CORRESPONDING GROUP AND MODULE SEQUENCES[1]

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§ 1. Statement of results

For convenience we consider throughout an arbitrary but fixed multiplicative group H. The integral group ring of H is denoted by ZH, and the homomorphism $\varepsilon : ZH \to Z$ is always the trivializer, or unit augmentation, defined by $\varepsilon h = 1$ for all $h \in H$. For any group extension of H, i.e., exact sequence of multiplicative groups

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\phi} H \longrightarrow 1, \qquad (1)$$

we shall construct an exact sequence of left ZH-modules

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\partial} ZH \xrightarrow{\varepsilon} Z \longrightarrow 0. \qquad (2)$$

(All unlabeled monomorphisms will be assumed to be inclusion mappings). We shall say that the module sequence (2) is *determined by* the group sequence (1).

Conversely, starting from an arbitrary exact sequence (2) of left ZH-modules, we shall construct a multiplicative group $\bar{A}$ and an abelian group extension

$$1 \longrightarrow B \longrightarrow \bar{A} \xrightarrow{\tau} H \longrightarrow 1 \qquad (3)$$

written multiplicatively

whose kernel is the additive group of the module B. We shall also say that the group sequence (3) is *determined by* the module sequence (2).

Two group extensions (1) will be called *equivalent* if there exists an isomorphism of one onto the other which is the identity on H. Similarly, two module sequences (2) are *equivalent* if there exist a ZH-isomorphism of one onto the other which is the identity on ZH. It will be obvious that equivalent group sequences determine equivalent module sequences and conversely.

Where K' is the commutator subgroup of K, a group sequence (1) induces the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K & \longrightarrow & G & \overset{\phi}{\longrightarrow} & H & \longrightarrow & 1 \\
& & \downarrow{\xi|K} & & \downarrow{\xi} & & \downarrow{\text{identity}} & & \\
1 & \longrightarrow & K/K' & \longrightarrow & G/K' & \overset{\zeta}{\longrightarrow} & H & \longrightarrow & 1
\end{array}
\qquad (4)
$$

We shall call the second row of the diagram, which is also exact, the *abelianization* of the first. The principal theorems of this paper are

(1.1) *The group sequence determined by the module sequence determined by a given group sequence* (1) *is equivalent to the abelianization of* (1).

And, conversely,

(1.2) *The module sequence determined by the group sequence determined by a given module sequence* (2) *is equivalent to* (2).

It is an immediate corollary that the module sequences determined by two group sequences are equivalent if and only if their abelianizations are equivalent.

The module sequence determined by a group sequence (1) will be shown in §6 to be the 0- and 1-dimensional part of the homology sequence of the pair $(G, 1)$ with coefficients in ZH as defined by Massey [7]. If G is given by a group presentation $(x_1, x_2, \ldots : r_1, r_2, \ldots)_{\varphi}$, the module sequence (2) may be calculated using the free differential calus of Fox [4, 5]. The matrix $\left\| \phi\varphi\left(\dfrac{\partial r_i}{\partial x_j}\right) \right\|$ is a relation matrix for the relative homology group $H_1(G, 1; ZH)$, which is the module A in the sequence (2). These facts and (1.1) imply Blanchfield's result [1] that, to quote Fox, "roughly speaking, the Jacobian class of G at ψ determines the structure of G modulo the commutator subgroup of the kernel of ψ''. Even though the approach and proofs are entirely different, many of Blanchfield's basic constructions translate directly into the ones used here. In §7 I have elaborated on the comparison.

I am greatly indebted to Hale F. Trotter of Queen's University, Kingston, Ontario, with whom I was associated in the NSF project[1]. His suggestions have resulted in substantial simplifications in the present treatment.

§ 2. Construction of the module sequence

Consider given a group extension (1). Let $\psi : ZG \to ZH$ be the linear ex-

tension to the group rings of the group homomorphism ψ, and let $\varepsilon: ZG \to Z$ be the trivializer of G. The ideals that are the kernels of ψ and ε we denote by $\Re$ and $\mathfrak{G}$, respectively. Where $\Re\mathfrak{G}$ is the product ideal and π is the factor homomorphism, we obtain the commutative diagram

$$\left. \begin{array}{c} 0 \longrightarrow \Re \longrightarrow \mathfrak{G} \xrightarrow{\psi|\mathfrak{G}} ZH \xrightarrow{\varepsilon} Z \longrightarrow 0 \\ \quad \downarrow \pi|\Re \quad\quad \downarrow \pi \quad\quad \downarrow \text{identity} \\ 0 \longrightarrow \Re/\Re\mathfrak{G} \longrightarrow \mathfrak{G}/\Re\mathfrak{G} \xrightarrow{\partial} ZH \xrightarrow{\varepsilon} Z \longrightarrow 0 \quad\quad (5) \end{array} \right\} \quad (6)$$

whose rows are easily seen to be exact. Observe that $\mathfrak{G}/\Re\mathfrak{G}$ is a left ZH-module with the operation of ZH well-defined by

$$(2.1) \qquad \psi u \cdot \pi v = \pi(uv), \quad \text{for all } u \in ZG \text{ and } v \in \mathfrak{G}.$$

Furthermore, $\Re/\Re\mathfrak{G}$ is a submodule and the mappings in the second row (5) are ZH-homomorphisms. The sequence (5) is by definition the module sequence (2) determined by (1).

§ 3. Construction of the group extension

Consider an arbitrary exact module sequence (2) (the mapping ε is always the trivializer). With respect to a product defined by

$$ab = a + (\partial a + 1) \cdot b, \quad \text{for all } a, b \in A,$$

the module A is a semi-group with the element 0 the identity.
Let $\bar{A}$ be the set of all $a \in A$ such that $\partial a + 1 \in H$. Then,

(3.1) *A is a multiplicative group* (the inverse of a is $a^{-1} = -(\partial a + 1)^{-1} \cdot a$).

Notice that $B \subset \bar{A}$ and, in addition,

$$(3.2) \qquad b_1 + b_2 = b_1 b_2, \quad \text{for any } b_1, b_2 \in B.$$

Thus B is a subgroup of $\bar{A}$. We define the mapping $\gamma: \bar{A} \to H$ by setting $\gamma a = \partial a + 1$, all $a \in \bar{A}$. It is straightforward to show that

(3.3) γ *is a group homomorphism of* $\bar{A}$ *onto* H *with kernel* B.

We therefore obtain the abelian extension (3). In any abelian extension there is defined an operation of the image on the kernel. In particular, the action of H on B is well defined by

$$\gamma a \circ b = aba^{-1}, \quad \text{for any } a \in \bar{A} \text{ and } b \in B.$$

This product can be extended in the obvious way so that B is a ZH-module with respect to $\circ$. Inasmuch as

$$\gamma a \circ b = ab(-(\gamma a)^{-1} \cdot a) = a(b - (\gamma a)^{-1} \cdot a) = a + \gamma a \cdot b - a = \gamma a \cdot b,$$

it follows that

(3.4)   *The induced module structure on B is the same as the original.*

§4. The principal mapping diagram

Consider a given group sequence (1), and let (5) be the module sequence which it determines. The principal mapping diagram is the commutive diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & K/K' & \longrightarrow & G/K' & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow{\theta} & & \downarrow{\eta} & & \downarrow{\beta} & & \\
0 & \longrightarrow & \Re/\Re\mathfrak{G} & \longrightarrow & \mathfrak{G}/\Re\mathfrak{G} & \longrightarrow & ZH & \longrightarrow & Z & \longrightarrow & 0
\end{array}
\tag{7}
$$

The mapping β is defined by $\beta h = h - 1$, for all $h \in H$. Let $\alpha : G \to \mathfrak{G}$ be given by $\alpha g = g - 1$, for all $g \in G$. The mapping η is defined by the commutativity relation

$$\eta \xi = \pi \alpha.$$

(The diagram chasing in the following lemmas is based on (4) and (6).)

(4.1) η *is well-defined.*

Proof. Observe first of all that α is a crossed homomorphism:

$$\alpha(g_1 g_2) = g_1 g_2 - 1 = (g_1 - 1) + g_1(g_2 - 1) = \alpha g_1 + g_1 \alpha g_2.$$

Hence, for any $k_1,\ k_2 \in K$,

$$\alpha k_1 k_2 k_1^{-1} k_2^{-1} = (1 - k_1 k_2 k_1^{-1}) \alpha k_1 + k_1 (1 - k_2 k_1^{-1} k_2^{-1}) \alpha k_2.$$

Thus, by (2.1), $\pi \alpha k_1 k_2 k_1^{-1} k_2^{-1} = 0$; whence it follows that $\pi \alpha K' = 0$. Next, suppose that $\xi g_1 = \xi g_2$. Then, $g_1 g_2^{-1} \in K'$ and so

$$0 = \pi \alpha g_1 g_2^{-1} = \pi(\alpha g_1 - g_1 g_2^{-1} \alpha g_2) = \pi \alpha g_1 - \pi \alpha g_2,$$

and the proof is complete.

(4.2) $\partial \eta = \beta \zeta.$

Proof. For any $g \in G$, $\partial \eta \xi g = \partial \pi \alpha g = \psi(g - 1)$ and $\beta \zeta \xi g = \beta \psi g = \psi g - 1$. ·

(4.3) *For any* r_1, $r_2 \in G/K'$, $\eta(r_1 r_2) = \eta r_1 + \zeta r_1 \cdot \eta r_2$, *i.e.*, η *is a crossed homomorphism.*

Proof. Choose counter-images $r_1 = \xi g_1$ and $r_2 = \xi g_2$. Then,

$$\eta(r_1 r_2) = \eta \xi g_1 g_2 = \pi \alpha g_1 g_2 = \pi(\alpha g_1 + g_1 \alpha g_2)$$
$$= \pi \alpha g_1 + \psi g_1 \cdot \pi \alpha g_2 = \eta \xi g_1 + \zeta \xi g_1 \cdot \eta \xi g_2$$
$$= \eta r_1 + \zeta r_1 \cdot \eta r_2.$$

Inasmuch as $\partial \eta r = \beta \zeta r = 1 - 1 = 0$ for any $r \in K/K'$, it follows that the image of the restriction $\theta = \eta \,|\, K/K'$ is contained in $\Re/\Re\mathfrak{G}$. Thus the existence of the principal mapping diagram (7) is established. It is a consequence of (4.3) that

$$\theta(r_1 r_2) = \theta r_1 + \theta r_2, \quad \text{for any } r_1, \ r_2 \in K/K'.$$

Hence, θ is a multiplicative-to-additive group homomorphism. In addition θ is a ZH-homomorphism. The operation of H on K/K' is defined by $h \cdot r = srs^{-1}$, for all $h \in H$, $r \in K/K'$, and s such that $\zeta s = h$ (just like the definition of $\circ$ in the paragraph preceding (3.4)). We have

$$\theta(h \cdot r) = \eta(srs^{-1})$$
$$= \eta s + \zeta s \cdot \eta r - \zeta(srs^{-1}) \cdot \eta s$$
$$= h \cdot \theta r,$$

which is sufficient to verify that θ is a ZH-homomorphism.

(4.4) *The mapping θ is a ZH-isomorphism onto.*

Proof. The proof is obtained by constructing the inverse mapping. Choose a system of coset representatives, i.e., for each $h \in H$, select $\bar{h} \in G$ such that $\psi \bar{h} = h$, and consider an arbitrary element $\sum_{g \in G} n_g g$ in the ideal $\Re$. Then

$$\sum_{g \in G} n_g g = \sum_{h \in H} \sum_{k \in K} n_{\bar{h}k} \, \bar{h}k$$

$$0 = \psi \Big(\sum_{g \in G} n_g g \Big) = \sum_{h \in H} \Big(\sum_{k \in K} n_{\bar{h}k} \Big) h$$

Since H is a basis for the free additive group ZH, $0 = \sum_{k \in K} n_{\bar{h}k}$. Hence,

$$\sum_{g \in G} n_g g = \sum_{h \in H} \sum_{k \in K} n_{\bar{h}k} \, \bar{h}(k - 1).$$

In addition, if $\sum_{h, \, k \neq 1} n_{\bar{h}k} \, \bar{h}(k - 1) = 0$, then all $n_{\bar{h}k} = 0$. It follows that the elements

$\bar{h}(k-1)$, for all $h \in H$ and $1 \neq k \in K$, constitute a basis for the free additive group $\mathfrak{R}$. We may therefore define an additive-to-multiplicative homomorphism $v: \mathfrak{R} \to K/K'$ by

$$v(\bar{h}(k-1)) = h \cdot \xi k, \qquad h \in H \text{ and } k \in K.$$

Consider arbitrary elements $g \in G$ and $k \in K$. Where $h = \psi g$, we have $g = \bar{h}k'$ for some $k' \in K$. Then,

$$g(k-1) = \bar{h}(k'k-1) - \bar{h}(k'-1),$$

and so

$$v(g(k-1)) = (h \cdot \xi k'k)(h \cdot \xi k')^{-1} = \psi g \cdot \xi k.$$

This implies

(4.5) $v(uv) = \psi u \cdot vv, \qquad$ for all $u \in ZG$ and $v \in \mathfrak{R}$.

Consider the identity

$$(k-1)(g-1) = g(g^{-1}kg-1) - (k-1), \qquad g \in G \text{ and } k \in K.$$

Inasmuch as $\zeta \xi g = \psi g$, we have, according to the definition of the operation of H on K/K',

$$v(g(g^{-1}kg-1)) = \psi g \cdot \xi(g^{-1}kg) = \xi(gg^{-1}kgg^{-1}) = \xi k.$$

Hence,

$$v(k-1)(g-1) = (\xi k)(\xi k)^{-1} = 1.$$

The ideal $\mathfrak{R}\mathfrak{G}$ is a left ZG-module generated by all elements $(k-1)(g-1)$, $k \in K$ and $g \in G$. As a result of the above equation and (4.5), we conclude that $v(\mathfrak{R}\mathfrak{G}) = 1$. Consequently, v induces a homomorphism $\omega: \mathfrak{R}/\mathfrak{R}\mathfrak{G} \to K/K'$, which is easily seen by (4.5) and (2.1) to be ZH-linear. Since ξK and $\{\pi(k-1)\}_{k \in K}$ generate the modules K/K' and $\mathfrak{R}/\mathfrak{R}\mathfrak{G}$, respectively, the equations

$$\theta \xi k = \pi(k-1)$$
$$\omega \pi(k-1) = v(k-1) = \xi k$$

show that $\omega\theta = $ identity and $\theta\omega = $ identity, and the proof is complete.

The preceding result, (4.4), is the key lemma of this paper. I had originally proved it using the free calculus of Fox, specifically, §4 of his paper [4]. Later K. Iwasawa suggested a proof independent of group presentations. The present proof was constructed in a discussion with Trotter and was motivated

by an analogous argument on page 190 of [2]. It is a corollary of (4.4) and the principal mapping diagram that *if $k \in K$ and $k - 1 \in \Re \mathfrak{G}$, then $k \in K'$.* This is a result proved by Fox for the case that G is a free group, cf. (4.9) in [4]. Fox's paper also contains references to earlier proofs by Schuman and Blanchfield.

§ 5. Proofs of (1.1) and (1.2)

The proof of (1.1) is an immediate corollary of (4.4) and the principal mapping diagram. We write the second row of the latter as $0 \to B \to A \xrightarrow{\partial} ZH \xrightarrow{\varepsilon} Z \to 0$, and consider the abelian extension $0 \to B \to \bar{A} \xrightarrow{\gamma} H \to 1$ which it determines. Diagram chasing around (7) yields

(5.1) $\bar{A} = $ image η *and* $\gamma \eta = \zeta$.

(5.2) η *is a group isomorphism of* G/K' *onto* $\bar{A}$.

Proof. That η is a homomorphism follows from

$$\eta(r_1 r_2) = \eta r_1 + \zeta r_1 \cdot \eta r_2$$
$$= \eta r_1 + \gamma \eta r_1 \cdot \eta r_2 = (\eta r_1)(\eta r_2).$$

Next suppose that $\eta r = $ identity for some $r \in G/K'$. Since the identity element of $\bar{A}$ is 0, we have

$$0 = \partial \eta r = \beta \zeta r = \zeta r - 1.$$

It follows that $r \in K/K'$ and thence that $0 = \eta r = \theta r$. Since, by (4.4), θ is a multiplicative to additive isomorphism, we conclude that $r = 1$. This completes the proof.

As a result, the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \dashrightarrow & K/K' & \longrightarrow & G/K' & \xrightarrow{\zeta} & H & \longrightarrow & 1 \\
& & \downarrow{\theta} & & \downarrow{\eta} & & \downarrow{\text{identity}} & & \\
1 & \longrightarrow & B & \longrightarrow & \bar{A} & \xrightarrow{\gamma} & H & \longrightarrow & 1
\end{array}
$$

exhibits the equivalence of the two rows, and theorem (1.1) is proved.

To prove (1.2), we start with an arbitrary module sequence (2). In order to avoid the necessity of introducing new notation in what follows, we write the group sequence (3) determined by (2) as $1 \to K \to G \xrightarrow{\phi} H \to 1$. This group

extension in turn detemines the module sequence (5), and it is our objective
to show that (2) and (5) are equivalent. The additive group of ZG is freely
generated by G. Since $G = \overline{A} \subset A$, the inclusion can be extended to an additive
group homomorphism $i\colon ZG \to A$.

(5.3) $i(uv) = (\psi u) \cdot (iv)$, *for any* $u \in ZG$ *and* $v \in \mathfrak{G}$.

Proof. It is sufficient to prove that $i(gv) = (\psi g) \cdot (iv)$ for any $g \in G$ and
$v \in \mathfrak{G}$. Since $v = \sum_{g \in G} n_g(g-1)$, it is actually enough to show

$$i(g_1(g_2 - 1)) = (\psi g_1) \cdot i(g_2 - 1), \qquad g_1,\ g_2 \in G.$$

The identity $1 \in G$ is the element $0 \in A$, Thus $i1 = 0$. Furthermore, $\psi = \gamma$, and
so according to the definition of multiplication in the semi-group A, the equa-
tion $g_1 g_2 = g_1 + \psi g_1 \cdot g_2$ holds in A. Therefore,

$$i(g_1(g_2 - 1)) = i(g_1 g_2 - g_1) = g_1 g_2 - g_1$$
$$= \psi g_1 \cdot g_2 = (\psi g_1) \circ i(g_2 - 1),$$

and this completes the argument.

(5.4) $i(\mathfrak{R}\mathfrak{G}) = 0$.

Proof. In view of (5.3), it suffices to verify the assertion on a generator
$(k-1)(g-1)$, $k \in K$ and $g \in G$.

$$i(k-1)(g-1) = i(kg - k - g + 1)$$
$$= kg - k - g = k + \psi k \cdot g - k - g = 0.$$

As a result, there exists the following commutative diagram of additive group
homomorphisms

$$
\begin{array}{ccc}
\mathfrak{G} & \xrightarrow{\ i\,|\,\mathfrak{G}\ } & A \\
\pi \downarrow & \nearrow{\scriptstyle j} & \\
\mathfrak{G}/\mathfrak{R}\mathfrak{G} & &
\end{array}
$$

(5.5) *j is a ZH-homomorphism.*

Proof. This amounts to checking

$$j(\psi u \cdot \pi v) = \psi u \cdot j\pi v, \qquad \text{for any } u \in ZG \text{ and } v \in \mathfrak{G}.$$

By (2.1), (5.3), and the above diagram

$$j(\psi u \cdot \pi v) = j\pi u v = i u v = \psi u \cdot i v = \psi u \cdot j\pi v.$$

Since in the present instance $B = K$, the commutator subgroup K' is trivial. This simplifies the fundamental mapping diagram, and we obtain $\eta: G \to \mathfrak{G}/\mathfrak{K}\mathfrak{G}$. For any $g \in G$, we have $\eta g = \pi(g-1)$, and so $j\eta g = j\pi(g-1) = i(g-1) = g$. That is,

(5.6)   *The composition $G \xrightarrow{\eta} \mathfrak{G}/\mathfrak{K}\mathfrak{G} \xrightarrow{j} A$ is the inclusion mapping.*

Inasmuch as $\eta \mid K$ is the isomorphism θ, it follows from (5.6) that j maps $\mathfrak{K}/\mathfrak{K}\mathfrak{G}$ onto $K = B$ and that the restriction of j to $\mathfrak{K}/\mathfrak{K}\mathfrak{G}$ is the inverse θ^{-1}. We therefore obtain the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{K}/\mathfrak{K}\mathfrak{G} & \longrightarrow & \mathfrak{G}/\mathfrak{K}\mathfrak{G} & \xrightarrow{\partial_1} & ZH & \xrightarrow{\varepsilon} & Z & \longrightarrow 0 \\
& & \downarrow{\theta^{-1}} & & \downarrow{j} & & \downarrow{\text{identity}} & \downarrow{\text{identity}} & & \\
0 & \longrightarrow & B & \longrightarrow & A & \xrightarrow{\partial_2} & ZH & \xrightarrow{\varepsilon} & Z & \longrightarrow 0
\end{array}
\tag{8}
$$

whose rows are the module sequences (5) and (2). To establish commutativity, it remains to show

(5.7) $\partial_2 j = \partial_1.$

Proof. We need $\partial_2 j\pi v = \partial_1 \pi v$, for any $v \in \mathfrak{G}$; so it suffices to prove $\partial_2 j\pi(g-1) = \partial_1\pi(g-1)$, for any $g \in G$. The mapping $\gamma = \psi$ is defined by $\partial_2 g + 1 = \gamma g = \psi g$. By (5), $\partial_1\pi(g-1) = \psi g - 1$. Hence, $\partial_2 j\pi(g-1) = \partial_2 i(g-1) = \partial_2 g = \psi g - 1 = \partial_1 \pi(g-1)$, and the proof is complete.

Thus (8) is a commutative diagram. Since θ^{-1} is an isomorphism, it follows from the "five" lemma that j is also. We conclude that the rows are equivalent, and theorem (1.2) is proved. It is interesting that according to (5.6) the mapping j^{-1} is an extension of η from $G = \overline{A}$ to A. If we could produce this extension directly, (1.2) would be an immediate corollary of the principal mapping diagram.
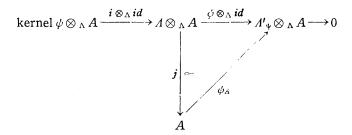
§ 6. The homology sequence

In this section we show that the module sequence (5) determined by a given group extension (1) is the 0- and 1-dimensional part of the homology sequence of the pair $(G, 1)$ with coefficients in ZH.

For any ring homomorphism $\phi\colon \varLambda \to \varLambda'$ (a ring is assumed to have an identity) and right $\varLambda'$-module B', we define the right $\varLambda$-module B'_ψ whose underlying additive group is the same as that of B' and whose scalar multiplication is defined by $b' \cdot \lambda = b' \cdot \psi\lambda$. For any left $\varLambda$-module A, we define the left $\varLambda'$-module $_{(\psi)}A = \varLambda'_\psi \otimes_\varLambda A$. The mapping $\psi_A\colon A \to {}_{(\psi)}A$ defined by $\psi_A(a) = 1' \otimes_\varLambda a$ obviously satisfied

$$\psi_A(\lambda \cdot a) = \psi\lambda \cdot \psi_A(a), \qquad \text{for all } \lambda \in \varLambda \text{ and } a \in A.$$

Assuming that ψ is onto and using the right exactness of the tensor product, we obtain from the exact sequence $0 \to \text{kernel } \psi \overset{i}{\longrightarrow} \varLambda \overset{\phi}{\longrightarrow} \varLambda'_\psi \to 0$ of right $\varLambda$-modules the commutative diagram

$$\text{kernel } \psi \otimes_\varLambda A \xrightarrow{\ i \otimes_\varLambda id\ } \varLambda \otimes_\varLambda A \xrightarrow{\ \phi \otimes_\varLambda id\ } \varLambda'_\psi \otimes_\varLambda A \longrightarrow 0$$

whose row is exact. The isomorphism j is the mapping $\lambda \otimes_\varLambda a \to \lambda \cdot a$, and the image of kernel $\psi \otimes_\varLambda A$ under $j(i \otimes_\varLambda id)$ is (kernel $\psi) \cdot A$. We conclude that

(6.1)   *If ψ is onto, then* $0 \to (\text{kernel } \psi) \cdot A \to A \overset{\psi_A}{\longrightarrow} {}_{(\psi)}A \to 0$ *is exact.*

Finally, we remark that any $\varLambda$-homomorphism $f\colon A_1 \to A_2$ of two left $\varLambda$-modules induces a $\varLambda'$-homomorphism $f'\colon {}_{(\psi)}A_1 \to {}_{(\psi)}A_2$ in the obvious way and that $\psi_{A_2}f = f'\psi_{A_1}$.

Our construction of the homology sequence of $(G, 1)$ follows Trotter's formulation [8] for a group system. The system here consists simply of the group G, the trivial subgroup $\{1\}$, and the inclusion $\{1\} \to G$. The module of coefficients is ZH_ψ, where $\psi\colon ZG \to ZH$ is the extension to the group rings of the group homomorphism ψ. Let X be a projective ZG-resolution of Z,

$$\longrightarrow X_2 \overset{d_2}{\longrightarrow} X_1 \overset{d_1}{\longrightarrow} ZG \overset{\varepsilon}{\longrightarrow} Z \longrightarrow 0.$$

The subcomplex Y corresponding to $\{1\}$ is

$$\longrightarrow 0 \longrightarrow 0 \longrightarrow ZG \overset{\varepsilon}{\longrightarrow} Z \longrightarrow 0.$$

The graded structure of the chain mapping $\psi_X\colon X \to {}_{(\psi)}X$ is exhibited in

$$\begin{array}{ccccccccc}
\longrightarrow & X_2 & \xrightarrow{d_2} & X_1 & \xrightarrow{d_1} & ZG & \xrightarrow{\varepsilon} & Z & \longrightarrow 0 \\
 & \downarrow{\psi_2} & & \downarrow{\psi_1} & & \downarrow{\psi} & & \downarrow{\text{identity}} & \\
\longrightarrow & {}_{(\psi)}X_2 & \xrightarrow{d_2'} & {}_{(\psi)}X_1 & \xrightarrow{d_1'} & ZH & \xrightarrow{\varepsilon} & Z & \longrightarrow 0
\end{array}$$

For $n \geq 1$, we have abbreviated ψ_{X_n} by ψ_n. The module $ZH_\psi \otimes_G ZG$ has been identified with ZH and, therefore, ψ_{ZG} is replaced by ψ. The homology sequence of $(G, 1)$ with coefficients in ZH is that of the pair $({}_{(\psi)}X, {}_{(\psi)}Y)$. That is, of

$$\left.\begin{array}{l}
\longrightarrow {}_{(\psi)}X_2 \xrightarrow{d_2'} {}_{(\psi)}X_1 \xrightarrow{d_1'} ZH \longrightarrow 0 \\[2mm]
\longrightarrow \quad 0 \quad \longrightarrow \quad 0 \quad \longrightarrow ZH \longrightarrow 0
\end{array}\right\}$$

It follows that the 0- and 1-dimensional part of this sequence is of the form $0 \to H_1(G;\ ZH) \to H_1(G, 1;\ ZH) \xrightarrow{\partial} ZH \xrightarrow{\varepsilon} Z \to 0$. In order to establish its equivalence with (5), it will suffice to establish a commutative diagram

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & & & & \\
 & \downarrow & & \downarrow & & & & & \\
 & \mathfrak{K}\mathfrak{G} & & \mathfrak{K}\mathfrak{G} & & & & & \\
 & \downarrow & & \downarrow & & & & & \\
0 \longrightarrow & \mathfrak{K} & \longrightarrow & \mathfrak{G} & \xrightarrow{\phi|\mathfrak{G}} & ZH & \xrightarrow{\varepsilon} & Z & \longrightarrow 0 \\
 & \downarrow{\bar{\pi}|\mathfrak{K}} & & \downarrow{\bar{\pi}} & & \downarrow{\text{identity}} & & & \\
0 \longrightarrow & H_1(G;\ ZH) & \longrightarrow & H_1(G, 1;\ ZH) & \xrightarrow{\partial} & ZH & \xrightarrow{\varepsilon} & Z & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & & & \\
 & 0 & & 0 & & & & &
\end{array} \qquad (9)$$

whose rows and columns are exact.

From the exact sequence $\to X_2 \xrightarrow{d_2} X_1 \xrightarrow{e} \mathfrak{G} \to 0$. where $ex = d_1 x$, for all $x \in X_1$, we obtain the commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathfrak{K}\mathfrak{G} & & \\
 & & & & \downarrow & & \\
X_2 & \xrightarrow{d_2} & X_1 & \xrightarrow{e} & \mathfrak{G} & \longrightarrow & 0 \\
\downarrow{\psi_2} & & \downarrow{\psi_1} & & \downarrow{\bar{\pi} = \phi_\mathfrak{G}} & & \\
{}_{(\psi)}X_2 & \xrightarrow{d_2} & {}_{(\psi)}X_1 & \xrightarrow{e'} & H_1(G, 1;\ ZH) & \longrightarrow & 0 \\
 & & & & \downarrow & & \\
 & & & & 0 & &
\end{array} \qquad (10)$$

The second row is exact by the right exactness of the tensor product and the column is exact by (6.1).

The remaining two lemmas serve to complete the proof of the existence of the mapping diagram (9).

(6.2) $\bar{\pi}\Re = H_1(G ;\ ZH)$.

Proof. An element of $H_1(G, 1 ;\ ZH)$ belongs to $H_1(G ;\ ZH)$ if and only if it has a counter-image under e' in kernel d_1'. Consider $v \in \Re$ and select $x \in X_1$ such that $ex = d_1 x = v$. Then $\bar{\pi}v = e'\psi_1 x$ and $d_1'\psi_1 x = \psi d_1 x = \psi v = 0$. Hence $\pi v \in H_1(G ;\ ZH)$. Conversely, consider $b \in H_1(G ;\ ZH)$. Select $x' \in$ kernel d_1' such that $e'x' = b$, and choose $x \in X_1$ such that $\psi_1 x = x'$. Since $\psi ex = \psi d_1 x = d_1'\psi_1 x = d_1'x' = 0$, we have $ex \in \Re$. Since $\bar{\pi}ex = e'\psi_1 x = e'x' = b$, we conclude $b \in \pi\Re$. This completes the proof.

(6.3) $\partial\bar{\pi} = (\psi\,|\,\mathfrak{G})$.

Proof. It follows from the definition of the boundary operator in an homology sequence that $\partial e' = d_1'$. Hence,

$$\partial\bar{\pi}e = \partial e'\psi_1 = d_1'\psi_1 = \psi d_1 = (\psi\,|\,\mathfrak{G})e.$$

Since e is onto, the proof is complete.

§ 7. Jacobian matrices and Blanchfield's senior thesis

Suppose G is defined by a group presentation $G = (x_1 x_2, \ldots :\ r_1, r_2, \ldots)_{\varphi}$. That is, there is given a free group F freely generated by $x_1, x_2, \ldots$ and a homomorphism φ of F onto G with kernel the consequences of $r_1, r_2, \ldots$ (cf. [4, 5]). We first show that the Jacobian matrix at ψ, i.e., $\left|\psi\varphi\left(\dfrac{\partial r_i}{\partial x_j}\right)\right|$, is a relation matrix for $H_1(G, 1 ;\ ZH)$. Following Trotter, we define the complex

$X_0 = ZG$

$X_1 =$ free ZG-module generated by $a_j \leftrightarrow x_j,\ j = 1, 2, \ldots$

$X_2 =$ free ZG-module generated by $b_i \leftrightarrow r_i,\ i = 1, 2, \ldots$

$d_1 :\ X_1 \to X_0$ defined by $d_1 a_j = \varphi x_j - 1,\ j = 1, 2, \ldots$

$d_2 :\ X_2 \to X_1$ defined by $d_2 b_i = \sum_j \varphi\left(\dfrac{\partial r_i}{\partial x_j}\right)\cdot a_j,\ i = 1, 2, \ldots .$

It is not hard to prove that the sequence $X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \xrightarrow{\varepsilon} Z \to 0$ is exact (cf. lemma 5.1 of [6]). It is therefore the 2-skeleton of a projective ZG-resolution

of Z. Consider the mapping diagram (10) in which X_1 and X_2 are assumed to be as described in this section. The ZH-modules $_{(\psi)}X_1$ and $_{(\psi)}X_2$ are free and generated by $\{\psi_1 a_1,\ \psi_1 a_2,\ \ldots\}$ and $\{\psi_2 b_1,\ \psi_2 b_2,\ \ldots\}$, respectively. Since

$$d_2' \psi_2 b_i = \psi_1 d_2 b_i = \sum_j \psi\varphi\Big(\frac{\partial r_i}{\partial x_j}\Big)\cdot\psi_1 a_j, \qquad i = 1, 2, \ldots ,$$

it follows that $\left|\psi\varphi\Big(\dfrac{\partial r_i}{\partial x_j}\Big)\right|$ is the matrix of d_2' and therefore a relation matrix of $H_1(G, 1; ZH)$.

Blanchfield proved that, for any finite presentation $(x_1, \ldots, x_n : r_1, \ldots, r_m)_\varphi = G$, the matrix $P = \left\|\psi\varphi\Big(\dfrac{\partial r_i}{\partial x_j}\Big)\right\|$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and the column matrix $Q = \|\psi\varphi x_j - 1\|$, $j = 1, \ldots, n$, determine the groups K/K' and G/K'. In view of the preceding paragraph, this result follows at once from our theorem (1.2). We have seen that specifying P is the same as specifying d_2'. Similarly, Q defines d_1' inasmuch as

$$d_1' \psi_1 a_j = \psi d_1 a_j = \psi\varphi x_j - 1.$$

Finally, the sequence

$$_{(\psi)}X_2 \xrightarrow{d_2'} {}_{(\psi)}X_1 \xrightarrow{d_1'} Z(H) \xrightarrow{\varepsilon} Z \longrightarrow 0$$

defines the homology sequence (2) which, by (1.1), determines

$$1 \to K/K' \to G/K' \to H \to 1.$$

Blanchfield proved (4.4) for finitely generated groups. That is, he has

$$K/K' \approx \text{kernel } d_1'/\text{image } d_2',$$

where d_1' and d_2' are defined by the matrices Q and P. Interestingly enough, he did not consider $H_1(G, 1; ZH) = A$ but a group which can be identified as the image of G/K' under η, i.e., our group $\bar{A}$.

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