ON A PRACTICAL WAY OF DESCRIBING
FORMAL DEDUCTIONS

KATUZI ONO

Introduction

Even though the logical structure of any formal deduction can be nicely expressed in a tree-form diagram, it is more practical to write it down in a series of propositions. In each step of inference, we usually deduce a proposition on basis of some foregoing propositions[1]. However, global aspects of mathematical theories show us that this is not always the case. For, in mathematical theories, theorems are usually stated before their proofs. In fact, also in proofs of theorems, it is often practical that we prove propositions after stating them. Accordingly, in our real way of thinking, we arrange propositions going back and forth in the logical order.

In formal deductions, some propositions are stated as temporary assumptions, definitions, or something like that. Furthermore, propositions such as "Take any object, say x, satisfying the conditions $\mathfrak{C}(x)$." nominate temporarily the variable x, having the assumption character. On the other hand, propositions such as "There is an object t satisfying the condition $\mathfrak{C}(t)$, so take any one of such objects and call it x." also nominate temporarily the variable x, having the assertion character. In such cases, we are tempted to use the universal or existential quantifiers, though confusions are hardly avoidable without introducing some devices.

In Section (1), we introduce new symbols of the forms $\forall x!$ and $\exists x!$ called nominating quantifiers to meet the demand. Namely, by making use of these quantifiers, we denote the former example as "$\forall x!\ \mathfrak{C}(x)$" and the latter one as "$\exists x!\ \mathfrak{C}(x)$".

In Section (2), we introduce a system of numbering of all the propositions in any formal deduction. We use Latin letters for numbering propositions in

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[1] Such is the case for the most of logical systems, e.g. Gentzen's systems [1]. Kuroda [2] introduced a system, in which inferences are described in the inverse direction.

a formal deduction which is usually a proof of a theorem, as it looks advisable to reserve numerals for numbering theorems and definitions.

Our system is properly designed so as to fulfill the following four conditions :

A. Propositions are arranged in the lexicographical order of their numbers. (Check this in the example of Section (3).)

B. One can see easily only by numbers, whether any proposition is qualified to be a logical basis of a given proposition. (Just behind every proposition, we indicate its logical basis by numbers following after the symbol "/". In the example of Section (3), the propositions A—j, ka, kbA, kbbA—kbbe are qualified to be basis of kbb, the propositions A—j, ka, kbA, kbb, kbc, kbd, kbe are qualified to be basis of kb, and the propositions A—j, ka, kb are qualified to be basis of k.)

C. For any one of temporary assumptions, definitions, and nominations, one can easily see only by numbers its scope in which it is valid. (In the example of Section (3), all the propositions from kac to kae i.e. the propositions kac, kacA—kacc, kad, kadA—kadc, kae are in the scope of the nomination kab).

D. For any propositon stated before its proof in the deduction, one can easily see only by numbers, what part of the deduction is devoted to the proof of the proposition. (In the example of Section (3), all the propositions from kbA to kbe i.e. the propositions kbA, kbb, kbbA—kbbe, kbc, kbcA—kbce, kbd, kbe form a proof of kb).

In Section (3), we explain our way of description by an example which asserts that a special case of the axiom of replacement of the Fraenkel set-theory implies that there is no set containing all the sets as members. Although it is sometimes desirable to express the leading idea of a formal deduction more vividly by an illustration in ordinary sentences, our way of description may be recognized by this example as being practical enough to take notes of mathematical proofs without being too long and always exposing their logical structure exactly.

(1) Nominating quantifiers

Let us express by Q_i either of the quantifiers $\forall$ or $\exists$. Symbols of the form $Q_1 x_1 \cdots Q_n x_n !$ are called *nominating quantifiers*, assuming that $x_1, \ldots, x_n$

are all distinct.   Nominating quantifiers can stand only at the top of pro-
positions. Such propositions are called *nominations* of the variables $x_1, \ldots, x_n$.

$\forall x!\ \mathbb{C}(x)$ means "Take any object, say x, satisfying the condition $\mathbb{C}(x)$.",
and $\exists x!\ \mathbb{C}(x)$ means "There is an object t satisfying the condition $\mathbb{C}(t)$, so
take any one of such objects and call it x".   Propositions of this kind are called
simple nominations.

General nomination of objects of the form $Q_1 x_1 \cdots Q_n x_n!\ \mathbb{C}(x_1, \ldots, x_n)^{2)}$
is an abbreviation of the series of the following simple nominations:

$$Q_1 x_1!\ [Q_2 x_2 \cdots Q_n x_n \cdot \mathbb{C}(x_1, \ldots, x_n)]_1,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$Q_{n-1} x_{n-1}!\ [Q_n x_n \cdot \mathbb{C}(x_1, \ldots, x_n)]_{n-1},$$

$$Q_n x_n!\ \mathbb{C}(x_1, \ldots, x_n),$$

where $[\mathfrak{A}]_k$ denotes $\mathfrak{T} \to \mathfrak{T}$ (a true proposition containing no free variables) or
$\mathfrak{A}$ according as Q_k denotes $\forall$ or $\exists$.

In any formal deduction, all the propositions which are qualified to be basis
(defined in Section (2)) of a nomination of variables, say $x_1, \ldots, x_n$, should stand
before the nomination in the lexicographical order, and these variables $x_1, \ldots,$
x_n should not occur in them as free variables[3].   Any nominating quantifier
nominates free variables temporarily in a part of the deduction called the scope
(defined in Section (2)) of the nomination.   In the scope of the nomination, these
variables are regarded as free.

(2) Numbering

To every proposition in a formal deduction, we give a *number* i.e. a finite
sequence of letters, any one of which except the last letter of the number of
any temporary assumption, definition, or something like that is a small letter.
If necessary, we supply a series of letters $\bar{a}, \bar{b}, \ldots$ coming after the letter z,
and also $\bar{\bar{a}}, \bar{\bar{b}}, \ldots$ coming after the letter $\bar{z}$, and so on.   The same also for
the capital letters standing at the end of the numbers of any temporary as-

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[2] Sometimes it is convenient to denote propositions of the form $Q_1 x_1 \cdots Q_n x_n!\ (\mathfrak{T} \to \mathfrak{T})$
simply by "$Q_1 x_1 \cdots Q_n x_n!$".

[3] We can impose this condition on all the propositions standing before the nomina-
tion. To be practical, however, we impose this condition only on propositions qualified
to be basis of it. Evidently, this condition corresponds to the so-called variable condition
for introducing $\forall$ and $\exists$ in ordinary logical systems.

sumptions or definitions.

All the propositions in a formal deduction are arranged in the lexicographical order of their numbers. The numbering should be given in the following way: If there is a number of the form $\alpha\iota$, there are also all the numbers of the forms α, αa (or αA), ..., $\alpha\iota$ where α denotes a number (including the vacant one) and ι denotes a letter. All the propositions having the numbers of the form $\alpha\S\varphi$ form a *proof* of α. If α is vacant, they form the whole deduction. (Here we use the word "a proof of α" in the sense that it is the whole series of propositions which proves α by making use of any propositions already proved before it).

Any proposition α is qualified to be a *basis* of another proposition β, if α is a number of the form $\beta\iota$, or if α and β are numbers of the forms $\gamma\S$ and $\gamma\iota\varphi$ respectively and $\S$ stands before ι in the alphabetical order. Notice that $\gamma\S\theta$ is not qualified to be a basis of $\gamma\iota\varphi$ unless θ is vacant. In other words, the logical order is not transtive. This may sound strange, but it is rather natural if we take into account that all the propositions having the numbers of the form $\gamma\S\iota\lambda$ form a proof of $\gamma\S$.

If α is qualified to be a basis of β and also stands before β in the lexicographical order, we say that β is in the *scope* of α. Temporary assumptions, definitions, and nominations are valid only in their scopes.

Any proposition having a number ending with a capital letter is a temporary assumption, definition, or something of that kind[4]. It is not a conclusion of an inference anyway, so we do not have to indicate its basis.

Any other proposition is a conclusion of an inference. Usually, they are deduced from some premises. We indicate these premises by their numbers just behind the proposition following after the symbol "/" (read "by"). They are theorems already proved, definitions, or propositions qualified to be basis of these propositions. If the proposition is proved by the propositions following after it, its number is marked by "))". (Otherwise, marked by ")".)

(3) Example

To show an example of a proof described in our way, we introduce an

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[4] Notice that nominations are not always propositions of this kind. Indeed, nominations of the form $\exists x!\ \mathfrak{C}(x)$ have the assertion character.

operator "$\{\ )$" (mapping binary relations to binary relations). Namely, $x\{\varGamma)y$ means $\forall t(t \in x \equiv t\varGamma y)$, where $\varGamma$ denotes any binary relation. Evidently, $x\{\varGamma)y$ is unique regarding it as a mapping y to x, if we assume the axiom of extensionality. Accordingly,

$$\forall m \exists p \cdot p\{\{\varGamma) \in)m^{5)}$$

can be regarded as a special case of the axiom of replacement.

By the assumption $\forall m \exists p \cdot p\{\{\varGamma) \in)m$ only, we can conclude $\neg \forall x \cdot x \in z^{6)}$. In the following, we show a formal proof of this theorem in our way of description.

Proof /reductio ad absurdum : $\mathbf{A} \rightarrow \mathbf{t}^{7)}$.

A) Assume $\forall z \exists p \cdot p\{\{\varGamma) \in)z$ and $\exists m! \ \forall t \cdot t \in m^{8)}$.

B) Define "$=$" by $x = y \cdot \overset{df}{\equiv} \forall s(s \in x \equiv s \in y)$.

c) $\forall x \cdot x = x$ /B.   **d)** $\forall xy(x \in = y \cdot \rightarrow x \in y)$ /B.

E) Define $\varGamma$ by $x\varGamma y \cdot \overset{df}{\equiv} (x \in y \wedge y \notin y)$.

f) $\exists p! \ p\{\{\varGamma) \in)m$ /A.   **g)** $\forall u(u \in p \equiv \exists z(u\{\varGamma)z \wedge z \in m))$ /f.

h) $\forall u(\exists z(u\{\varGamma)z \wedge z \in m) \equiv : \exists z \cdot u\{\varGamma)z)$ /A.

i) $\forall u(u \in p \equiv : \exists z \cdot u\{\varGamma)z)$ /g, h.

j) $\forall u(u \in p \equiv \exists z \forall t(t \in u \equiv \cdot t \in z \wedge z \notin z))$ /i, E.

k)) $\forall u(u \in p \equiv \cdot \exists z(u = z \wedge z \notin z) \vee (\exists z \cdot z \in z : \wedge : \forall t \cdot t \notin u))$ /ka, kb$^{9)}$.

ka)) $\forall u(u \in p \rightarrow \cdot \exists z(u = z \wedge z \notin z) \vee (\exists z \cdot z \in z : \wedge : \forall t \cdot t \notin u))$
        /kaA $\rightarrow$ kae$^{10)}$.

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[5] In our new object theory, which will be published in the near future, $\exists P \cdot P\{\{\varGamma)\sigma)M$ is the only axiom scheme, where σ is a binary relation defined by $\in$, the only primitive notion of the system. The relation σ is slightly deviated from the satellite notion of our former object theory [3]. However, it is defined in such a way that $\forall XY(X \in Y \rightarrow X\sigma Y)$ holds. Our former system is, so to say, an object theory in the manner of the Zermelo set-theory, whereas our new system can be regarded as an object theory in the manner of the Fraenkel set-theory.

[6] The proof of this example is a deformation of the Russell paradox. This simple example is one of the clues which lead to our new object theory. Indeed, one can conclude easily by this example proof that $\forall M \exists P \cdot P\{\{\varGamma)\sigma)M$ implies $\neg \forall X \cdot X \in Z$, if we assume $\forall XY(X \in Y \rightarrow X\sigma Y)$.

[7] This means that the whole deduction from $\mathbf{A}$ to $\mathbf{t}$ forms a proof of this theorem by reductio ad absurdum. (Compare with the proof of l.)

[8] This is an abbreviation of the following two steps. First step: Assume $\exists m \forall t \cdot t \in m$. Second step: $\exists m! \ \forall t \cdot t \in m$ /first step. (Compare with **kbbA** and **kbcA**.)

[9] This means that $\mathbf{k}$ is proved on the basis of $\mathbf{ka}$ and $\mathbf{kb}$.

[10] This means that $\mathbf{ka}$ is proved by the whole deduction from $\mathbf{kaA}$ to $\mathbf{kae}$ (Compare with $\mathbf{kac}$, for example).

kaA) $\forall u!\ u \in p.$ kab) $\exists w!\ \forall t(t \in u \equiv (t \in w \wedge w \not\in w))$ /kaA, j[11].

kac)) $w \not\in w \to \exists z(u = z \wedge z \not\in z)$ /kacA → kacc.

kacA) Assume $w \not\in w$[12].   kacb) $u = w$ /kab, kacA, B.

kacc) $\exists z(u = z \wedge z \not\in z)$ /kacb, kacA.

kad)) $w \in w \to (\exists z \cdot z \in z \wedge \forall t \cdot t \not\in u)$ /kadA → kadc.

kadA) Assume $w \in w.$ kadb) $\forall t \cdot t \not\in u$ /kab, kadA.

kadc) $\exists z \cdot z \in z : \wedge : \forall t \cdot t \not\in u$ /kadA, kadb.

kae) $\exists z(u = z \wedge z \not\in z) \vee (\exists z \cdot z \in z : \wedge : \forall t \cdot t \not\in u)$ /kac, kad.

kb)) $\exists z(u = z \wedge z \not\in z) \vee (\exists z \cdot z \in z : \wedge : \forall t \cdot t \not\in u) \cdot \to u \in p$ /kbA → kbe.

kbA) Assume $\exists z(u = z \wedge z \not\in z) \vee (\exists z \cdot z \in z : \wedge : \forall t \cdot t \not\in u).$

kbb)) $\exists z(u = z \wedge z \not\in z) \to \exists z(u\langle \Gamma \rangle z \wedge z \in m)$ /kbbA → kbbe.

kbbA) Assume $\exists z!\ (u = z \vee z \not\in z).$

kbbb) $\forall t(t \in u \equiv t \in z)$ /kbbA.

kbbc) $\forall t(t \in u \equiv (t \in z \wedge z \not\in z))$ /kbbb, kbbA.

kbbd) $u\langle \Gamma \rangle z$ /kbbc, E.   kbbe) $\exists z(u\langle \Gamma \rangle z \wedge z \in m)$ /kbbd, A.

kbc)) $(\exists z \cdot z \in z : \wedge : \forall t \cdot t \not\in u) \to \exists z(u\langle \Gamma \rangle z \wedge z \in m)$ /kbcA → kbce.

kbcA) Assume $\exists z!\ z \in z$ and $\forall t \cdot t \not\in u.$

kbcb) $\forall t \cdot \neg (t \in z \wedge z \not\in z)$ /kbcA, B.

kbcc) $\forall t(t \in u \equiv \cdot t \in z \wedge z \not\in z)$ /kbcA, kbcb.

kbcd) $u\langle \Gamma \rangle z$ /kbcc, E.   kbce) $\exists z(u\langle \Gamma \rangle z \wedge z \in m)$ /kbcd, A.

kbd) $\exists z(u\langle \Gamma \rangle z \wedge z \in m)$ /kbA, kbb, kbc   kbe) $u \in p$ /kbd, g.

l)) $p \in p$ /reductio ad absurdum : lA → ld.   lA)   Assume $p \not\in p.$

lb) $p = p \wedge p \not\in p$ /c, lA.   lc) $\exists z(p = z \wedge z \not\in z)$ /lb.

ld) $p \in p$ /k, lc.

m) $\exists t \cdot t \in p$ /l.   n) $\exists q!\ (p = q \wedge q \not\in q)$ /k, l, m.

o) $q = q \wedge q \not\in q$ /c, n.   p) $\exists z(q = z \wedge z \not\in z)$ /o.

q) $q \in p$ /k, p.   r) $q \in = q$ /q, n.   s) $q \in q$ /r, d.

t) Contradiction /n, s.

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[11] This nomination is valid until **kae.**
[12] The assumption is valid until **kacc.**

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Mathematical Institute,

Nagoya University.