

FINITE GROUPS WHICH CONTAIN A SELF-CENTRALIZING SUBGROUP OF ORDER 3

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Dedicated to RICHARD BRAUER on his sixtieth birthday

§1. Introduction

The polyhedral group (l, m, n) is defined in [3] by the presentation

$$(l, m, n) = \langle x, y, z \mid x^l = y^m = z^n = xyz = 1 \rangle.$$

It is known ([3] page 68) that (l, m, n) is finite if and only if

$$\frac{1}{l} + \frac{1}{m} + \frac{1}{n} > 1.$$

The groups $(2, 2, n)$ and $(1, n, n)$ are respectively the dihedral group of order $2n$ and the cyclic group of order n . Using the above mentioned criterion it can be shown that the list of finite polyhedral groups is completed by including

$$\mathfrak{A}_4 = (2, 3, 3), \mathfrak{S}_4 = (2, 3, 4) \text{ and } \mathfrak{A}_5 = (2, 3, 5).$$

Let G be a finite group. If C_1, C_2, C_3 are three conjugate classes of G which contain elements of order l, m, n respectively and if K_1, K_2, K_3 are the corresponding class sums in the group ring of G , a moment's reflection reveals that in order to compute the multiplicity of K_3 in K_1K_2 by group theoretic methods as distinct from character theoretic methods it is necessary to deal with factor groups of (l, m, n) . R. Brauer and K. A. Fowler [1] first realized the importance of this idea for studying finite groups. They were only concerned with the groups $(2, 2, n)$ but these were sufficient to prove some powerful results about groups of even order. Using the groups $(2, 2, n)$ this idea has been used by many authors in recent years and has proved very fruitful for

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the study of groups of even order. It is unlikely that knowledge about the other polyhedral groups can be utilized as widely as that for the groups $(2, 2, n)$. However, the other polyhedral groups can surely play a role in group theory which is not totally eclipsed by the groups $(2, 2, n)$.

The purpose of this paper is to illustrate how the above mentioned method can be used with the group $(3, 3, 3)$. By the result referred to above the group $(3, 3, 3)$ is infinite. However, it is manageable since, as is shown in section 2, it has an abelian commutator subgroup.

The following result will be proved in this paper.

THEOREM. *Let G be a finite group which contains a self-centralizing subgroup of order 3. Then one of the following statements is true.*

(I) *G contains a nilpotent normal subgroup N such that G/N is isomorphic to either \mathfrak{A}_3 or \mathfrak{S}_3 .*

(II) *G contains a normal subgroup N which is a 2-group such that G/N is isomorphic to \mathfrak{A}_5 .*

(III) *G is isomorphic to $PSL(2, 7)$.*

As an immediate consequence of this theorem we get

COROLLARY. *Let G be a non-cyclic simple group which contains a self-centralizing subgroup of order 3. Then G is isomorphic to \mathfrak{A}_5 or $PSL(2, 7)$.*

If A is a subset of the group G then $C(A)$, $N(A)$, $\langle A \rangle$, $|A|$ will denote respectively the centralizer of A , normalizer of A , group generated by A and the number of elements in A . $H \triangleleft G$ means that H is a normal subgroup of G . If p is a prime then a S_p subgroup of G is a Sylow p -subgroup of G . Elements of order two are called involutions. For any subgroup H of G , 1_H denotes the principal character of H . If α is a class function of H then α^* denotes the class function of G induced by α .

§2. The Group $(3, 3, 3)$.

THEOREM 1. *The group $(3, 3, 3)$ possesses a normal abelian subgroup of index 3.*

Proof. Let

$$(3, 3, 3) = \langle x, y \mid x^3 = y^3 = (xy)^3 = 1 \rangle.$$

The relation $(xy)^3 = 1$ can be rewritten as

$$xyx = y^{-1}x^{-1}y^{-1}.$$

Since $y^{-2} = y$ and $x^{-2} = x$ this implies that

$$xy^{-1}y^{-1}x = y^{-1}xxy^{-1}.$$

Hence xy^{-1} and $y^{-1}x$ commute. Conjugating this relation by x and x^2 yields

$$\begin{aligned} y^{-1}x \cdot x^{-1}y^{-1}x^{-1} &= x^{-1}y^{-1}x^{-1} \cdot y^{-1}x \\ x^{-1}y^{-1}x^{-1} \cdot xy^{-1} &= xy^{-1} \cdot x^{-1}y^{-1}x^{-1}. \end{aligned}$$

Thus $H = \langle xy^{-1}, y^{-1}x, x^{-1}y^{-1}x^{-1} \rangle$ is abelian. Since x permutes the three elements xy^{-1} , $y^{-1}x$, $x^{-1}y^{-1}x^{-1}$ cyclically x normalizes H . Hence y also normalizes H as $xy^{-1} \in H$. Thus H is a normal subgroup of $(3, 3, 3)$. Since $(3, 3, 3)$ can be mapped homomorphically onto a non abelian group of order 27, H is a proper subgroup. As $xy^{-1} \in H$ and $x \notin H$, H has index 3 as required.

§ 3. Proof of the Theorem

Throughout this section let G be a counter-example of minimum order to the theorem stated in section 1. Let x be an element of G such that $x^3 = 1$ and $C(x) = \langle x \rangle$. It is easily seen that $\langle x \rangle$ is a S_3 subgroup of G . We will eventually derive a contradiction from the assumed existence of G . This will be done in a series of Lemmas.

LEMMA 1. *G is a non-cyclic simple group.*

Proof. Suppose this is not the case and let H be a minimal normal subgroup of G . Suppose that 3 divides $|H|$. Then the Sylow theorems imply that $G = N(\langle x \rangle)H$. Thus $[G:H] = 2$ and $N(\langle x \rangle) \cap H = \langle x \rangle$. Hence by Burnside's transfer theorem H contains a normal 3-complement H_0 . Thus $H_0 \triangleleft G$ and the minimality of H implies that $H_0 = 1$. Consequently G is isomorphic to \mathbb{S}_3 contrary to assumption.

Assume now that 3 divides $[G:H]$. Then $\langle x \rangle H$ is a Frobenius group. Thus H is nilpotent ([2], page 91). It is easily seen that G/H satisfies the hypotheses of the theorem stated in section 1. Thus by induction G/H satisfies condition (I), (II), or (III). Therefore G contains a normal subgroup N such that G/N is isomorphic to \mathcal{U}_3 , \mathbb{S}_3 , \mathcal{U}_6 or $PSL(2, 7)$. In any case $\langle x \rangle N$ is a Frobenius group and N is nilpotent ([2], page 91). If G/N is isomorphic to \mathcal{U}_3 or \mathbb{S}_3 nothing remains to be proved.

Let p be a prime dividing $|N|$. We will show that $p=2$ if G/N is isomorphic to \mathfrak{A}_5 while $|N|=1$ if G/N is isomorphic to $PSL(2, 7)$. By induction it may be assumed that N is an elementary abelian p -group. Suppose that A is a subgroup of G such that $A \cap N=1$, $A \cong A'$, $[A : A']=3$ and p does not divide $|A|$. Then A is a Frobenius group acting on N . Since x has no fixed points on N we get that A' acts trivially on N . Thus $N \subset C(N)$. Since $C(N) \triangleleft G$ and $\mathfrak{A}_5, PSL(2, 7)$ are simple, this yields that $C(N)=G$. Thus $N \subseteq C(x)$ or $N=1$. As both \mathfrak{A}_5 and $PSL(2, 7)$ contain a subgroup A which is isomorphic to \mathfrak{A}_4 this implies that $p=2$. As $PSL(2, 7)$ contains a nonabelian subgroup of order 21 we get that $N=1$ in this case. The proof is complete in all cases.

LEMMA 2. G contains only one conjugate class of elements of order three, and $|G|$ is even.

Proof. Lemma 1 and Burnside's transfer theorem imply that $N(\langle x \rangle) \cong \langle x \rangle$. Thus $|N(\langle x \rangle)|=6$. The result is immediate.

LEMMA 3. There exist exactly two non-principal irreducible characters θ, χ of G which do not vanish on x . They can be chosen so that $\theta(x)=1, \chi(x)=-1$ and $1+\theta(y)-\chi(y)=0$ for y not conjugate to x .

Proof. Let λ be a nonprincipal irreducible character of $\langle x \rangle$. Let α be the generalized character of $N(\langle x \rangle)$ induced by $\lambda_{\langle x \rangle}$. Then it is easily seen that $\|\alpha^*\|^2=3$ and

$$(1) \quad \begin{aligned} \alpha^*(x) &= \alpha(x) = 3 \\ \alpha^*(y) &= 0 \quad \text{for } y \text{ not conjugate to } x. \end{aligned}$$

Consequently $\alpha^* = 1_G + \theta - \chi$, where χ, θ are distinct nonprincipal irreducible characters of G . Furthermore $1 + \theta(y) - \chi(y) = 0$ for y non conjugate to x . Furthermore by (1)

$$1 = (\theta, \alpha^*) = \frac{1}{6} 3(\theta(x^{-1}) + \theta(x)) = \theta(x)$$

Thus by (1) $\chi(x) = -1$. Consequently

$$|C(x)| = 3 = 1 + |\theta(x)|^2 + |\chi(x)|^2.$$

Hence the orthogonality relations imply that every irreducible character of G ,

distinct from 1_G , χ and θ vanishes on x . The proof is complete.

The next lemma is due to R. Brauer and M. Suzuki. We are indebted to them for informing us of the result.

LEMMA 4. *G contains exactly one class of involutions.*

Proof. Any two involutions which normalize a subgroup of G of order 3 are conjugate. Suppose that G contains two classes of involutions then there is one class containing involutions such that uv is not conjugate to x for any u, v in that class. Let C be the group algebra sum of this class of involutions and let K be the group algebra sum of the elements of order 3. Thus the coefficient of K in C^2 is zero. Hence by a well-known formula ([2], page 316)

$$\frac{|G|}{|C(u)|^2} \left[\sum \frac{\zeta_i(u)^2 \overline{\zeta_i(x)}}{\zeta_i(1)} \right] = 0,$$

where ζ_i ranges over all the irreducible characters of G . In view of Lemma 3 this implies that

$$1 + \frac{\theta(u)^2}{\theta(1)} - \frac{\{\theta(u) + 1\}^2}{\theta(1) + 1} = 0.$$

Therefore

$$\theta(1)^2 + \theta(1) + \theta(1)\theta(u)^2 + \theta(u)^2 - \theta(1)\theta(u)^2 - 2\theta(1)\theta(u) - \theta(1) = 0,$$

or equivalently

$$\theta(1)^2 - 2\theta(1)\theta(u) + \theta(u)^2 = 0.$$

Thus $\{\theta(1) - \theta(u)\}^2 = 0$ and $\theta(1) = \theta(u)$. This implies that u lies in a proper normal subgroup of G contrary to Lemma 1. The proof is complete.

Throughout the rest of this paper the following notation will be used.

K is the group algebra sum of all elements of order 3 in G .

C is the group algebra sum of all involutions in G .

u is a fixed involution in G .

M_1, \dots, M_{s+m} is a complete set of representatives of the conjugate classes of maximal solvable subgroups of G whose order is divisible by 3. By induction each M_i contains a normal nilpotent subgroup N_i . The notation is chosen so that

$$M_i/N_i \text{ is isomorphic to } \mathfrak{A}_3 \quad \text{for } 1 \leq i \leq k$$

M_i/N_i is isomorphic to \mathfrak{S}_3 for $k < i \leq s + m$

where $|N_i|$ is odd for $k + 1 \leq i \leq s$ and $|N_i|$ is even for $s + 1 \leq i \leq s + m$.

Let $N_i = H_i \times T_i$, where $|H_i|$ is odd and T_i is a 2-group. Define

$$h_i = |H_i|, t_i = |T_i| \quad \text{for } 1 \leq i \leq s + m.$$

LEMMA 5. H_i is a Hall subgroup of G for $1 \leq i \leq s + m$ and $(h_i, h_j) = 1$ for $1 \leq i < j \leq s + m$. N_i is a Hall subgroup of G for $1 \leq i \leq k$ and $(|N_i|, |N_j|) = 1$ for $1 \leq i < j \leq k$.

Proof. Let P be a S_p subgroup of N_i for some prime p . Lemma 1 and the maximality of M_i imply by induction that $N(P) = M_i$ if $p > 2$ or if $1 \leq i \leq k$. Thus in these cases P is a S_p subgroup of G . Hence H_i, N_i are Hall subgroups for $1 \leq i \leq s + m, 1 \leq i \leq k$ respectively. If one of the other statements of the Lemma is false it may be assumed by taking conjugates that for some S_p subgroup P of G , $P \subseteq H_i \cap H_j, i \neq j$, or $P \subseteq N_i \cap N_j$ and $1 \leq i < j \leq k$. Hence in either case $\langle M_i, M_j \rangle \subseteq N(P)$. By the first part of the lemma this implies that $M_i = M_j$ contrary to the definition of the groups M_i .

Lemma 5 yields that

$$(2) \quad g = |G| = 3 \cdot 2^n g_0 \prod_{i=1}^{s+m} h_i, \quad (g_0, 6) = 1$$

Furthermore $t_i \neq 1$ for at most one value of i with $1 \leq i \leq k$. Choose the notation so that

$$(3) \quad \begin{aligned} t_1 &= 1 \text{ or } t_1 = 2^n \\ t_i &= 1 \text{ for } 2 \leq i \leq k \\ t_i &\neq 1 \text{ for } s + 1 \leq i \leq s + m. \end{aligned}$$

$$(4) \quad h_{s+1} \geq h_i \quad \text{for } s + 1 \leq i \leq s + m.$$

LEMMA 6.

$$(5) \quad \frac{g}{9} < \frac{g}{9} \left\{ 1 + \frac{1}{\theta(1)} - \frac{1}{\theta(1)+1} \right\} \leq 1 + 2 \sum_{i=1}^k (h_i t_i - 1) + \sum_{i=k+1}^{s+m} (h_i t_i - 1)$$

Proof. The first inequality is trivial. By Lemma 3 the second term in (5) is the multiplicity of K in K^2 . Thus the second term in (5) is the number of ordered pairs (y, z) with $yz = x$ and y, z of order 3. Since $\langle y, z \rangle$ is a homomorphic image of $(3, 3, 3)$ it is solvable by Theorem 1. Thus for every such pair, $\langle y, z \rangle$ is contained in a conjugate of some $M_i, 1 \leq i \leq s + m$.

Suppose that $x \in M_i \cap w^{-1}M_iw$ for some $w \in G$. Then $wxw^{-1} \in M_i$. There exists $w_1 \in M_i$ such that $w_1\langle x \rangle w_1^{-1} = w\langle x \rangle w^{-1}$. Hence it may be assumed that $w \in N(\langle x \rangle)$. This implies that x is contained in exactly one conjugate of M_i for $k+1 \leq i \leq s+m$ and x is contained in exactly two conjugates of M_i for $1 \leq i \leq k$. The number of ordered pairs (y, z) with $yz = x$, y, z of order 3 and $y, z \in M_i$ is easily seen to be $h_i t_i$. If the pair (x^2, x^2) is counted just once the second inequality in (5) follows.

LEMMA 7. *Let a be the multiplicity of C in K^2 . Then*

$$a \geq \sum_{i=s+1}^{s+m} \frac{|C(\mathbf{u})|}{2h_i t_i} h_i t_i.$$

Proof. Let (y, z) be an ordered pair of elements of order 3 such that $yz = \mathbf{u}$. Then $\langle y, z \rangle$ is isomorphic to $(3, 3, 2) = \mathfrak{A}_4$.

Suppose that $\langle y, z \rangle$ is contained in two distinct subgroups which are respectively conjugate to M_i and M_j with $s+1 \leq i < j \leq s+m$. By changing notation it may be assumed that $\langle y, z \rangle \subseteq M_i \cap M_j$, where $M_i \cap M_j$ is maximal among all such intersections. Let $D = N_i \cap N_j$, then $N(\langle y \rangle) \subseteq N(D)$. Since $[\langle y, z \rangle : \langle y, z \rangle'] = 3$ it follows that $\langle y, z \rangle' \subseteq D$. Define

$$L_i = N(D) \cap N_i, \quad L_j = N(D) \cap N_j.$$

Then $\langle L_i, L_j \rangle \subseteq N(D)$. Thus by Lemma 1 $\langle L_i, L_j \rangle \cong G$. Furthermore

$$N(\langle y \rangle) \subseteq N(L_i) \cap N(L_j) \subseteq N(\langle L_i, L_j \rangle).$$

Let M be a maximal solvable subgroup of G which contains $N(\langle y \rangle)\langle L_i, L_j \rangle$ and let N be the maximal normal nilpotent subgroup of M . By induction M/N is isomorphic to \mathfrak{S}_3 . Since $N(\langle y \rangle) \cap L_i = \langle 1 \rangle$ this implies that $L_i \subseteq N$. Since $N_i \not\cong N_j$ we have that $D \cong N_i$. Thus $D \cong L_i$ as N_i is nilpotent. Therefore $M_i \cap M_j \subseteq M_i \cap M$. A similar argument shows that $M_i \cap M_j \subseteq M_j \cap M$. As M cannot be conjugate to both M_i and M_j one of these inclusions contradicts the maximal nature of $M_i \cap M_j$. Thus $\langle y, z \rangle$ is not contained in two subgroups which are conjugate M_i, M_j respectively with $s+1 \leq i < j \leq s+m$.

If $\langle y, z \rangle \subseteq M_i \cap w^{-1}M_iw$ for $w \in G$ then $N(\langle y \rangle) \subseteq M_i \cap w^{-1}M_iw$. This implies that $M_i = w^{-1}M_iw$. Let u lie in exactly m_i subgroups conjugate to N_i . Since M_i contains at least $h_i t_i$ ordered pairs (y, z) with $y^3 = z^3 = 1, yz = u$, this implies that

$$a \geq \sum_{i=s+1}^{s+m} h_i t_i m_i.$$

Clearly $m_i \geq [C(u) : C(u) \cap M_i] \geq \frac{|C(u)|}{2h_i t_i}$. The lemma follows.

LEMMA 8.

$$\sum_{i=s+1}^{s+m} \frac{|C(u)|}{2h_i t_i} h_i t_i \leq \frac{g}{3}.$$

Proof. Let a be the multiplicity of C in K^2 . Then by Lemma 3

$$a = \frac{g}{9} \left\{ 1 + \frac{\theta(u)}{\theta(1)} + \frac{\chi(u)}{\chi(1)} \right\} \leq \frac{g}{3}.$$

The result now follows from Lemma 7.

LEMMA 9. $|C(u)| = 2^n h$ with $h \neq 1$.

Proof. By Lemma 4, u is in the center of a S_2 -subgroup of G . Suppose that $h = 1$. Then ([4], p. 870, [5], Theorem 3) G is isomorphic to $PSL(2, 9)$, $PSL(3, 4)$ or $PSL(2, q)$ for q a prime or a power of 2. Since 9 does not divide g the first two possibilities cannot occur. If q is odd $PSL(2, q)$ contains cyclic subgroups of order $\frac{q+1}{2}$ and $\frac{q-1}{2}$. Thus one of $p, \frac{p-1}{2}, \frac{p+1}{2}$ equals 3. Hence $p = 3, 5, 7$. Since $PSL(2, 3), PSL(2, 5)$ are respectively isomorphic to $\mathcal{A}_4, \mathcal{A}_5$ these possibilities cannot occur. If q is a power of 2, then $q \pm 1 = 3$ and so $q = 2$ or 4. As $PS(2, 2)$ is not simple and $PSL(2, 4)$ is isomorphic to \mathcal{A}_6 we get that $h \neq 1$.

The proof of the main Theorem is now divided into three cases.

- Case I. $h = h_{s+1}, t_1 \neq 2^n$
- Case II. $h = h_{s+1}, t_1 = 2^n$
- Case III. $h \neq h_{s+1}$.

In cases I and II $h_i = 1$ for $i > s + 1$. In case II $h_1 = 1$. Thus in cases I and II Lemmas 6, 7 and 8 and equation (2) yield that

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{3} \leq 2 \left\{ h_1 t_1 + \sum_{i=2}^k h_i \right\} + h t_{s+1} + \frac{1}{h} \left\{ 2^n g_0 \prod_{i=1}^{s+m} h_i \right\}.$$

Since $(h, 6) = 1$ and $h \neq 1$ by Lemma 9 we get that $h \geq 5$. Thus in cases I or II we get

$$(6) \quad \frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2} \left(\frac{1}{3} - \frac{1}{h} \right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq h_1 t_1 + \sum_{i=2}^k h_i + h t_{s+1}.$$

Hence in Case I we get

$$(7) \quad \frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2} \left(\frac{1}{3} - \frac{1}{h} \right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq \sum_{i=1}^k h_i + h 2^n.$$

Since $t_{s+1} \leq 2^{n-1}$ we get in case II that

$$h_1 t_1 + h t_{s+1} \leq 2^n + 2^{n-1} h < 2^n h.$$

Thus in case II

$$(8) \quad \frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2} \left(\frac{1}{3} - \frac{1}{h} \right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq \sum_{i=2}^k h_i + h 2^n.$$

In case III let h_0 be the minimum value of h/h_i for $s+1 \leq i \leq s+m$. Hence $h_0 \geq 5$. Thus

$$\frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{3} \leq 2 \left\{ h_1 t_1 + \sum_{i=2}^k h_i \right\} + \frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{h_0}$$

or in case III

$$(9) \quad \frac{2^n g_0 \prod_{i=1}^{s+m} h_i}{15} \leq \frac{1}{2} \left(\frac{1}{3} - \frac{1}{h_0} \right) 2^n g_0 \prod_{i=1}^{s+m} h_i \leq h_1 t_1 + \sum_{i=2}^k h_i.$$

For convenience the following notation is now introduced.

Case I $q = k+1$, $z = h$

$$\{x_1, \dots, x_q\} \text{ is the set } \{h_1, \dots, h_k, h 2^n\}$$

in ascending order, and

$$y = \frac{1}{h} g_0 \prod_{i=k+1}^s h_i.$$

Case II $q = k$, $z = h$

$$\{x_1, \dots, x_q\} \text{ is the set } \{h_2, \dots, h_k, h 2^n\}$$

in ascending order, and

$$y = \frac{1}{h} g_0 \prod_{i=k+1}^s h_i.$$

Case III $q = k$, $z = h_0$

$$\{x_1, \dots, x_q\} \text{ is the set } \{h_1 t_1, h_2, \dots, h_k\}$$

in ascending order, and

$$\begin{aligned} y &= g_0 2^n \prod_{i=k+1}^{s+m} h_i & \text{if } t_1 = 1 \\ &= g_0 \prod_{i=k+1}^{s+m} h_i & \text{if } t_1 = 2^n. \end{aligned}$$

In all cases we get that x_1, \dots, x_q, y, z are integers such that

$$(10) \quad g = 3y \prod_{i=1}^q x_i$$

$$(11) \quad (x_i, x_j) = 1 \quad \text{for } 1 \leq i < j \leq q$$

$$(12) \quad (3, y) = (x_i, y) = 1 \quad \text{for } 1 \leq i \leq q$$

If $x_i \not\equiv 1 \pmod{3}$ then $x_i = h2^n$ in cases I or II. Thus $x_i > 4h \geq 20$.

Therefore

$$(13) \quad x_i \equiv 1 \pmod{3} \text{ or } x_i > 20 \text{ for } 1 \leq i \leq q.$$

The inequalities (7), (8) and (9) become

$$(14) \quad \frac{y \prod_{i=1}^q x_i}{15} \leq \frac{1}{2} \left(\frac{1}{3} - \frac{1}{z} \right) y \prod_{i=1}^q x_i \leq \sum_{i=1}^q x_i$$

LEMMA 10. $q \leq 2$. If $q = 2$ then $y = 1$.

Proof. If $x_1 > 4$ then by (13) $x_1 \geq 7$. Hence (14) yields that

$$7^{q-1} x_q \leq 15 \sum_{i=1}^q x_i \leq 15 q x_q.$$

Thus $7^{q-1} \leq 15q$ and so $q \leq 2$ in this case. If $x_1 \leq 4$ then $x_1 = 4$ and

$$4^{q-1} x_q \leq 15 \sum_{i=1}^q x_i \leq 15 q x_q$$

Thus $4^{q-1} \leq 15q$ and $q \leq 3$. Hence $q = 3$ and by (14)

$$4x_2 x_3 < 15(x_1 + x_2 + x_3) < 45x_3.$$

Hence $x_2 < 12$. Thus by (11) and (12) $x_2 = 7$ and so $28x_3 < 15(4 + 7 + x_3)$ or $13x_3 < 165$. Hence $x_3 < 13$ contrary to $7 < x_3$, (11) and (12). Thus $q \leq 2$.

Suppose that $q = 2$. Then (14) yields that $yx_1 x_2 \leq 15(x_1 + x_2)$. If $y \geq 5$ this implies that $x_1 x_2 \leq 3(x_1 + x_2) < 6x_2$. Thus $x_1 = 4$ and $4x_2 \leq 12 + 3x_2$. Hence $x_2 = 7$. Therefore $28y \leq 15(11) = 165$ and $y < 6$. Therefore $y = 5$ and by (10) $g = 3 \cdot 4 \cdot 5 \cdot 7 = 420$. This is impossible since there is no simple group of order

420. Thus $y < 5$. If $y \neq 1$, then $y = 2$ or $y = 4$. If $y = 2$, then $x_1 x_2$ is odd and by (10) 4 does not divide g contrary to the simplicity of G . Thus $y = 4$. Hence $x_1 x_2$ is odd and so $x_1 \geq 7$. If $x_1 > 7$ then $x_1 \geq 13$ and $52 x_2 \leq 4 x_1 x_2 \leq 15(x_1 + x_2) < 30 x_2$ which is not the case. If $x_1 = 7$ then $28 x_2 \leq 15(7 + x_2)$ or $13 x_2 < 15.7$. Hence $x_2 < 13$ which is not the case. The lemma is proved in all cases.

LEMMA 11. *In case I or case II*

$$\frac{11}{75} y \prod_{i=1}^q x_i \leq \sum_{i=1}^q x_i.$$

Proof. H_{s+1} admits \mathfrak{S}_3 as a group of automorphisms, thus H_{s+1} is not cyclic. Hence $z = h = |H_{s+1}| \geq 25$ and the result follows from (14).

LEMMA 12. $q = 2, y = 1$.

Proof. Suppose that $q = 1$. Assume first that we have case I or II. Then Lemma 11 implies that $y < 7$. Furthermore $|C(u)| = x_1$ and $[G : C(u)] = 3y$. Thus $y \neq 1$ and so by (12) $y = 5$. Hence in case I (6) becomes

$$\frac{11}{75} \cdot 5 \cdot 2^n h \leq h t_{s+1} \leq h 2^{n-1},$$

or $\frac{22}{15} \leq 1$ which is not the case. In case II $|N(H_1)| = 3 x_1$ and so $[G : N(H_1)] = 5$, thus G is isomorphic to a subgroup of \mathfrak{S}_5 . Hence G is isomorphic to \mathfrak{A}_5 contrary to assumption.

Assume now that $q = 1$ and we are in case III. Then (14) implies that $y \leq 15$. Since G is simple $4 | g$. Thus by (10) and (12) either y is odd or $4 | y$. Hence $y = 4, 8, 5, 7, 11$ or 13 and $g = 3 x_1 y$. If x_1 is even then $x_1 | |C(u)|$ and $x_1 \neq |C(u)|$. Since in this case $y = 5, 7, 11$ or 13 , it is a prime. Hence $|C(u)| = x_1 y$ and $[G : C(u)] = 3$ which is impossible. If x_1 is odd then $x_1 \equiv 1 \pmod{3}$, $y = 4$ or 8 and $[G : N(H_1)] = y$. Thus $y = 8$ and G is isomorphic to subgroup of \mathfrak{S}_8 . As H_1 is nilpotent the Sylow theorems imply that the only prime dividing x_1 is 7. As 49 does not divide $8!$ this implies that $x_1 = 7$. Hence $g = 3 \cdot 7 \cdot 8$ and G is isomorphic to $PSL(2, 7)$ contrary to assumption.

Hence $q = 2$ and by Lemma 10 $y = 1$.

The proof of the main Theorem will now be completed.

By Lemma 12 $g = 3 x_1 x_2$. In case I or II H_{s+1} is not cyclic. Thus $z = h = |H_{s+1}| \geq 25$. By (14) we get that

$$\frac{11}{75} x_1 x_2 \leq x_1 + x_2 < 2 x_2.$$

Hence $x_1 < 14$. Thus x_1 is odd and $x_1 \equiv 1 \pmod{3}$. This implies that $x_1 = 7$ or $x_1 = 13$. If $x_1 = 13$ then $\frac{11 \cdot 13}{75} x_2 \leq 13 + x_2$ or $25 \leq h \leq x_2 \leq \frac{75}{68} \cdot 13$ which is not the case. Suppose that $x_1 = 7$. In case I (7) implies that

$$2^n h < \frac{11}{75} 7 \cdot 2^n h \leq 7 + 2^{n-1} h.$$

Hence $25 < 2^{n-1} h \leq 7$ which is not the case. In case II (6) implies that $2^n h < \frac{11}{75} 7 \cdot 2^n h \leq 2^n + 7 + h t_{s+1} \leq 2^n + 7 + 2^{n-1} h$. Hence $25 \leq 2^{n-1} h \leq 2^n + 7$. So that $2^n > 7$ and $2^{n-1} \cdot 25 \leq 2^{n-1} h < 2^{n+1}$ which is not the case.

Assume now that we have Case III. Then $x_1 \equiv x_2 \equiv 1 \pmod{3}$, and by (14) $\frac{x_1 x_2}{15} \leq x_1 + x_2 < 2 x_2$. Hence $x_1 < 30$. If x_1 is even then $x_1 \equiv 0 \pmod{4}$. If x_1 is a prime then $|C(u)| = x_1 x_2$ and $[G : C(u)] = 3$ which is not the case. Thus $x_1 = 4, 16, 25, 28$. $[G : N(H_2)] = x_1$, thus $x_1 \nmid 4$. If $x_1 = 28$ then (14) yields that $\frac{28}{15} x_2 \leq 28 + x_2$ or $x_2 \leq \frac{28 \cdot 15}{13} < 33$. Hence $x_2 = 31$ is a prime. Thus $|C(u)| = x_1 x_2$ and $[G : C(u)] = 3$ which is not the case. If $x_1 = 25$ then $\frac{25}{15} x_2 \leq 25 + x_2$ or $x_2 \leq \frac{3}{2} 25 < 38$. Since $x_2 \equiv 0 \pmod{4}$, $x_2 = 28$. The Sylow theorems now imply that some divisor d of 25 satisfies $d \equiv 1 \pmod{7}$ which is not the case. Assume finally that $x_1 = 16$. If P is a Sylow subgroup of H_2 for some prime p then $|N(P)| = 3 x_2$ as $N(H_2)$ is a maximal solvable subgroup of G . Thus $16 \equiv 1 \pmod{p}$. Hence $p = 5$. Since $x_2 \equiv 1 \pmod{3}$, $x_2 = 5^{2a}$ for some integer a . By (14) $\frac{16}{15} x_2 \leq 16 + x_2$ or $x_2 \leq 240$. Thus $x_2 = 25$ and $g = 3 \cdot 16 \cdot 25 = 1200$. There is no simple group of order 1200.

This final contradiction establishes the main theorem of the paper.

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