ON A RING ISOMORPHISM INDUCED BY QUASICONFORMAL MAPPINGS

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Introduction

The purpose of this paper is to study the relationship between a certain isomorphism of some rings of functions on Riemann surfaces and a quasi-conformal mapping.

It is well known that two compact Hausdorff spaces are topologically equivalent if and only if their rings of continuous functions are isomorphic. We shall establish an analougous result concerning a function ring on a Riemann surface and the quasi-conformal equivalence.

As one of the important properties of quasi-conformal mapping is its absolute continuity in the sense of Tonelli, it is natural to consider the ring of functions which are absolutely continuous in the sense of Tonelli. On the other hand, we can show that this ring, with an additional condition, is coincident with a normed ring considered by Royden [10], which we shall call Royden's ring.

This leads us to study of the correspondence of the ideal boundary defined by using the above normed ring under a quasi-conformal mapping.

Our main results are, roughly speaking, as follows.

Two Riemann surfaces are quasi-conformally equivalent if and only if thier Royden's rings are isomorphic in some sense.

This can be considered as a ring-theoretic characterization of quasi-conformality.

A quasi-conformal mapping between two Riemann surfaces can be continuously extended to their "ideal boundaries" in an appropriate manner.

This includes the invariance of the classes O_G and O_{HD} of Riemann surfaces by a quasi-conformal mapping.

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1. Royden's ring $M(R)$ and Royden's compactification R^*

1.1. Let R be an open or closed Riemann surface and let BD be the class of all complex-valued bounded continuous functions on R, each of which has piecewise continuous[1] derivatives of the first order and has the finite Dirichlet integral over R. We introduce the norm $\|f\|$ of an element f in BD by

$$\|f\| = \|f\|_\infty + \sqrt{D[f]}, \tag{1.1}$$

where $\|f\|_\infty$ denotes the least upper bound of values $|f(P)|$ when P varies over R and $D[f]$ is the Dirichlet integral of f over R, i.e.,

$$D[f] = \iint_R \left(\left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) dxdy,$$

where $z = x + iy$ is a local parameter.

By the usual algebraic operations, BD is an algebra over the complex number field C with an adjoint operation $f \to f^*$ defined by $f^*(P) = \bar{f}(P)$, where the bar denotes the complex conjugate number. We denote by $M(R)$ the abstract completion of BD by the norm-topology defined by (1.1). As $\|f^*\| = \|f\|$, not only the structure of algebra but also the adjoint operation in BD can be extended to $M(R)$. Thus $M(R)$ is a commutative Banach algebra over the complex number field C, or a so-called normed ring with the adjoint operation.

This ring $M(R)$ was first considered by Royden [10], so we shall call $M(R)$ "*Royden's ring*" of R for brevity.

1.2. Let $z = x + iy$ be a local parameter in R valid for a domain D in R and let Δ be a domain contained in D with its closure mapped onto the plane rectangle $z(\Delta)$: $a < x < b$, $c < y < d$ by the local parameter z. Then we call (Δ, z) the rectangular domain on R.

A complex-valued function $f(P)$ on R is called *absolutely continuous in the sense of Tonelli* (abbreviated as a.c.T.), if, for every rectangular domain (Δ, z), $f(x, y)$ is continuous on the closure $\overline{z(\Delta)}$: $a \leqq x \leqq b$, $c \leqq y \leqq d$ and absolutely continuous with respect to x in the usual sense for almost all values of y in

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1) The precise meaning of piecewise continuity is as follows. Let $R = \cup \Delta_i$ be a triangulation of R. We suppose that the boundary of Δ_i consists of a finite number of analytic arcs. If a complex-valued function $f(P)$ on R is continuous in each of the interior of Δ_i, we say that $f(P)$ is piecewise continuous on R.

$c \leqq y \leqq d$, and absolutely continuous with respect to y in the usual sense for almost all values of x in $a \leqq x \leqq b$, and if further the Dirichlet integral of f, whose existence can be easily seen, is finite over $z(\Delta)$.

Let $\mathfrak{M}(R)$ be the totality of complex-valued bounded continuous functions on R, each of which is a.c.T. on R and has the finite Dirichlet integral over R.

Let the norm and the adjoint operation in $\mathfrak{M}(R)$ be defined by $\|f\| = \|f\|_\infty + \sqrt{D[f]}$ and $f^*(P) = \overline{f(P)}$ as in BD.

We shall deal with the relationship of $M(R)$ and $\mathfrak{M}(R)$. First we can prove the following

LEMMA 1.1. *$\mathfrak{M}(R)$ is a normed ring.*

Proof. We can easily verify that $\mathfrak{M}(R)$ satisfies the condition of the normed ring except its completeness. Hence we have only to prove the completeness of $\mathfrak{M}(R)$.

Let $\{f_n\}$ be a Cauchy sequence in $\mathfrak{M}(R)$. As the uniform norm of an element in $\mathfrak{M}(R)$ is smaller than the norm of $\mathfrak{M}(R)$, $\{f_n\}$ is a Cauchy sequence with respect to the uniform norm. Hence we have a bounded continuous function $f(P)$ on R such that

$$\lim_{n\to\infty} \|f_n - f\|_\infty = 0. \tag{1.2}$$

Denote by $\Gamma(R)$, the totality of differentials μ of the first order on R satisfying

$$\iint_R \mu \wedge {}^*\bar{\mu} < +\infty,$$

where ${}^*\mu$ is a dual differential of μ, ${}^*\bar{\mu}$ is a complex-conjugate of ${}^*\mu$ and $\wedge$ denotes the exterior product. The $\Gamma(R)$ is a Hilbert space with an inner product (μ, ν) defined by

$$(\mu, \nu) = \iint_R \mu \wedge {}^*\bar{\nu},$$

where μ and ν are in $\Gamma(R)$. As we have

$$\sqrt{(df_n - df_m, df_n - df_m)} = \sqrt{D[f_n - f_m]} \leqq \|f_n - f_m\|,$$

where $df_n = \frac{\partial f_n}{\partial x} dx + \frac{\partial f_n}{\partial y} dy$, $\{df_n\}$ is a Cauchy sequence in the sense of strong convergence in $\Gamma(R)$. Hence we have $\alpha = a(z)dx + b(z)dy$ in $\Gamma(R)$ such that

df_n converges strongly to $\alpha \in \Gamma(R)$.

Let (Δ, z) be any rectangular domain on R such that $z(\Delta)$ be a rectangle: $a<x<b$, $c<y<d$ and let (Δ_0, z) be another rectangular domain on R such that Δ_0 contains the closure of Δ and corresponds to $z(\Delta_0)$: $a_0<x<b_0$, $c_0<y<d_0$. Let $\theta(z)$ be a function of the class C^1 with compact carrier contained in Δ_0 and $\theta(z)\equiv 1$ on Δ. We put

$$\tilde{f}_n(z) = \theta(z)f_n(z),\ \tilde{f}(z) = \theta(z)\cdot f(z),$$
$$\tilde{a}(z) = \theta(z)\cdot a(z) + \frac{\partial}{\partial x}\theta(z)\cdot f(z)$$

and

$$\tilde{f}_0(x, y) = \int_{a_0}^{x} \tilde{a}(X, y)dX.$$

Then we see that $\frac{\partial}{\partial x}\tilde{f}_n(z)$ converges strongly to $\tilde{a}(z)$ in the Hilbert space $L^2(\Delta_0)$.

For almost all values of y in $c_0 \leqq y \leqq d_0$, we have

$$\tilde{f}_n(x, y) = \int_{a_0}^{x} \frac{\partial}{\partial x}\tilde{f}_n(X, y)dX \quad (n = 1, 2, 3, \ldots).$$

Using this and the Schwarz inequality, we get

$$|\tilde{f}_n(z) - \tilde{f}_0(z)|^2 \leqq (b_0 - a_0)\int_{a_0}^{b_0}\left|\frac{\partial}{\partial X}\tilde{f}_n(X, y) - \frac{\partial}{\partial X}\tilde{f}_0(X, y)\right|^2 dX,$$

and hence

$$\iint_{z(\Delta_0)} |\tilde{f}_n(z) - \tilde{f}_0(z)|^2\, dxdy \leqq (b_0 - a_0)^2\int_{c_0}^{d_0}\int_{a_0}^{b_0}\left|\frac{\partial}{\partial X}\tilde{f}_n(X, y) - \tilde{a}(X, y)\right|^2 dXdy.$$

This implies that $\tilde{f}_n$ converges strongly to $\tilde{f}_0$ in $L^2(\Delta_0)$. Selecting the subsequence, if necessary, we may assume that $\tilde{f}_n(z)$ converges to $\tilde{f}_0(z)$ almost everywhere in $z(\Delta_0)$. On the other hand, $\tilde{f}_n(z)$ converges uniformly $\tilde{f}(z)$. Hence $\tilde{f}(z) - \tilde{f}_0(z) = 0$ almost everywhere in $z(\Delta_0)$. As $\tilde{f}(z) - \tilde{f}_0(z)$ is continuous with respect to x, $\tilde{f}(z) - \tilde{f}_0(z) \equiv 0$ for almost all values of y in $c_0<y<d_0$,[2] i.e.,

$$\tilde{f}(z) = \int_{a_0}^{x} \tilde{a}(X, y)\,dX$$

[2] As the set $\{(x, y);\ \tilde{f}(x, y) - \tilde{f}_0(x, y) \neq 0\}$ is of 2-dimensional measure zero, we see that the set $\{x;\ \tilde{f}(x, y) - \tilde{f}_0(x, y) \neq 0\}$ is of 1-dimensional measure zero for almost every value of y by Fubini's theorem. For such y, $\tilde{f}(x, y) \equiv \tilde{f}_0(x, y)$ $(a \leqq x \leqq b)$ by the continuity in x.

for almost all values of y in $c_0<y<d_0$. From this and by the definition of $\theta(z)$, we see that $f(x, y)$ is absolutely continuous with respect to x in $a\leqq x\leqq b$ for almost all values of y in $c\leqq y\leqq d$ and further

$$\frac{\partial}{\partial x}f(z) = a(z).$$

Similarlly, $f(x, y)$ is absolutely continuous with respect to y in $c\leqq y\leqq d$ for almost all values of x in $a\leqq x\leqq b$ and

$$\frac{\partial}{\partial y}f(z) = b(z).$$

Thus we have $df=\alpha$ and f is a.c.T. As $\alpha=df$ is the strong limit point of the sequence $\{df_n\}$ in $\Gamma(R)$, it holds that

$$\lim_{n\to\infty} \sqrt{D[f_n-f]} = 0. \tag{1.3}$$

From these, we can say that f is in $\mathfrak{M}(R)$ and f is the limit point of the sequence $\{f_n\}$ in $\mathfrak{M}(R)$. Thus $\mathfrak{M}(R)$ is complete. q.e.d.

Next we prove

LEMMA 1.2. *Suppose that an element f in $C^n\cap\mathfrak{M}(R)$*[3] *has compact carrier car. f in a rectangular domain $(\varDelta, z)$ and that a positive number ε is given. Then we can find an element g in $C^{n+1}\cap\mathfrak{M}(R)$ such that car. g is contained in $\varDelta$ and $\|f-g\|\leqq\varepsilon$.*

Proof. We may assume without loss of generality that R is the complex z-plane and $\varDelta$ is the rectangle $a<x<b$, $c<y<d$.

Let the distance between car. f and the boundary of $\varDelta$ be $3d$ and choose a number ρ satisfying $0<\rho<d$. Denote by $M_\rho f(z)$ the integral

$$M_\rho f(z) = \frac{1}{\pi\rho^2}\iint_{|Z|<\rho} f(z+Z)\,dX\,dY$$

over ρ-disc with the center z, where $z=x+iy$ and $Z=X+iY$. It is well known that $M_\rho f(z)$ is of class C^{n+1}. It is easy to see that car. $M_\rho f$ is contained in $\varDelta$. By the uniform continuity of $f(z)$, we get

$$\lim_{\rho\downarrow 0} \|M_\rho f - f\|_\infty = 0. \tag{1.4}$$

[3] The number n is an integer $\geqq 0$. Here C^0 denotes the class of continuous functions.

Next we have

$$\frac{\partial}{\partial x} M_\rho f(z) = M_\rho \frac{\partial}{\partial x} f(z). \tag{1.5}$$

In fact, since $f(x, y)$ is a.c.T., by Fubini's theorem

$$\begin{aligned}\int_a^x M_\rho \frac{\partial}{\partial x} f(z)dx &= \frac{1}{\pi\rho^2}\iint_{|Z|<\rho} dXdY \int_a^x \frac{\partial}{\partial x} f(z+Z)dx \\ &= \frac{1}{\pi\rho^2}\iint_{|Z|<\rho} f(z+Z)dXdY \\ &= M_\rho f(z).\end{aligned}$$

From this and the continuity of $M_\rho \frac{\partial}{\partial x} f(z)$, we have (1.5).

As $\frac{\partial}{\partial x} f(z)$ is in L^2, it is well known that $M_\rho \frac{\partial}{\partial x} f(z)$ converges to $\frac{\partial}{\partial x} f(z)$ strongly in $L^2(R)$ as $\rho \downarrow 0$. Hence $\frac{\partial}{\partial x} M_\rho f(z)$ converges strongly to $\frac{\partial}{\partial x} f(z)$ in $L^2(R)$ as $\rho \downarrow 0$. The similar argument holds for $\frac{\partial}{\partial y}$. Hence we obtain

$$\lim_{\rho\downarrow 0} \sqrt{D[M_\rho f - f]} = 0. \tag{1.6}$$

From (1.4) and (1.6), we have only to put $g = M_\rho f$ for a sufficiently small number ρ. q.e.d.

By the iterated use of Lemma 1.2, we obtain the following

LEMMA 1.3. *Suppose that an element f in $\mathfrak{M}(R)$ has compact carrier in a rectangular domain (Δ, z) and that a positive number ε and an integer $n (\geqq 0)$ are given. Then there exists an element g in $C^n \cap \mathfrak{M}(R)$ such that car. g is contained in Δ and $\|f-g\| \leqq \varepsilon$.*

Now we omit the restriction on the carrier of f, namely,

LEMMA 1.4. *For any f in $\mathfrak{M}(R)$, a positive number ε and an integer n, there exists an element g in $C^n \cap \mathfrak{M}(R)$ such that*

$$\|f-g\| \leqq \varepsilon.$$

Proof. Let $\{\Delta_j\}_{j=1}^\infty$ be a locally finite covering[4] of R consisting of rectangular domains Δ_j in R and let $\{\phi_i\}_{i=1}^\infty$ be a resolution of unity satisfying the

[4] If R is compact, we consider that Δ_j are empty except a finite number.

following: (i) ϕ_i is in class C^n, (ii) *car.* ϕ_i is contained in a Δ_j and (iii) $\{car.\ \phi_i\}$ is locally finite.

Put $f_i = f\phi_i$. Then, for any compact set K, there exists a number N_K such that

$$f(P) = \sum_{i=1}^{N_K} f_i(P)$$

at any point P in K. As f_i satisfies the requirment in Lemma 1.3, we can find g_i in $C^n \cap \mathfrak{M}(R)$ such that *car.* g_i is contained in Δ_j and

$$\|f_i - g_i\| < \frac{\varepsilon}{2^{i+1}}\,. \tag{1.6}$$

It is easy to see that $\{car.\ g_i\}$ is locally finite and thus there exists a number N'_K for any compact set K such that

$$g(P) = \sum_{i=1}^{\infty} g_i(P) = \sum_{i=1}^{N'_K} g_i(P)$$

at any P in K. Thus g is in class C^n.

Let $\{R_m\}$ be an exhaustion of R. For a fixed m, we can find a number N such that for all P in R_m

$$f(P) - g(P) = \sum_{i=1}^{N} (f_i(P) - g_i(P))$$

and

$$\frac{\partial}{\partial x} f(z(P)) - \frac{\partial}{\partial x} g(z(P)) = \sum_{i=1}^{N} \Big(\frac{\partial}{\partial x} f_i(z(P)) - \frac{\partial}{\partial x} g_i(z(P)) \Big).$$

From these and (1.6), we get

$$\|f-g\|_{\infty, R_m} \leqq \frac{\varepsilon}{2} \text{ and } \sqrt{D_{R_m}[f-g]} \leqq \frac{\varepsilon}{2}\,.$$

Letting $m \to \infty$, we have

$$\|f-g\|_{\infty} \leqq \frac{\varepsilon}{2} \text{ and } \sqrt{D[f-g]} \leqq \frac{\varepsilon}{2}\,.$$

Thus we see that g is in $\mathfrak{M}(R) \cap C^n$ and $\|f-g\| \leqq \varepsilon$. q.e.d.

1.3. It is easily seen that

$$C^1 \cap \mathfrak{M}(R) \subset BD \subset \mathfrak{M}(R).^{5)}$$

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[5] cf. foot note 1).

By Lemmas 1.1, 1.4 and the definition of $M(R)$, it holds $\mathfrak{M}(R)=M(R)$ and thus we obtain the following

THEOREM 1. *(i) Royden's ring on a Riemann surface R is the totality of complex-valued bounded continuous functions on R each of which is absolutely continuous in the sense of Tonelli on R and has finite Dirichlet integral over R.*

(ii) The norm of f in $M(R)$ is given by (1.1).

(iii) The set $C^n \cap M(R)$ is dense in $M(R)$ $(n=1,2,3,\ldots)$.

1.4. Besides the norm-topology defined by (1.1), we use another topologies in $M(R)$. They are as follows.

(a) *B-topology*: the sequence $\{f_i\}$ in $M(R)$ converges to 0 in B-topology if the sequence $\{\|f_i\|_\infty\}$ is bounded and $f_i(P)$ converges to 0 uniformly on every compact set.

(b) *D-topology*: the sequence $\{f_i\}$ in $M(R)$ converges to 0 in D-topology if the sequence $\{D[f_i]\}$ converges to 0.

(c) *Uniform topology*: the sequence $\{f_i\}$ in $M(R)$ converges to 0 in the uniform topology if the sequence $\{\|f_i\|_\infty\}$ converges to 0.

(d) *BD-topology*: the sequence $\{f_i\}$ in $M(R)$ converges to 0 in the BD-topology if $\{f_i\}$ converges to 0 in B-topology and also in D-topology.

(e) *Norm topology*: The sequence $\{f_i\}$ in $M(R)$ converges to 0 in the norm topology if $\{\|f_i\|\}$ converges to 0; the relation between these topologies are stated as follows:

$$\begin{array}{ccccc} B\text{-topology} & \longrightarrow & \text{uniform topology} & \searrow & \\ & \searrow & & & \text{norm topology,} \\ D\text{-topology} & \longrightarrow & BD\text{-topology} & \nearrow & \end{array}$$

where $\rightarrow$ means "is weaker than".

Obviously $M(R)$ is not complete with respect to B- or D- or uniform topology. But we can prove the following.

LEMMA 1.5. *$M(R)$ is complete with respect to the BD-topology.*

The proof of Lemma 1.1 can be applied nearly verbatim to this case and so we omit the proof.

1.5. We denote by $M_0(R)$ the totality of elements in $M(R)$ with the compact carrier and by $M_1(R)$ the closure of $M_0(R)$ in the BD-topology. It is easily verified that $M_0(R)$ and $M_1(R)$ are ideals of $M(R)$ and that these are closed under the adjoint operation.

By Theorem 1 and Royden's decomposition of BD (see [9]), we get the following

LEMMA 1.6. *(Royden) $M(R) = HBD \oplus M_1(R)$, that is, any element f of $M(R)$ has the unique decomposition*

$$f = u + g, \ \|u\| \leqq \|f\| \text{ and } (du, dg) = 0 \text{ in } \Gamma(R),^{6)}$$

where u is in HBD (the class of harmonic functions in BD) and g is in $M_1(R)$.

1.6. A *character* of $M(R)$ is a functional on $M(R)$ satisfying

(1. 7) $\quad \chi(f+g) = \chi(f) + \chi(g), \ \chi(\alpha f) = \alpha\chi(f),$

(1. 8) $\quad \chi(fg) = \chi(f)\,\chi(g),$

(1. 9) $\quad \chi(f^*) = \overline{\chi(f)},$

(1.10) $\quad \chi(1) = 1.$

From (1.8) and (1.9), the character is positive, i.e.,

(1.11) $\quad f \geqq 0$ implies $\chi(f) \geqq 0.$

As $\|f\|_\infty^2 - ff^*$ is non-negative, we get

(1.12) $\quad |\chi(f)| \leqq \|f\|_\infty \leqq \|f\|$

by virtue of (1.11).

Thus the character space R^* (the totality of characters of $M(R)$) is a subspace of the conjugate space $M(R)^*$ of $M(R)$ as a Banach space.

The point of R can be considered as a character by defining

$$P(f) = f(P) \qquad (f \in M(R)).$$

We call this character $P(f)$ a point character. In this sense R is embedded in R^*.

We introduce a topology in R^* by the induced topology of the weak topology $\sigma(M(R)^*, M(R))$ (see Bourbaki [3]) in $M(R)^*$. For the brevity this is denotod by $\sigma(R^*, M(R))$. Then R^* is a compact Hausdorff space[7] and the

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6) For $\Gamma(R)$, cf. the proof of Lemma 1.1.

7) By (1.10) and (1.12), R^* is in the surface of the unit sphere in $M(R)^*$. It is clear that R^* is $\sigma(M(R)^*, M(R))$-closed in $M(R^*)$. Hence, by the well known theorem in the theory of Banach spaces, R^* is compact subset of $M(R)^*$.

point character space R is an open subset of R^* which is dense in R^* and is homeomorphic to the Riemann surface R. In fact, $R^* - R$ is closed in R^* as $R^* - R$ is the totality of characters which vanish on $M_0(R)$, and that by the general theory of normed ring, the semi-simple normed ring with an adjoint operation is represented as the dense subset of the continuous function space $C(R^*)$ (Loomis [4]).

This R^* is first introduced by Royden to investigate the ideal boundary of R (Royden [10]). So we shall call R^* "*Royden's compactification*" of R for brevity. It is clear that every function in $M(R)$ can be extended continuously to whole R^* uniquely.

The set $I(R) = R^* - R$ is called the *ideal boundary* of R. As stated already, the kernel $N_\chi = \{f \in M(R);\ \chi(f) = 0\}$ of the character χ in $I(R)$ contains $M_0(R)$. The totality of χ such that N_χ contains $M_1(R)$, if exists, will be denoted by $I_1(R)$ and the remainder set $I(R) - I_1(R)$ will be denoted by $I_2(R)$. The set $I_1(R)$ is called the *harmonic boundary* (Royden [10]) and $I_2(R)$ is called the non-harmonic boundary.

2. The Normal isomorphism induced by quasiconformal mapping

2.1. Let R and R' be two open or closed Riemann surfaces and let $M(R)$ and $M(R')$ be Royden's rings of R and R' respectively. Suppose that the one-to-one mapping φ of $M(R)$ onto $M(R')$ satisfies the following conditions:

(2.1) φ is an isomorphism of the C-algebra[8] $M(R)$ onto the C-algebra $M(R')$ preserving the adjoint operation.

(2.2) φ is bicontinuous with respect to the B-topology,

and

(2.3) φ is bicontinuous with respect to the D-topology.

Then we shall call φ the ***normal isomorphism*** of $M(R)$ onto $M(R')$.

We denote the totality of functions on R by $\mathfrak{F}(R)$. For a one-to-one mapping T of R onto R', we can define the one-to-one mapping φ_T of $\mathfrak{F}(R)$ onto $\mathfrak{F}(R')$ by

$$\varphi_T f(P) = f(T^{-1}P) \qquad (P \in R',\ f \in \mathfrak{F}(R)).$$

Then φ_T is an isomorphism between C-algebras $\mathfrak{F}(R)$ and $\mathfrak{F}(R')$. The φ_T is

[8] I.e. an algebra over the complex-number field C.

called the induced isomorphism by T of $\mathfrak{F}(R)$ onto $\mathfrak{F}(R')$.

2.2. A homeomorphism T of R onto R' is called the quasi-conformal mapping if the ratio of moduli of corresponding quadrilaterals by T is bounded. This definition is due to Pfluger and Ahlfors (see [1]).

The following simple fact seems to have some applications besides its own interest.

THEOREM 2. *If T is a quasi-conformal mapping of R onto R', then the induced mapping φ_T restricted on $M(R)$ is the normal isomorphism of $M(R)$ onto $M(R')$.*

Proof. Let the maximal dilatation of T (and hence of T^{-1}) be K. Let the local equation of T^{-1} be

$$u = u(x, y),\ v = v(x, y),$$

where $w = u + iv$ and $z = x + iy$ are local parameters in R and R' respectively. It is known that T^{-1} is a measurable mapping and further we have

$$m(T^{-1}e) = \iint_e J(z)\, dxdy.$$

Here e is a measurable set in R' for which the local parameter z is valid and the local parameter w is valid for $T^{-1}e$ in R and $m(T^{-1}e)$ denotes the measure of the measurable set $T^{-1}e$ and finally

$$J(z) = \frac{\partial}{\partial x} u(z) \frac{\partial}{\partial y} v(z) - \frac{\partial}{\partial y} u(z) \frac{\partial}{\partial x} v(z)$$

almost everywhere in e. The same as above holds also for T (c.f. Bers [2]). Using this fact and the Schwarz inequality, we obtain

$$K^{-1}D[f] \leqq D[\varphi_T f] \leqq KD[f]. \tag{2.4}$$

It is easily seen that φ_T is bicontinuous in the B-topology in $\mathfrak{F}(R)$.

If f is in $C^1 \cap M(R)$, then $\varphi_T f$ is in $M(R')$. Theorem 1 implies that, for any f in $M(R)$, we can find a sequence $\{f_n\}$ in $C^1 \cap M(R)$ such that f_n converges to f in the norm topology (and hence in the BD-topology). As we have seen that φ_T is continuous in the BD-topology, the sequence $\{\varphi_T f_n\}$ in $M(R')$ converges to $\varphi_T f$ in the BD-topology. By Lemma 1.5, $M(R)$ is complete in the BD-topology and hence $\varphi_T f$ is in $M(R')$. Therefore, we have that

$\varphi_T M(R) \subset M(R')$. Similarly, it holds that $\varphi_{T^{-1}} M(R') \subset M(R)$. Hence we get $M(R) = M(R')$.

It is seen by the above argument that the conditions (2.1)-(2.3) for φ_T are satisfied. Hence φ_T is normal. Thus we obtain our theorem.

3. Quasi-conformal mapping induced by normal isomorphism

3.1. First we give the proof of the following.

LEMMA 3.1. *Let φ be a one-to-one mapping of $M(R)$ onto $M(R')$.*

(i) *If φ satisfies* (2.1), *then φ is isometric with respect to the uniform norm, i.e.,*

$$\|\varphi f\|_\infty = \|f\|_\infty. \tag{3.1}$$

(ii) *If φ satisfies* (2.1) *and* (2.3), *then there exists a positive constant K such that*

$$K^{-1}\sqrt{D[f]} \leqq \sqrt{D[\varphi f]} \leqq K\sqrt{D[f]} \tag{3.2}$$

and

$$K^{-1}\|f\| \leqq \|\varphi f\| \leqq K\|f\|. \tag{3.3}$$

Hence the normal isomorphism is bicontinuous in the norm topology.

Proof. For the uniform norm, we can prove

$$\|f\|_\infty = \sup\{|\lambda|;\ f-\lambda \text{ has no inverse in } M(R)\}. \tag{3.4}$$

To show this, we denote by a the right side of (3.4). If $\|f\|_\infty < |\lambda|$, it is easily seen that $1/(f(P)-\lambda)$ is the function in $M(R)$ and actually the inverse of $f-\lambda$. Hence we have $a \leqq \|f\|_\infty$. On the other hand, $f-f(P)$ is not invertible in $M(R)$ for fixed P in R. Thus we see $|f(P)| \leqq a$, or $\|f\|_\infty \leqq a$. So we have (3.4).

Similarly we get

$$\|\varphi f\|_\infty = \sup\{|\lambda|;\ \varphi f-\lambda \text{ has no inverse in } M(R')\}. \tag{3.5}$$

As $\varphi f - \lambda = \varphi(f-\lambda)$ is invertible if and only if $f-\lambda$ is invertible, the right sides of (3.4) and (3.5) are coincident, which proves (i).

To prove (3.2), we have only to show that the existence of a positive constant k such that $D[\varphi f] \leqq k\ D[f]$. If this is false, we can choose a sequence $\{f_n\}$ in $M(R)$ such that $D[\varphi f_n] > nD[f_n]$ as usual. Then $g_n = f_n/n\ D[f]$ satisfies $D[g_n] < 1/n$ and $D[\varphi g_n] > 1$, which contradicts (2.3).

From (3.1) and (3.2) follows (3.3). q.e.d.

REMARK. The following problem seems important and is still open: Can we conclude that φ is normal if φ satisfies merely (2.1) and is continuous (and so bicontinuous by Banach's theorem) with respect to the norm topology?[9]

Next we prove the following

LEMMA 3.2. *If φ is a normal isomorphism of $M(R)$ onto $M(R')$, then there exists the unique homeomorphism T^* of R^* onto R'^* satisfying the followings:*

(i) $$T^*R = R',\ T^*I_i(R) = I_i(R') \qquad (i = 1, 2)$$

and

(ii) $$\varphi = \varphi_T,$$

where T is the restriction of T^* on R.

Proof. For χ in R^*, we define a functional $T^*\chi$ on $M(R')$ by the formula

$$T^*\chi(f') = \chi(\varphi^{-1}f').$$

Then it is quite easy to verify that $T^*\chi$ satisfies the conditions (1.7)-(1.10). Thus T^* defines a mapping of R^* into R'^*. It is one-to-one and onto. By the definition of $\sigma(R^*, M(R))$ and $\sigma(R'^*, M(R'))$, we see that T^* is bicontinuous.

Now we show that $\chi' = T^*\chi$ is in R' if and only if χ is in R. Assume that $\chi = P$ is in R. The image of the kernel $N_P = \{f \in M(R);\ P(f) = f(P) = 0\}$ of χ by φ is the kernel $N_{\chi'}$ of $\chi' = T^*\chi$. Here we notice that N_P is closed in the B-topology in $M(R)$. In fact, if the sequence $\{f_i\}$ in N_P converges to f in $M(R)$ in the B-topology, then $f_i(P) = 0$ implies $f(P) = 0$. Hence we have f in N_P.

Suppose that $\chi' = T^*\chi$ is not in R'. Then $N_{\chi'}$ contains $M_0(R')$. Hence we can select a sequence $\{f'_i\}$ in $N_{\chi'}$ such that f'_i converges to 1 in the B-topology. By (2.2), $N_{\chi'} = \varphi N_\chi$ is closed in the B-topology in $M(R')$ along with N_χ. Hence it holds that $N_{\chi'}$ contains 1, which contradicts the condition (1.10). Hence χ' is in R'.

[9] The converse of Lemma 3.1.

Similarly we can prove that χ is in R if $P' = T^*\chi$ is in R'.

Next we show that (ii) holds. Let P' be in R'. Then we have

$$\begin{aligned}\varphi f(P') &= \varphi f(TT^{-1}P') = TT^{-1}P'(\varphi f)\\ &= T^*T^{-1}P'(\varphi f) = T^{-1}P'(\varphi^{-1}\cdot\varphi f)\\ &= T^{-1}P'(f) = f(T^{-1}P')\\ &= \varphi_T f(P'),\end{aligned}$$

which proves $\varphi = \varphi_T$.

By (ii) and $T^*R = R'$, we can see that $\varphi_T f$ is of compact carrier if and only if f is of compact carrier, i.e.,

$$\varphi(M_0(R)) = M_0(R').$$

As φ is bicontinuous in the BD-topology, we have

$$\varphi(M_1(R)) = M_1(R').$$

From this and the definition of $I_1(R)$, we can conclude that

$$T^*I_1(R) = I_1(R').$$

By (ii), such T^* is uniquely determined on R. On the other hand, R and R' are dense in R^* and R'^* respectively. Hence by the bicontinuity of T^*, T^* is uinquely determined. q.e.d.

3.2. By the annulus $\Omega = (C_0, C_1)$ on R, we mean the subset of R which is conformally equivalent to the plane domain: $1 < |z| < e^{\mu}$, where the Jordan curves C_0 and C_1 correspond to $|z| = e^{\mu}$ and $|z| = 1$ respectively. We shall assume that the 2-dimensional measure of $C_0 \cup C_1$ is zero.

The uniquely determined number μ is called the modulus of Ω and is denoted by mod.Ω.

Let the harmonic measure of C_1 with respect to Ω be $\omega(P, \Omega)$, i.e., the harmonic function in Ω which is equal to zero on C_0 and to 1 on C_1. Then it holds that

$$2\pi(\text{mod. }\Omega)^{-1} = D_\Omega[\omega] = \iint_\Omega \Big[\Big(\frac{\partial w}{\partial x}\Big)^2 + \Big(\frac{\partial w}{\partial y}\Big)^2 \Big] dxdy. \tag{3.6}$$

Now we suppose that a homeomorphism T of R onto R' satisfies the following condition;

(3.7) *for any annulus Ω in R, the inequality*

$$\text{mod. } \Omega \leqq K \text{ mod. } T\Omega \leqq K^2 \text{ mod. } \Omega$$

holds, where K is a positive constant depending only on T.

If T is a quasi-conformal mapping, then it is known that T satisfies (3.7) for the maximal dilation K of T (Ahlfors [1], Mori [5]). Conversely, if a homeomorphism T satisfies (3.7), it is quasi-conformal (cf. Yûjôbô [12]). We state this as follows.

LEMMA 3.3. *A homeomorphism T of R onto R' satisfying* (3.7) *is a quasi-conformal mapping.*

Proof. As the quasi-conformality of a mapping is a local property, we may assume without loss of generality that R and R' are unit discs in the complex plane. Moreover, we may assume T is sense-preserving, for, if it is sense-reversing, then we may replace T by $\bar{T}$.

From merely (3.7), it holds that

$$M(r)/m(r) \leqq e^{\pi K},$$

where $M(r)$ and $m(r)$ are the maximum and the minimum distance of the image curve of the circle $|z| = r$ from the image point of the center $z = 0$ for a sufficiently small positive number r (A. Mori [5]).

From this, T is totally differentiable almost everywhere in R (A. Mori [5]). At the point z where T is totally differentiable, we get

$$\left|\frac{\dfrac{\partial T}{\partial z} + e^{-2i\theta_1}\dfrac{\partial T}{\partial \bar{z}}}{\dfrac{\partial T}{\partial z} + e^{-2i\theta_2}\dfrac{\partial T}{\partial \bar{z}}}\right| = \lim_{r\to 0}\left|\frac{\dfrac{T(z+re^{i\theta_1}) - T(z)}{r}}{\dfrac{T(z+re^{i\theta_2}) - T(z)}{r}}\right|$$

$$= \lim_{r\to 0}\frac{|T(z+re^{i\theta_1}) - T(z)|}{|T(z+re^{i\theta_2}) - T(z)|}$$

$$\leqq \lim_{r\to 0} M(r)/m(r) \leqq e^{\pi K},$$

where $\dfrac{\partial}{\partial z}$ and $\dfrac{\partial}{\partial \bar{z}}$ are formal complex derivatives. For suitably chosen θ_1 and θ_2, we get

$$\frac{\left|\dfrac{\partial T}{\partial z}\right| + \left|\dfrac{\partial T}{\partial \bar{z}}\right|}{\left|\dfrac{\partial T}{\partial z}\right| - \left|\dfrac{\partial T}{\partial \bar{z}}\right|} \leqq e^{\pi K},$$

or

$$\frac{\partial T}{\partial \bar{z}} \Big/ \frac{\partial T}{\partial z} \leqq k = \frac{e^{\pi K}-1}{e^{\pi K}+1} < 1.$$

Then T is a homeomorphic solution of the Bertrami equation

$$w_{\bar{z}} = \mu w_z,$$

where $\mu = \frac{\partial T}{\partial \bar{z}} \Big/ \frac{\partial T}{\partial z}$. Thus T is quasi-conformal in the sense of Pfluger-Ahlfors (Bers [2]).

Using this characterization of quasi-conformality, we get

LEMMA 3.4. *T in Lemma 3.2 is a quasi-conformal mapping.*

Proof. Let $\tilde{\Omega} = (\tilde{C}_0, \tilde{C}_1)$ be an annulus in R and let $\Omega = (C_0, C_1)$ be the image of $\tilde{\Omega}$ by T. We suppose that $\tilde{\Omega}$ and Ω are contained in rectangular domains in R and R' respectively.

Let $\tilde{u}$ be harmonic in $\tilde{\Omega}$ and be identically equal to 1 inside $\tilde{C}_1$ and to 0 outside $\tilde{C}_0$ and, further, be continuous on R. We also consider the similar function u for Ω. Then $\tilde{u}$ and u are harmonic measures of $\tilde{\Omega}$ and Ω when they are restricted on $\tilde{\Omega}$ and Ω respectively. From (3.6), we have

$$2\pi(\text{mod. } \tilde{\Omega})^{-1} = D[\tilde{u}] \text{ and } 2\pi(\text{mod. } \Omega)^{-1} = D[u]. \tag{3.8}$$

For the function $\varphi\tilde{u}$ in $M(R')$, we can select a sequence $\{v_n\}$ such that $v_n \in C^1 \cap M(R')$ and

$$\lim_{n\to\infty} \|v_n - \varphi\tilde{u}\| = 0. \tag{3.9}$$

This follows from Theorem 1. In particular, from (3.9),

$$\lim_{n\to\infty} D[v_n] = D[\varphi\tilde{u}]. \tag{3.10}$$

Let u_n be identical with v_n outside Ω and with the harmonic function inside Ω defined by the boundary value v_n on C_0 and C_1. Then, by Dirichlet principle, we get

$$D[u_n] \leqq D[v_n]. \tag{3.11}$$

By (3.9), it holds that $u(P) = \varphi\tilde{u}(P) = \lim_{n\to\infty} v_n(P) = \lim_{n\to\infty} u_n(P)$ on $C_0 \cup C_1$. So we have

$$\frac{\partial}{\partial x} u(P) = \lim_{n\to\infty} \frac{\partial}{\partial x} u_n(P), \quad \frac{\partial}{\partial y} u(P) = \lim_{n\to\infty} \frac{\partial}{\partial y} u_n(P) \tag{3.12}$$

inside Ω by Harnack's theorem. This holds also outside Ω almost everywhere by selecting subsequence, if necessary.

Moreover, we get

$$D[u] \leqq \lim_{n\to\infty} D[u_n]. \tag{3.13}$$

In fact,

$$\begin{aligned} D[u] &= \iint_R \left[\left| \frac{\partial}{\partial x} u(z) \right|^2 + \left| \frac{\partial}{\partial y} u(z) \right|^2 \right] dx\,dy \\ &= \iint_R \varliminf_{n\to\infty} \left[\left| \frac{\partial}{\partial x} u_n(z) \right|^2 + \left| \frac{\partial}{\partial y} u_n(z) \right|^2 \right] dx\,dy \\ &\leqq \varliminf_{n\to\infty} \iint_R \left[\left| \frac{\partial}{\partial x} u_n(z) \right|^2 + \left| \frac{\partial}{\partial y} u_n(z) \right|^2 \right] dx\,dy \\ &= \varliminf_{n\to\infty} D[u_n] \end{aligned}$$

by (3.12) and by Fatou's lemma. Noticing Lemma 3.1, we have

$$D[\varphi\tilde{u}] \leqq K\, D[\tilde{u}]. \tag{3.14}$$

Using (3.13), (3.11), (3.10) and (3.14), we see that

$$\begin{aligned} D[u] &\leqq \varliminf_{n\to\infty} D[u_n] \leqq \varliminf_{n\to\infty} D[v_n] \\ &= \lim_{n\to\infty} D[v_n] = D[\varphi\tilde{u}] \\ &\leqq K\, D[\tilde{u}]. \end{aligned}$$

From this and (3.8), we have

$$\text{mod.}\tilde{\Omega} \leqq K \,\text{mod.}\Omega.$$

By Lemma 3.3, we can see that T is quasi-conformal. q.e.d.

3.3. By Lemma 3.2 and Lemma 3.4, we obtain the following

THEOREM 3. *If Royden's rings $M(R)$ and $M(R')$ of Riemann surfaces R and R' are normally isomorphic by the correspondence φ, then there exists a unique homeomorphism T^* between the Royden's compactifications R^* and R'^* of R and R' satisfying*

(i) $$T^*R = R', \ T^*I_i(R) = I_i(R') \qquad (i = 1, 2)$$

and

(ii) *T is a quasi-conformal mapping of R onto R' such that*

$$\varphi = \varphi_T,$$

where $I_1(R)$ *and* $I_2(R)$ *are harmonic and non-harmonic ideal boundary of* R *and* T *is the restriction of* T^* *on* R *and* φ_T *is the induced isomorphism of* T.

By Theorems 2 and 3, we can say that

THEOREM 4. *Two Riemann surfaces are quasi-conformally equivalent if and only if their Royden's rings are normally isomorphic.*

4. Correspondence between ideal boundaries by quasi-conformal mappings

4.1. Let T be a quasi-conformal mapping of R onto R'. Then φ_T is a normal isomorphism of $M(R)$ onto $M(R')$ by Theorem 2. Using Theorem 3, we can find a unique homeomorphism T^* of R^* onto R'^* such that $T^* = T$ on R and T^* carries $I_1(R)$ and $I_2(R)$ onto $I_1(R')$ and $I_2(R')$ respectively.

Thus we get the following

THEOREM 5. *For a quasi-conformal mapping* T *of* R *onto* R', *there exists a unique homeomorphism* T^* *of the Royden's compactification* R^* *of* R *onto the Royden's compactification* R'^* *of* R' *such that*

(i) $$T^* = T \text{ on } R$$

and

(ii) $$T^*I_i(R) = I_i(R') \qquad (i = 1, 2),$$

where $I_1(R)$ *and* $I_2(R)$ (*or* $I_1(R')$ *and* $I_2(R')$) *are the harmonic and the non-harmonic ideal boundary of* R(*or* R') *respectively.*

As a direct consequence of Theorem 5, we get the following

THEOREM 6. *The properties on Riemann surfaces, depending only on*

(a) *the set theoretical structure*[10] *of the harmonic ideal boundary*

or

(b) *the topological structure of the harmonic ideal boundary, are preserved by the quasi-conformal mapping.*

As an example, we state a property on Riemann surfaces depending only on (a) or (b).

[10] I.e. the cardinal number.

4.2. As usual, we denote by O_G the class of Riemann surfaces without Green's function and by O_{HD} the class of Riemann surfaces on which no non-constant harmonic function with the finite Dirichlet integral exists.

The O_G(or O_{HD})-property depends only on the cardinal number of the harmonic ideal boundary, for we have the following simple lemma (cf. Royden [10], S. Mori [6]).

LEMMA 4.1. *Let R be a Riemann surface.*

(i) *R belongs to the class O_G if and only if $I_1(R)$ is empty.*

(ii) *R belongs to the class $O_{HD} - O_G$ if and only if $I_1(R)$ consists of only one point.*

Proof. Royden [8] has given a characterization of the class O_G as follows; using our notation,

R belongs to O_G if and only if 1 *belongs to $M_1(R)$.*

The fact $1 \in M_1(R)$ means that there exists no character vanishing on $M_1(R)$, or $I_1(R)$ is empty, which proves (i).

By Lemma 1.6, R belongs to $O_{HD} - O_G$ if and only if

$$M(R) = C \oplus M_1(R).$$

In this case, $M(R)/M_1(R) = C$. This shows that the character vanishing on $M_1(R)$ is only the canonical homeomorphism of $M(R)$ onto $M(R)/M_1(R)$. Hence $I_1(R)$ is one point, which proves (ii). q.e.d.

By Theorem 6 and Lemma 4.1, *a quasi-conformal mapping preserves the class O_G and the class $O_{HD} - O_G$,* which gives an alternating proof of a theorem of Pfluger [8] and Royden [11] on the invariance of the classes O_G and O_{HD} by the quasi-conformal mappings.

4.3. Next we state a property on Riemann surfaces depending only on the topological structure of the harmonic ideal boundaries.

Let $\mathfrak{N}$ be a class of some real-valued functions on an abstract space $\mathfrak{N}$. A non-negative and non-zero function $f(P)$ in $\mathfrak{N}$ is called *minimal* in $\mathfrak{N}$ if for any g in $\mathfrak{N}$ satisfying

$$f(P) \geqq g(P) \geqq 0 \qquad (P \in \mathfrak{N})$$

we can find a non-negative real number c_g such that

$$g(P) = c_g f(P) \qquad (P \in \mathfrak{N}).$$

We denote by $M_{\mathfrak{N}}$ the class of Riemann surfaces, on each of which there exists a minimal function in the class $\mathfrak{N}$ of some real-valued functions. We also denote by HBD^* the completion of HBD in the uniform-topology. Hence HBD^* is a class of harmonic functions on a Riemann surface R which is continuous on R^*. Here we treat only M_{HBD^*}. Clearly we have $M_{HBD^*} \supset O_{HD} - O_G$.

LEMMA 4.2. *Let $\mathfrak{A}$ be a compact Hausdorff space. The continuous function space $C(\mathfrak{A})$ contains a minimal function if and only if $\mathfrak{A}$ contains an isolated point.*

A minimal function in $C(\mathfrak{A})$ is a characteristic function of an isolated point in $\mathfrak{A}$.

Proof. Suppose P_0 is an isolated point in $\mathfrak{A}$. Then the function $e(P, P_0)$ defined by

$$e(P, P_0) = \begin{cases} 1 \text{ on } P_0 \\ 0 \text{ on } \mathfrak{A} - P_0 \end{cases}$$

is a continuous function and minimal in $C(\mathfrak{A})$.

Conversely, suppose that $g(P)$ is minimal in $C(\mathfrak{A})$. We can find a point P_0 in $\mathfrak{A}$ such that $g(P_0) = 2\rho > 0$. Let $P_1 (\neq P_0)$ be a point in $\mathfrak{A}$. We can find a neighbourhood U of P_0 such that $P_1 \notin U$ and $g(P) > \rho$ in U. Choose a function $f(P)$ in $C(\mathfrak{A})$ such that $f(P_0) = \rho$, $f(P) = 0 (P \in \mathfrak{A} - U)$ and $0 \leqq f(P) \leqq \rho$. Then $0 \leqq f(P) \leqq g(P)$ $(P \in \mathfrak{A})$ and $f(P_0) = 1/2\, g(P_0)$. Hence, by the minimality of g, we have $2f(P) = g(P)$. In particular, we have $g(P_1) = 2f(P_1) = 0$.

Thus $g(P) = 0$ except $P = P_0$, or $g(P) = e(P, P_0)$. Thus P_0 is an isolated point in $\mathfrak{A}$. q.e.d.

LEMMA 4.3. *A Riemann surface R belongs to the class M_{HBD^*} if and only if the harmonic ideal boundary $I_1(R)$ of R contains at least one isolated point with respect to $I_1(R)$.*

Proof. Suppose there is given a continuous function $g(\chi)$ on $I_1(R)$. It is extended continuously to whole R^*. We denote it again $g(\chi)$. Then $g(\chi)$ is approximated uniformly by the sequence $\{g_n(\chi)\}$ in $M(R)$. Then the harmonic part (cf. Lemma 1.6) $u_n(\chi)$ of $g_n(\chi)$ converges uniformly to a function u in HBD^*. Clearly $u(\chi) = g(\chi)$ $(\chi \in I_1(R))$. Such u is determined uniquely by the distribution $g(Z)$ on $I_1(R)$, for functions in HBD takes its minimum and maximum on $I_1(R)$ (cf. S. Mori and M. Ôta [7]). Thus HBD^* is isometrically C-module isomorphic to the continuous function space $C(I_1(R))$ preserving the positiveness. Hence HBD^* contains minimal function if and only if $C(I_1(R))$ does so. Thus we get our assertion from Lemma 4.2. q.e.d.

By Theorem 6 and Lemma 4.3, *a quasi-conformal mapping preserves the class* M_{HBD^*}.

REMARK. We state an interesting problem being still open: Can we conclude that two Riemann surfaces are quasi-conformally equivalent when their Royden's compactifications are homeomorphic in such a way as the harmonic and non-harmonic ideal boundaries correspond each other?

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