Phi_{\mathfrak{g}_k}(q(\mathfrak{g}_j))\Big)^p$$

On the other hand, by virtue of (22) in Lemma 2, we have $\Phi_{\mathfrak{g}_i}(uq(\mathfrak{g}_j)) = \Phi_{\mathfrak{g}_i+\mathfrak{g}_j}(u)/\Phi_{\mathfrak{g}_j}(u)$. Hence it follows

$$\Phi_{\mathfrak{g}_i+\mathfrak{g}_j}(u)\Phi_{\mathfrak{g}_i^{1/2}}(u)^{p-2}\Phi_{\mathfrak{g}_i^{1/2}+\mathfrak{g}_j}(1)^{p-2}\Big(\sum_{h,k=1}^{p^r} c_{\mathfrak{g}_h,\mathfrak{g}_k}\Phi_{\mathfrak{g}_h}(u)\Phi_{\mathfrak{g}_k+\mathfrak{g}_j}(1)\Big)^p = \Phi_{\mathfrak{g}_j}(u)\Phi_{\mathfrak{g}_j}(1)\cdot\Big(\sum_{h,k=1}^{p^r} c_{\mathfrak{g}_h-\mathfrak{g}_i^{1/2},\mathfrak{g}_k-\mathfrak{g}_i^{1/2}}\Phi_{\mathfrak{g}_h}(u)\Phi_{\mathfrak{g}_k+\mathfrak{g}_j}(1)\Big)^p.$$

This proves Corollary 3.

§ 2. Invariant derivations and invariant differentials

2.1. We shall denote by $\mathfrak{e}_1, \dots, \mathfrak{e}_r$ the unit vectors and denote by $x_i(\mathfrak{m})$ the i-th coordinate madulo p of $\mathfrak{m}$ in $\mathfrak{M}_Q$, i.e. $\mathfrak{m} \equiv \sum_{i=1}^{r} x_i(\mathfrak{m})\mathfrak{e}_i \bmod p$.[3] Let us denote by D_i' the Ω-linear mapping of the field $\Omega(u) = \Omega(\{u(\mathfrak{m}) \mid \mathfrak{m} \in \mathfrak{M}\})$ into itself such that

$$D_i'(u(\mathfrak{m})) = x_i(\mathfrak{m})\, u(\mathfrak{m}), \qquad (\mathfrak{m} \in \mathfrak{M}), \tag{20}$$

$$D_i'(\xi) = 0, \qquad (\xi \in \Omega). \tag{21}$$

LEMMA 1. *$D_1', \dots, D_r'$ are derivations of $\Omega(u)/\Omega$.*

Proof. From the definition of D_i', it follows for any $\mathfrak{m}, \mathfrak{n} \in \mathfrak{M}$

$$\begin{aligned} D_i'(u(\mathfrak{m})u(\mathfrak{n})) &= D_i'(u(\mathfrak{m}+\mathfrak{n})) = x_i(\mathfrak{m}+\mathfrak{n})u(\mathfrak{m}+\mathfrak{n}) = (x_i(\mathfrak{m}) + x_i(\mathfrak{n}))u(\mathfrak{m})u(\mathfrak{n}) \\ &= x_i(\mathfrak{m})u(\mathfrak{m})u(\mathfrak{n}) + u(\mathfrak{m})x_i(\mathfrak{n})u(\mathfrak{n}) = D_i'(u(\mathfrak{m}))u(\mathfrak{n}) + u(\mathfrak{m})D_i'(u(\mathfrak{n})). \end{aligned}$$

This proves Lemma 1.

The derivations $D_1', \dots, D_r'$ are naturally extended to the derivations of the field $\Omega(u)(\vartheta[\mathfrak{g}_1, 0](q \mid u), \dots, \vartheta_p[\mathfrak{g}_{p^r}, 0](q \mid u))$ such that $D_i'(\vartheta_p[\mathfrak{g}_j, 0])(q \mid u) = x_i(p\mathfrak{g}_j)\vartheta_p[\mathfrak{g}_j, 0](q|u)$, $(1 \le i \le r;\ 1 \le j \le p^r)$. We denote by the same notation the extended derivation.

LEMMA 2. $(u(e_i)^{-1}du(e_i))(D_j') = \delta_{ij}, \qquad (1 \le i, j \le r).$

Proof. From the definition of the differential $du(\mathfrak{e}_i)$ follows

$$(u(\mathfrak{e}_i)^{-1}du(\mathfrak{e}_i))(D_j') = u(\mathfrak{e}_i)^{-1}D_j'(u(\mathfrak{e}_i)) = x_j(\mathfrak{e}_i) = \delta_{ij}.$$

Let us operate D_i' on the theta's $\{\vartheta_p[\mathfrak{g}_i, 0](q \mid u)\}$.

LEMMA 3. $D_i'(\vartheta_p[\mathfrak{g}_j, 0](q \mid u)) = x_i(p\mathfrak{g}_j)\, \vartheta_p[\mathfrak{g}_j, 0](q \mid u),$

$$(1 \le i \le r;\ 1 \le j \le p^r).$$

Proof. Since $\vartheta_p[\mathfrak{g}_j, 0](q \mid u) = \sum_{\mathfrak{m} \in \mathfrak{M}} q(p(\mathfrak{m}+\mathfrak{g}_j), \mathfrak{m}+\mathfrak{g}_j)\, u(p\mathfrak{m}+p\mathfrak{g}_j)$ and $x_i(p\mathfrak{m}) = 0$ for $\mathfrak{m}$ in $\mathfrak{M}$, it follows

$$\begin{aligned} D_i'(\vartheta_p[\mathfrak{g}_j, 0](q \mid u) &= \sum_{\mathfrak{m} \in \mathfrak{M}} x_i(p\mathfrak{m}+p\mathfrak{g}_j)\, q(p(\mathfrak{m}+\mathfrak{g}_j), \mathfrak{m}+\mathfrak{g}_i)\, u(p\mathfrak{m}+p\mathfrak{g}_j) \\ &= x_i(p\mathfrak{g}_j)\, \vartheta_p[\mathfrak{g}_j, 0](q \mid u). \end{aligned}$$

[3] We consider $x_i(\mathfrak{m})$ $(i = 1, 2, \dots, r)$ as elements of the Galois Feld $GF(p)$.

LEMMA 4. $D_i'(\Phi_{\mathfrak{g}_j}(u)) = x_i(p\mathfrak{g}_j)\Phi_{\mathfrak{g}_j}(u), \quad (1 \leq i \leq r;\ 1 \leq j \leq p^r)$.

Proof. From Lemma 3 follows

$$\begin{aligned} D_i'(\Phi_{\mathfrak{g}_j}(u)) &= \vartheta_p[0,\ 0]^{-2}(q|u)(D_i'(\vartheta_p[\mathfrak{g}_j,\ 0](q|u)\vartheta_p[0,\ 0](q|u) \\ &\quad - \vartheta_p[\mathfrak{g}_j,\ 0](q|u)D_i'(\vartheta_p[0,\ 0](q|u)) \\ &= \vartheta_p[0,\ 0](q|u)^{-2}x_i(p\mathfrak{g}_j)\vartheta_p[\mathfrak{g}_j,\ 0](q|u)\vartheta_p[0,\ 0](q|u) \\ &= x_i(p\mathfrak{g}_j)\Phi_{\mathfrak{g}_j}(u). \end{aligned}$$

We shall now express $u(\mathfrak{e}_i)^{-1}du(\mathfrak{e}_i)$ by means of $\Phi_{\mathfrak{g}_j}$.

LEMMA 5.

$$\Phi_{\mathfrak{g}_j}(u)^{-1}d\Phi_{\mathfrak{g}_j}(u) = \sum_{i=1}^{r} x_i(p\mathfrak{g}_j)u(\mathfrak{e}_i)^{-1}du(\mathfrak{e}_i), \tag{22}$$

$$\Phi_{p^{-1}\mathfrak{e}_i}(u)^{-1}d\Phi_{p^{-1}\mathfrak{e}_i}(u) = u(\mathfrak{e}_i)^{-1}du(\mathfrak{e}_i), \quad (1 \leq i \leq r;\ 1 \leq j \leq p^r). \tag{23}$$

Proof. By virtue of Lemma 2 and 4 it follows

$$(\Phi_{\mathfrak{g}_j}(u)^{-1}d\Phi_{\mathfrak{g}_j}(u))^{-1}(D_i') = x_i(p\mathfrak{g}_j) = \sum_{i=1}^{r} x_i(p\mathfrak{g}_j)((u(\mathfrak{e}_i)^{-1}d(u(\mathfrak{e}_i))(D_i'), \quad (1 \leq i \leq r;\ 1 \leq j \leq p^r).$$

Since $(D_1', \ldots, D_r')$ is the $\Omega(u)$-base of all the derivations of $\Omega(u)/\Omega$, we have $\Phi_{\mathfrak{g}_j}(u)^{-1}d\Phi_{\mathfrak{g}_i}(u) = \sum_{i=1}^{r} x_i(p\mathfrak{g}_j)u(\mathfrak{e}_i)^{-1}du(\mathfrak{e}_i)$.

COROLLARY.

$$\Phi_{\mathfrak{g}_j}(u)^{-1}d\Phi_{\mathfrak{g}_j}(u) = \sum_{i=1}^{r} x_i(p\mathfrak{g}_j)\Phi_{p^{-1}\mathfrak{e}_i}(u)^{-1}d\Phi_{p^{-1}\mathfrak{e}_i}(u), \quad (1 \leq i \leq r;\ 1 \leq j \leq p^r).$$

2.2. By virtue the first part of the proof of Theorem 3 in §3(I) the field K_q of abelian functions is separably algebraic over $\Omega(\Phi_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r})$, and thus any derivation of $\Omega(\Phi_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r})$ is uniquely extended to a derivation of K_q.

By virtue of Lemma 4 $D_1', \ldots, D_r'$ map $\Omega(\Phi_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r})$ into itself, and thus the restrictions $D_1^*, \ldots, D_r^*$ of $D_1', \ldots, D_r'$ on $\Omega(\Phi)_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r})$ are derivations of $\Omega(\Phi_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r})$. We denote by $D_1, \ldots, D_r$ the extensions of $D_1^*, \ldots, D_r^*$ to K_q, respectively. Since $D_i(\Phi_{p^{-1}\mathfrak{e}_i}) = \delta_{ij}\Phi_{p^{-1}\mathfrak{e}_j}(u)$, $(1 \leq i \leq r)$, $\{D_1, \ldots, D_r\}$ is a K_q-base of derivations of K_q/Ω.

For any multiplicative function χ on $\mathfrak{M}_Q$[4] we denote by T_χ the mapping

[4] We mean by a multiplicative function x on $\mathfrak{M}_Q$ a function on M_Q valued in the multiplicatnie group $\Omega^\times = \Omega - \{0\}$ of Ω such that $\chi(\mathfrak{m}+\mathfrak{n}) = \chi(\mathfrak{m})\chi(\mathfrak{n})$, $(\mathfrak{m},\ \mathfrak{n} \in \mathfrak{M}_Q)$.

of K_q defined by

$$(T_\chi f)(u) = f(u\chi^{-1}), \qquad (f \in K_q).$$

T_χ is extended to $K_q(u)$ such that $T_\chi u(\mathfrak{m}) = u(\mathfrak{m})\chi^{-1}(\mathfrak{m})$.

If a derivation D of K_q/Ω satisfies

$$T_{\chi^{-1}} \circ D \circ T_\chi = D \qquad (\text{for every } \chi), \tag{25}$$

we call D an invariant derivation of K_q/Ω. If a differential $\omega = \sum_{i=1}^{r} f_i(u) dg_i(u)$ of K_q/Ω satisfies

$$\sum_{i=1}^{r} f_i(u)\, dg_i(u) = \sum_{i=1}^{r} f_i(u\chi^{-1}) dg_i(u\chi^{-1}) \qquad (\text{for every } \chi), \tag{25'}$$

we call ω an invariant differential (of degree one).

THEOREM 3. *$\{D_1, \ldots, D_r\}$ is a base of invariant derivations of K_q/Ω such that*

$$D_i(\Phi_{p^{-1}\mathfrak{e}_j}) = \delta_{ij}\Phi_{p^{-1}\mathfrak{e}_j}, \qquad (1 \le i, j \le r). \tag{26}$$

Proof. The last assertion is nothing but Lemma 4. We shall show that D_i is invariant. For any χ it follows

$$\begin{aligned}(T_{\chi^{-1}} \circ D_i' \circ T_\chi)(u(\mathfrak{m})) &= T_{\chi^{-1}} \circ D_i'(\chi(\mathfrak{m})^{-1} u(\mathfrak{m})) \\ &= T_{\chi^{-1}}\chi(\mathfrak{m})^{-1}\chi_i(\mathfrak{m})\, u(\mathfrak{m}) \\ &= x_i(\mathfrak{m})\, u(\mathfrak{m}) = D_i'(u(\mathfrak{m})).\end{aligned}$$

Hence we have $(T_{\chi^{-1}} \circ D_i' \circ T_\chi)(f) = D_i(f)$ for every $f \in K_q$, and thus D_i is an invariant derivation of K_q/Ω. Conversely assume $D = \sum_{i=1}^{r} h_i(u) D_i$ is an invariant derivation of K_q/Ω. Then it follows

$$\begin{aligned}T_{\chi^{-1}} \circ D_i \circ T_\chi &= T_{\chi^{-1}} \circ \sum_{i=1}^{r} h_i(u) D_i \circ T_\chi \\ &= \sum_{i=1}^{r} h(u\chi)(T_{\chi^{-1}} \circ D_i \circ T) = \sum_{i=1}^{r} h_i(u\chi) D_i.\end{aligned}$$

Since $\{D_1, \ldots, D_r\}$ is a K_q-base of all the derivations of K_q/Ω, we have $h_i(u\chi) = h_i(u)$ $(1 \le i \le r)$ for any χ. This shows that $\{h_i(u)\}$ are constants $\{c_i\}$ in Ω, and thus we conclude that $\{D_1, \ldots, D_r\}$ is a base of invariant derivations of K_q/Ω.

Let us now translate Theorem 3 in the language of invariant differentials.

THEOREM 4. *$\{\Phi_{p^{-1}\mathfrak{e}_1}^{-1} d\Phi_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r}^{-1} d\Phi_{p^{-1}\mathfrak{e}_r}\}$ is a base of invariant differ-*

entials of K_q/Ω.

As a consequence of Theorem 3 we can characterize invariant derivations as follows:

THEOREM 5. *A derivation* D *of* K_q/Ω *is an invariant derivation if and only if* $D(\Phi_{p^{-1}e_i}) = c_i\Phi_{p^{-1}e_i}$, $(1\leq i\leq r)$, *with constants* $\{c_i\}$ *in* Ω.

Proof. Assume $D(\Phi_{p^{-1}e_j}) = c_i\Phi_{p^{-1}e_i}$ with $c_i\in\Omega$, $(1\leq i\leq r)$. Then $\left(D-\sum_{i=1}^{r}c_i D_i\right)(\Phi_{p^{-1}e_j})=0$, $(1\leq j\leq r)$. Since $K_q/\Omega(\Phi_{p^{-1}e_1},\ldots,\Phi_{p^{-1}e_r})$ is separably algebraic, we have $D=\sum_{i=1}^{r}c_iD_i$. Conversely if D is an invariant derivation of K_q/Ω. Then by virtue of Theorm 3 we have $D=\sum_{i=1}^{r}c_iD_i$ with c_i in Ω, and thus $D(\Phi_{p^{-1}e_i})=c_i$, $(1\leq i\leq r)$.

§ 3. Υ-functions of characteristic $p>0$ and p-Kummer generators

3.1. Let $\{D_1,\ldots,D_r\}$ be the canonical base of invariant derivation of K_q/Ω in § 2 such that $D_i(\Phi_{p^{-1}e_i}(q\mid u)) = \delta_{ij}\Phi_{p^{-1}e_j}(q\mid u)$, $(1\leq i,\ j\leq r)$, where we indicate q in $\Phi_{p^{-1}e_i}(q\mid u)$ explicitely instead of $\Phi_{p^{-1}e_i}(u)$ in § 2. We shall denote by $D_{\mathfrak{g}_l}$ the invariant derivation defined by

$$D_{\mathfrak{g}_l}=\sum_{i=1}^{r}x_i(p\mathfrak{g}_l)D_i,\qquad (1\leq l\leq p^r). \tag{27}$$

Let $D'_1,\ldots,D'_r$ be the derivations in § 2 such that $D'_i(u(\mathfrak{m}))=x_i(\mathfrak{m})u(\mathfrak{m})$ for $\mathfrak{m}\in\mathfrak{M}$, and $D'_{\mathfrak{g}_l}$ be the derivation $\sum_{i=1}^{r}x_i(p\mathfrak{g}_l)D'_i$. We shall denote by q^p the positive symmetric bimultiplicative function on $\mathfrak{M}_Q$ defined by $q^p(\mathfrak{m},\mathfrak{n})=q(p\mathfrak{m},\mathfrak{n})$, $(\mathfrak{m},\mathfrak{n}\in\mathfrak{M}_Q)$. We shall denote by $D'_{\mathfrak{g}}(\vartheta[0,0])(q\mid u)$ the $D'_{\mathfrak{g}}$-derivative of $\vartheta[0,0](q\mid u)$. Following Bolza,[3] we shall define Υ-functions as follows:

$$\Upsilon_{\mathfrak{g}_i,\mathfrak{g}_j}(q\mid u) = -D'_{\mathfrak{g}_i}\left(\frac{D'_{\mathfrak{g}_j}(\vartheta[0,0])(q\mid u)}{\vartheta[0,0](q\mid u)}\right) \tag{28}$$

$$=\frac{D'_{\mathfrak{g}_i}(\vartheta[0,0])(q\mid u)D'_{\mathfrak{g}_j}(\vartheta[0,0])(q\mid u)-D'_{\mathfrak{g}_i}D'_{\mathfrak{g}_j}(\vartheta[0,0])(q\mid u)\vartheta[0,0](q\mid u)}{\vartheta[0,0](q\mid u)^2},$$

$$(1\leq i,\ j\leq p^r).$$

We shall first show that $\Upsilon_{\mathfrak{g}_i,\mathfrak{g}_j}(q\mid u)$ is an abelian function in K_q.

LEMMA 1. $D'_{\mathfrak{g}_i}(\vartheta[0,0])(q\mid u)\,D'_{\mathfrak{g}_j}(\vartheta[0,0])(q\mid u)$
$-D'_{\mathfrak{g}_i}D'_{\mathfrak{g}_j}(\vartheta[0,0])(q\mid u)\,\vartheta[0,0](q\mid u)$, $\quad(1\leq i,\ j\leq p^r)$,

are theta functions of type (2, 1).

Proof. We denote briefly by $\{\phi_{\mathfrak{g}_i,\mathfrak{g}_j}(u)\}$ the quantities in Lemma 1. Since $\phi_{\mathfrak{g}_i,\mathfrak{g}_j}$ is bilinear in $\mathfrak{g}_i$ and $\mathfrak{g}_j$, it is sufficient to prove that $\phi_{p^{-1}\mathfrak{e}_i,p^{-1}\mathfrak{e}_j}$, $(1\leq i, j\leq r)$, are theta functions of type (2,1). Since $D_i'(u(\mathfrak{m}))=x_i(\mathfrak{m})u(\mathfrak{m})$, $(\mathfrak{m}\in\mathfrak{M})$, for every $\mathfrak{n}\in\mathfrak{M}$ we have

$$\begin{aligned}D_i'(\vartheta[0,\ 0])(q\,|\,q(\mathfrak{n})u)&=\sum_{\mathfrak{m}\in\mathfrak{M}}x_i(\mathfrak{m})\,q(\mathfrak{m},\ \mathfrak{m})\,q(\mathfrak{n},\ \mathfrak{m})^2u(\mathfrak{m})\\&=q(\mathfrak{n},\ \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\{\sum_{\mathfrak{m}\in\mathfrak{M}}x_i(\mathfrak{m}+\mathfrak{n})\,q(\mathfrak{m}+\mathfrak{n},\ \mathfrak{m}+\mathfrak{n})\,u(\mathfrak{m}+\mathfrak{n})\\&\quad-x_i(\mathfrak{n})\sum_{\mathfrak{m}\in\mathfrak{M}}q(\mathfrak{m}+\mathfrak{n},\ \mathfrak{m}+\mathfrak{n})\,u(\mathfrak{m}+\mathfrak{n})\}\\&=q(\mathfrak{n},\ \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\{D_i'(\vartheta[0,\ 0])(q\,|\,u)-x_i(\mathfrak{n})\vartheta[0,\ 0](q\,|\,u)\},\end{aligned}$$

and

$$\begin{aligned}D_i'D_j'(\vartheta[0,\ 0])(q\,|\,q(\mathfrak{n})u)&=\sum_{\mathfrak{m}\in\mathfrak{M}}x_i(\mathfrak{m})\,x_j(\mathfrak{m})\,q(\mathfrak{m},\ \mathfrak{m})\,q(\mathfrak{n},\ \mathfrak{m})^2u(\mathfrak{m})\\&=q(\mathfrak{n},\ \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\{\sum_{\mathfrak{m}\in\mathfrak{M}}x_i(\mathfrak{m}+\mathfrak{n})\,x_j(\mathfrak{m}+\mathfrak{n})\}\,q(\mathfrak{m}+\mathfrak{n},\ \mathfrak{m}+\mathfrak{n})\,u(\mathfrak{m}+\mathfrak{n})\\&\quad-x_i(\mathfrak{n})\sum_{\mathfrak{m}\in\mathfrak{M}}x_j(\mathfrak{m}+\mathfrak{n})\,q(\mathfrak{m}+\mathfrak{n},\ \mathfrak{m}+\mathfrak{n})\,u(\mathfrak{m}+\mathfrak{n})\\&\quad-x_j(\mathfrak{n})\sum_{\mathfrak{m}\in\mathfrak{M}}x_i(\mathfrak{m}+\mathfrak{n})\,q(\mathfrak{m}+\mathfrak{n},\ \mathfrak{m}+\mathfrak{n})\,u(\mathfrak{m}+\mathfrak{n})\\&\quad+x_i(\mathfrak{n})\,x_j(\mathfrak{n})\sum_{\mathfrak{m}\in\mathfrak{M}}q(\mathfrak{m}+\mathfrak{n},\ \mathfrak{m}+\mathfrak{n})\,u(\mathfrak{m}+\mathfrak{n})\}\\&=q(\mathfrak{n},\ \mathfrak{n})^{-1}u(\mathfrak{n})^{-1}\{D_i'D_j'(\vartheta[0,\ 0])(q\,|\,u)-x_i(\mathfrak{n})D_j'(\vartheta[0,\ 0])(q\,|\,u)\\&\quad-x_j(\mathfrak{n})D_i'(\vartheta[0,\ 0])(q\,|\,u)+x_i(\mathfrak{n})\,x_j(\mathfrak{n})\,\vartheta[0,\ 0](q\,|\,u)\}.\end{aligned}$$

This shows that for every $\mathfrak{n}\in\mathfrak{M}$

$$\phi_{p^{-1}\mathfrak{e}_i,p^{-1}\mathfrak{e}_j}(q\,|\,q(\mathfrak{n})u)=q(\mathfrak{n},\ \mathfrak{n})^{-2}u(\mathfrak{n})^{-2}\phi_{p^{-1}\mathfrak{e}_i,p^{-1}\mathfrak{e}_j}(q\,|\,u),\qquad(1\leq i,\ j\leq r).$$

Hence we have proved Lemma 1.

Dividing $\phi_{\mathfrak{g}_i,\mathfrak{g}_j}(q\,|\,u)$ by $\vartheta[0,\ 0](q\,|\,u)^2$, we have

PROPOSITION 1. $\Upsilon_{\mathfrak{g}_i,\mathfrak{g}_j}(q\,|\,u)$, $(1\leq i,\ j\leq p^r)$, *are abelian functions in* K_q *such that* $\vartheta[0,\ 0](q\,|\,u)^2\ \Upsilon_{\mathfrak{g}_i,\mathfrak{g}_j}(q\,|\,u)$ *are theta functions of type* (2,1).

We denote by $\Upsilon_{\mathfrak{g}_{i}\cdots\mathfrak{g}_{i_s}}(q\,|\,u)$ the higher derivatives $D_{\mathfrak{g}_i}\cdots D_{\mathfrak{g}_{i_{s-2}}}(\Upsilon_{\mathfrak{g}_{i_{s-1}},\mathfrak{g}_{i_s}})$, $(1\leq i_1,\ \ldots,\ i_s\leq p^r)$. Since $D_{\mathfrak{g}_i}'(f)=D_{\mathfrak{g}_i}(f)$ for every f in $\Omega(\Phi_{p^{-1}\mathfrak{e}_1},\ \ldots,\ \Phi_{p^{-1}\mathfrak{e}_r})$, $\Upsilon_{\mathfrak{g}_{i_1}\cdots\mathfrak{g}_{i_s}}(q\,|\,u)$ is independent of the order of the indecis $i_1,\ \ldots,\ i_s$.

COROLLARY. $\Upsilon_{\mathfrak{g}_{i_1}\cdots\mathfrak{g}_{i_s}}(q\,|\,u)$, $(1\leq i_1,\ \ldots,\ i_s\leq p^r;\ s\geq 2)$, *are abelian functions in* K_q *such that* $\vartheta[0,\ 0](q\,|\,u)^s\ \Upsilon_{\mathfrak{g}_{i_1}\cdots\mathfrak{g}_{i_s}}(q\,|\,u)$ *are theta functions of type* (s, 1).

Proof. Since $\Upsilon_{\mathfrak{g}_{i_{s-1}}\mathfrak{g}_{i_s}}$ is an abelian function in K_q such that $\phi_{\mathfrak{g}_{i_{s-1}}\mathfrak{g}_{i_s}}(u)=\vartheta[0,\ 0](q\,|\,u)^2\Upsilon_{\mathfrak{g}_{i_{s-1}}\mathfrak{g}_{i_s}}(q\,|\,u)$ is a theta function of type (2,1), we see that

$\mathfrak{r}_{\mathfrak{g}_{i_1},\ldots,\mathfrak{g}_{i_s}}$ is an abelian function in K_q and $\vartheta[0, 0](q\,|\,u)^s \mathfrak{r}_{\mathfrak{g}_{i_1}\cdots\mathfrak{g}_{i_s}}(q\,|\,u) = \vartheta[0, 0](q\,|\,u)^s(D_{\mathfrak{g}_{i_1}} D_{\mathfrak{g}_{i_2}} \cdots D_{\mathfrak{g}_{i_{s-2}}})(\vartheta[0, 0](q\,|\,u)^{-2}\phi_{\mathfrak{g}_{i_{s-1}}\mathfrak{g}_{i_s}}(u))$ is expressed as a series $\sum_{\mathfrak{m}\in\mathfrak{M}} c_{\mathfrak{m}} u(\mathfrak{m})$. Hence $\vartheta[0, 0](q\,|\,u)^s \mathfrak{r}_{\mathfrak{g}_{i_1}\cdots\mathfrak{g}_{i_s}}(q\,|\,u)$ is a theta function of type $(s, 1)$. This complete the proof of Corollary.

3.2. We shall now prove the following simple formula on a derivation of a field of characteristic p, and shall apply it to the invariant derivations of K_q and the theta function $\vartheta[0, 0](q\,|\,u)$.

LEMMA 2. *If D is a derivation of a field of characteristic p, we have*

$$D^{p-1}\left(\frac{D(y)}{y}\right) = \frac{D^p(y)}{y} - \frac{D(y)^p}{y^p} \tag{29}$$

for every non-zero y.

Proof. It is sufficient to prove for y such that $y, D(y), \ldots, D^p(y)$ are independent over the prime field. Since $y^p(D^{p-1}(y^{-1}D(y)))$ is a polynomial in $y, D(y), \ldots, D^p(y)$, we may put

$$D^{p-1}\left(\frac{D(y)}{y}\right) = \frac{D^p(y)}{y} + \frac{\sum_{i_1\le\cdots\le i_p} c_{i_1\ldots i_p} D^{i_1}(y)\cdots D^{i_p}(y)}{y^p}\,.$$

where the summation $\sum_{i_1\le\cdots\le i_p}$ runs over $\{i_1, \ldots, i_p\,|\,0\le i_1\le\cdots\le i_p,\ \sum_{l=1}^{p} i_l = p\}$. Since D^p is also a derivation, operating D on $D^{p-1}(y^{-1}D(y))$ and $y^{-1}D^p(y)$, we have

$$D\left(D^{p-1}\left(\frac{D(y)}{y}\right)\right) = D^p\left(\frac{D(y)}{y}\right) = \frac{D^{p+1}(y)y - D(y)D^p(y)}{y^2}$$

and

$$D\left(\frac{D^p(y)}{y}\right) = \frac{D^{p+1}(y) - D^p(y)D(y)}{y^2}$$

This shows that

$$D\Big(\sum_{i_1\le\cdots\le i_p} c_{i_1\ldots i_p} D^{i_1}(y)\cdots D^{i_p}(y)\Big) = 0.$$

Let us introduce the lexical order in the set $\{(i_1, \ldots, i_p)\,|\,0\le i_1\le\cdots\le i_p\le p;\ \sum_{l=1}^{p} i_l = p\}$. Let $D^{j_1}(y)\cdots D^{j_p}(y)$ be the first term in the lexical order such that $c_{j_1\cdots j_p}\neq 0$. Then $D^{j_1}(y)\cdots D^{j_{p-1}}(y)D^{j_p+1}(y)$ is the first term to appear in $D(\sum_{i_1\le\cdots\le i_p} c_{i_1\ldots i_p} D^{i_1}(y)\cdots D^{i_p}(y))$. Since $D(\sum_{i_1\le\cdots\le i_p} c_{i_1\ldots i_p} D^{i_1}(y)\cdots D^{i_p}(y)) = 0$, the multiplicity of $D^{j_p}(y)$ in $D^{j_1}(y)\cdots D^{j_p}(y)$ must be divided by p. On the othe hand $\sum_{l=1}^{p} j_l = p$, hence $D^{j_1}(y)\cdots D^{j_p}(y) = D(y)^p$. Since $D(y)^p$ is the first term in $\sum_{i_1\leq\cdots\leq i_p} c_{i_1\ldots i_p} D^{i_1}(y)\cdots D^{i_p}(y)$, other terms $D^{i_1}(y)\cdots D^{i_p}(y)$ satisfy $1\leq i_1\leq\cdots\leq i_p$. From $\sum_{l=1}^{p} i_l = p$ it follows $\sum_{i_1\leq\cdots\leq i_p} c_{i_1\ldots i_p} D^{i_1}(y)\cdots D^{i_p}(y) = cD^p(y)$ with a constant c. By the simple culculation we see $c = (-1)^{p-1}(p-1)!$. Since $(p-1)! \equiv -1 \bmod p$, we have $c = -1$. This shows

$$D^{p-1}\left(\frac{D(y)}{y}\right) = \frac{D^p(y)y^{p-1} - D(y)^p}{y^p}$$

Lemma 3. *If $D^p = D$, we have*

$$-D^{p-1}\left(\frac{D(y)}{y}\right) = \left(\frac{D(y)}{y}\right)^p - \frac{D(y)}{y} \tag{30}$$

This is an immediate consequence from Lemma 2.

3.3. Let us now apply the formula (29) to $\vartheta[0, 0](q|u)$ and $D'_{\mathfrak{g}_l}$, $(1\leq l\leq p^r)$.

Lemma 4. $D^p_{\mathfrak{g}_l} = D_{\mathfrak{g}_l}$, $D'^p_{\mathfrak{g}_l} = D'_{\mathfrak{g}_l}$, $(1\leq l\leq p^r)$.

Proof. Since $D_i(\Phi_{p^{-1}\mathfrak{e}_j}(q|u)) = \delta_{ij}\Phi_{p^{-1}\mathfrak{e}_j}(q|u)$, we have $D_i^p(\Phi_{p^{-1}\mathfrak{e}_j}(q|u)) = \delta_{ij}\Phi_{p^{-1}\mathfrak{e}_j}(q|u) = D_i(\Phi_{p^{-1}\mathfrak{e}_j}(q|u))$, $(1\leq i, j\leq p^r)$. This shows that the derivations $D_i^p - D_i$ vanish on $\Omega(\Phi_{p^{-1}\mathfrak{e}_1}, \ldots, \Phi_{p^{-1}\mathfrak{e}_r})$, and thus they vanish on K_q, Hence $D_i^p = D_i$, $(1\leq i\leq r)$. Therefore $D^p_{\mathfrak{g}_l} = \left(\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)D_i\right)^p = \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)^p D_i^p = \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)D_i = D_{\mathfrak{g}_l}$. Since D'_i can be considered the extension of D_i to the derivation of $K_q(u)$ such that $D'_i(u(\mathfrak{m})) = x_i(\mathfrak{m})u(\mathfrak{m})$ for $\mathfrak{m}\in\mathfrak{M}$, by the same reason as D_i we have $D'^p_i = D'_i$, $(1\leq i\leq r)$, and thus $D'^p_{\mathfrak{g}_l} = D'_{\mathfrak{g}_l}$.

Let us now culculate $D'_{\mathfrak{g}_l}(\vartheta[0, 0])(q|u)$.

Lemma 5.

$$D'^{\nu}_{\mathfrak{g}_l}(\vartheta[0, 0])(q|u) = \sum_{h=1}^{p^r}\left(\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_h)^{\nu}\right)\vartheta_p[\mathfrak{g}_h, 0](q^p|u), \tag{31}$$

$$D'^{\nu(p-1)}_{\mathfrak{g}_l}(\vartheta[0, 0])(q|u) = D'^{\nu}_{\mathfrak{g}_l}(\vartheta[0, 0])(q|u), \tag{32}$$

$$(1\leq 1\leq p^r;\ \nu = 0, 1, 2, \ldots).$$

Proof. (32) is an immediate consequence from (31). Since $D_i(u(\mathfrak{m}') = x_i(\mathfrak{m})u(\mathfrak{m})$ for $\mathfrak{m}\in\mathfrak{M}$ and $D'_{\mathfrak{g}_l} = \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)D'_i$, it follows

$$D'^{\nu}_{\mathfrak{g}_l}(\vartheta[0,0])(q\,|\,u)$$
$$=\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)D'^{\nu}_i\Big(\sum_{h=1}^{p^r}\sum_{\mathfrak{m}\in\mathfrak{M}} q(p(\mathfrak{m}+\mathfrak{g}_h),\ p(\mathfrak{m}+\mathfrak{g}_h))u(p(\mathfrak{m}+\mathfrak{g}_h))\Big)$$
$$=\sum_{h=1}^{p^r}\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\,x_i(p\mathfrak{g}_h)^{\nu}\sum_{\mathfrak{m}\in\mathfrak{M}} q(p(\mathfrak{m}+\mathfrak{g}_h),\ p(\mathfrak{m}+\mathfrak{g}_h))u(p(\mathfrak{m}+\mathfrak{g}_h))$$
$$=\sum_{h=1}^{p^r}\Big(\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\,x_i(p\mathfrak{g}_h)^{\nu}\Big)\vartheta_p[\mathfrak{g}_h,\ 0](q^p\,|\,u).$$

LEMMA 6. $\{D'^{\nu}_i(\vartheta[0,0])(q\,|\,u)\,|\,1\le i\le r;\ 0\le\nu\le p-1\}$ *form a base of theta functions of type* $(p,1)$ *with period* q^p.

Proof. By virtue of Lemma 5 we see

$$D'^{\nu}_i(\vartheta[0,0])(q\,|\,u)=\sum_{h=1}^{p^r} x_i(p\mathfrak{g}_h)\vartheta_p[\mathfrak{g}_h,\ 0](q^p\,|\,u).$$

Since the (p^r, p^r)-matrix $(x_i(p\mathfrak{g}_h)^{\nu})$ of which $((i,\nu),h)$-element is $x_i(p\mathfrak{g}_h)^{\nu}$ is non-singular, we see that $\{D'^{\nu}_i(\vartheta[0,0])(q\,|\,u)\}$ form a base of theta function of type $(p, 1)$ with period q^p.

Since $\vartheta[0,0](q\,|\,u)=\sum_{h=1}^{p^r}\vartheta_p[\mathfrak{g}_h,\ 0](q^p\,|\,u)$, putting $D=D_{\mathfrak{g}_h}$ and $y=\vartheta[0,0](q\,|\,u)$ in (30), we have

THEOREM 6.

$$\gamma_{\underbrace{\mathfrak{g}_l\ldots\mathfrak{g}_l}_{p}}(q\,|\,u)=X_{\mathfrak{g}_l}(q^p\,|\,u)^p-X_{\mathfrak{g}_l}(q^p\,|\,u),\tag{33}$$

where

$$X_{\mathfrak{g}_l}(q^p\,|\,u)=\frac{\sum_{h=1}^{p^r}\Big(\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\,x_i(p\mathfrak{g}_h)\Big)\vartheta_p[\mathfrak{g}_h,\ 0](q^p\,|\,u)}{\sum_{h=1}^{p^r}\vartheta_p[\mathfrak{g}_h,\ 0](q^p\,|\,u)},\tag{34}$$

$(1\le l\le p^r)$.

COROLLARY 1.

$$\gamma_{\underbrace{\mathfrak{g}_l\ldots\mathfrak{g}_l}_{p+\nu}}(q\,|\,u)=\gamma_{\underbrace{\mathfrak{g}_l\ldots\mathfrak{g}_l}_{\nu+1}}(q\,|\,u),\qquad(1\le l\le p^r;\ \nu=1,2,\ldots).\tag{35}$$

Proof. It is sufficient to prove

$$\gamma_{\underbrace{\mathfrak{g}_l\ldots\mathfrak{g}_l}_{p+1}}(q\,|\,u)=\gamma_{\mathfrak{g}_l\mathfrak{g}_l}(q\,|\,u),\qquad(1\le l\le p^r).\tag{36}$$

Since $\gamma_{\underbrace{\mathfrak{g}_l\ldots\mathfrak{g}_l}_{p+1}}(q\,|\,u)=D_{\mathfrak{g}_l}(\gamma_{\underbrace{\mathfrak{g}_l\ldots\mathfrak{g}_l}_{p}}(q\,|\,u))$, by virtue of Theorem 6, we have

$$\underbrace{\mathfrak{r}_{\mathfrak{g}_l \ldots \mathfrak{g}_l}}_{p+1}(q\,|\,u) = D'_{\mathfrak{g}_l}\Big(\frac{D'_{\mathfrak{g}_l}(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)} - \Big(\frac{D'_{\mathfrak{g}_l}(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}\Big)^p\Big)$$

$$= D'_{\mathfrak{g}_l}\Big(\frac{D'_{\mathfrak{g}_l}(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}\Big) = \mathfrak{r}_{\mathfrak{g}_l \mathfrak{g}_l}(q\,|\,u).$$

COROLLARY 2.

(37) $$\underbrace{\mathfrak{r}_{\mathfrak{g}_l \ldots \mathfrak{g}_l}}_{p}(q\,|\,u) = \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\, \mathfrak{r}_{\underbrace{p^{-1}\mathfrak{e}_i \ldots p^{-1}\mathfrak{e}_i}_{p}}(q\,|\,u). \qquad (l \leq 1 \leq p^r).$$

Proof. By virtue of Theorem 6, we have

$$\underbrace{\mathfrak{r}_{\mathfrak{g}_l \ldots \mathfrak{g}_l}}_{p}(q\,|\,u) = \Big(\frac{D'_{\mathfrak{g}_l}(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}\Big)^p - \frac{D'_{\mathfrak{g}_l}(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}$$

$$= \left(\frac{\sum_{i=1}^{r} x_i(p\mathfrak{g}_l) D'_i(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}\right)^p - \frac{\sum_{i=1}^{r} x_i(p\mathfrak{g}_l) D'_i(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}$$

$$= \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)^p\, \mathfrak{r}_{\underbrace{p^{-1}\mathfrak{e}_i \ldots p^{-1}\mathfrak{e}_i}_{p}}(q\,|\,u).$$

Since $x_i(p\mathfrak{g}_l)$, $(1 \leq i \leq r)$, belong to the prime-field $GF(p)$,, we get (37).

We shall explain the meaning of the formula (33) in theory of separable unramified covering of the abelian variety A_q with period $(E,\ q)$.

THEOREM 7.

(38) $$X_{\mathfrak{g}_l}(q^p\,|\,q^p(\mathfrak{g}_h)u) = X_{\mathfrak{g}_l}(q^p\,|\,u) - \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_h), \qquad (1 \leq 1,\ h \leq p^r).$$

Proof. From (34) it follows

$$X_{\mathfrak{g}_l}(q^p\,|\,q^p(\mathfrak{g}_h)u) = \frac{\sum_{i=1}^{r}\sum_{k=1}^{p^r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_h)\, \vartheta_p[\mathfrak{g}_h,\ 0](q^p\,|\,q^p(\mathfrak{g}_h)u)}{\vartheta[0,\ 0](q\,|\,q^p(\mathfrak{g}_h)u)}$$

$$= \frac{\sum_{i=1}^{r}\sum_{k=1}^{p^r} x_i(p\mathfrak{g}_l) x_i(p\mathfrak{g}_k) \sum_{\mathfrak{m} \in \mathfrak{M}} q(p(\mathfrak{m}+\mathfrak{g}_k),\ p(\mathfrak{m}+\mathfrak{g}_k)) q(p\mathfrak{g}_h,\ p(\mathfrak{m}+\mathfrak{g}_k) u(p(\mathfrak{m}+\mathfrak{g}_k))}{\vartheta[0,\ 0](q\,|\,u)}$$

$$= \frac{\sum_{i=1}^{r}\sum_{k=1}^{p^r} x_i(p\mathfrak{g}_l) x_i(p\mathfrak{g}_k) q(p\mathfrak{g}_h,\ p\mathfrak{g}_h)^{-1} u(p\mathfrak{g}_h)^{-1} \vartheta_p[\mathfrak{g}_k + \mathfrak{g}_h,\ 0](q^p\,|\,u)}{q(p\mathfrak{g}_h,\ p\mathfrak{g}_h)^{-1} u(p\mathfrak{g}_h)^{-1} \vartheta[0,\ 0](q\,|\,u)}$$

$$= \frac{\sum_{i=1}^{r}\sum_{k=1}^{p^r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_k - p\mathfrak{g}_h)\, \vartheta_p[\mathfrak{g}_k,\ 0]\,(q^p\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}$$

$$= \frac{\sum_{i=1}^{r}\sum_{k=1}^{p^r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_k)\, \vartheta_p[\mathfrak{g}_k,\ 0](q^p\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}$$

$$-\left(\sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_h)\right)\frac{\sum_{k=1}^{p^r}\vartheta_p[\mathfrak{g}_k,\ 0](q^p\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}$$
$$= X_{\mathfrak{g}_l}(q^p\,|\,q(\mathfrak{g}_h)u) - \sum_{i=1}^{r} x_i(p\mathfrak{g}_l)\, x_i(p\mathfrak{g}_h).$$

We denote by K_{q^p} the field of abelian functions with coefficients in Ω and with period q^p, and denote by A_{q^p} the abelian variety with period q_p. Then from Theorem 6 and 7 it follows

Theorem 8. *K_{q^p}/K_q is a separable abelian extension of type $(\overbrace{p, \ldots, p}^{r})$ generated by the quantities $X_{p^{-1}\mathfrak{e}_1}(q_p\,|\,u), \ldots, X_{p^{-1}\mathfrak{e}_r}(q^p\,|\,u)$ satisfying*

$$X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u)^p - X_{p^{-1}\mathfrak{e}_i}(q\,|\,u) = \Upsilon_{\underbrace{p^{-1}\mathfrak{e}_i,\ldots,p^{-1}\mathfrak{e}_i}_{p}}(q\,|\,u),$$
$$X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,q^p(\mathfrak{g}_l)u) = X_{p^{-1}\mathfrak{e}_i}(q^q\,|\,u) - x_i(p\mathfrak{g}_l),$$
$$(1\leq i\leq r;\ 1\leq l\leq p^r).$$

Proof. Let $\{X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u)\}$ be the quantities in Theorem 6. It is sufficient to prove $K_{q^p}=K_q(X_{p^{-1}\mathfrak{e}_1}(q^p\,|\,u), \ldots, X_{p^{-1}\mathfrak{e}_r}(q^p\,|\,u))$. Since $X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u) = D_i'(\vartheta[0,\ 0])(q\,|\,u)\vartheta[0,\ 0](q\,|\,u)^{-1}$, we have

$$\frac{D_i'^2(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)} = D_i'\left(\frac{D_i'(\vartheta[0,\ 0](q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}\right) + \left(\frac{D_i'(\vartheta[0,\ 0])(q\,|\,u)}{\vartheta[0,\ 0](q\,|\,u)}\right)^2$$
$$= D_i'(X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u)) + X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u)^2$$
$$= -D_i'(\Upsilon_{p^{-1}\mathfrak{e}_i,\ldots,p^{-1}\mathfrak{e}_i}(q\,|\,u)) + X_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u)^2.$$

This shows that $D_i'^2(\vartheta[0,\ 0])(q\,|\,u)\vartheta[0,\ 0](q\,|\,u)^{-1}$, $(1\leq i\leq r)$, belong to $K_q(X_{p^{-1}\mathfrak{e}_1}(q^p\,|\,u), \ldots, X_{p^{-1}\mathfrak{e}_r}(q^p\,|\,u))$. Hence by virtue of Lemma 6 it follows that

$$\Phi_{p^{-1}\mathfrak{e}_i}(q^p\,|\,u) = \vartheta_p[p^{-1}\mathfrak{e}_i,\ 0](q^p\,|\,u)/\vartheta_p[0,\ 0](q^p\,|\,u),\ (1\leq i\leq r),$$

belong to $K_q(X_{p^{-1}\mathfrak{e}_1}(q^p\,|\,u), \ldots, X_{p^{-1}\mathfrak{e}_r}(q^p\,|\,u))$. By virtue of Theorem 3 in §3 (I) K_{q^p} is separable over $\Omega(\Phi_{p^{-1}\mathfrak{e}_1}(q^p\,|\,u), \ldots, \Phi_{p^{-1}\ r}(q^p\,|\,u))$. Hence $K_{q^p}/K_q(X_{p^{-1}\ 1}(q^p\,|\,u), \ldots, X_{p^{-1}\ r}(q^p\,|\,u))$ is separable, and thus K_{q^p}/K_q is seperable. On the other hand the natural homomorphism λ of A_{q^p} onto A_q induced by the identity map $\chi\to\chi$ of multiplicative functions on $\mathfrak{M}_Q$ has the kernel $\lambda^{-1}(0)$ with p^r-elements corresponding to the multiplicative functions $q^p(\mathfrak{g}_1), \ldots, q^p(\mathfrak{g}_{p^r})$. This means that the separable degree of K_{q^p} over K_q is

p^r, and thus $[K_{q^p}: K_q] = p^r$. By virtue of Theorem 7 it follows $[K_q(X_{p^{-1}\,1}(q^p\,|\,\boldsymbol{u}), \ldots, X_{p^{-1}\,r})q^p\,|\,\boldsymbol{u})) : K_q] \geq p^r$. Hence we conclude $K_{q^p} = K_q(X_{p^{-1}\,1}(q^p\,|\,\boldsymbol{u}), \ldots, X_{p^{-1}\,r}(q^p\,|\,\boldsymbol{u}))$.

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