ON THE GROUPS OF COBORDISM Ω^k

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Introduction

In the papers [11] and [18] Rohlin and Thom have introduced an equivalence relation into the set of compact orientable (not necessarily connected) differentiable manifolds, which, roughly speaking, is described in the following manner: two differentiable manifolds are equivalent (*cobordantes*), when they together form the boundary of a bounded differentiable manifold. The equivalence classes can be added and multiplied in a natural way and form a graded algebra Ω relative to the addition, the multiplication and the dimension of manifolds. The precise structures of the groups of cobordism Ω^k of dimension k are not known thoroughly. Thom [18] has determined the free part of Ω and also calculated explicitly Ω^k for $0 \leqq k \leqq 7$.

The purpose of the present paper is to determine explicitly the groups Ω^k for $8 \leqq k \leqq 12$. Our method is analogous to that of Thom [18] and we shall calculate Ω^k using Serre's C-theory.

In § 1 we explain shortly some general results on the Eilenberg-MacLane complexes, Serre's C-theory and the Grassmann manifold, which will be used later. In § 2 the homotopy groups of the Thom complex $M(SO(n))$ associated with the rotation group are calculated. In § 3 we determine the groups of cobordism Ω^k for $8 \leqq k \leqq 12$, and discuss some problems related to Ω^k.

Some of the results contained in this paper have been announced in the note [1].

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§ 1. Preliminaries

Before we approach the determination of the homotopy groups of the Thom complex $M(SO(n))$ associated with the rotation group, it is necessary to recall

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some general results on the reduced powers of Steenrod, the Eilenberg-MacLane complexes, Serre's C-theory and the Grassmann manifolds.

We shall denote by Z and Z_p the ring of integers and of integers modulo p respectively.

A. Reduced powers of Steenrod

Let p be a prime number. The *Bockstein homomorphism* $\alpha_p : H^i(X, Z_p) \to H^{i+1}(X, Z)$ is identical with $(-1)^{i+1}\delta$, where δ is the coboundary homomorphism of the cohomology exact sequence of a space X relative to the exact sequence of coefficient groups

$$0 \to Z \to Z \to Z_p \to 0.$$

The *Bockstein homomorphism* $\beta_p : H^i(X, Z_p) \to H^{i+1}(X, Z_p)$ is defined by the composition of α_p and the natural homomorphism

$$\rho_p : H^{i+1}(X, Z) \to H^{i+1}(X, Z_p).$$

Let a be an integer $\geqq 0$ congruent to 0 or 1 mod $2p-2$. We define the homomorphism

$$St_p^a : H^i(X, Z_p) \to H^{i+a}(X, Z_p)$$

in the following manner: if $p=2$, we put $St_p^a = Sq^a$; if $p>2$ and $a=2k(p-1)$, k an integer, we put $St_p^a = P_p^k$; if $p>2$ and $a=2k(p-1)+1$, $St_p^a = \beta_p \circ P_p^k$. For a sequence $I=(a_1, a_2, \dots, a_r)$ of integers $a_i \geqq 0$, congruent to 0 or 1 mod $2p-2$, we denote the composed operation by

$$St_p^I = St_p^{a_1} \circ St_p^{a_2} \circ \dots \circ St_p^{a_r}.$$

The following formulas are often used in §2.

$$P_p^k(u \cdot v) = \sum_{i=0}^{k} P_p^i(u) P_p^{k-i}(v),$$

$$P_p^k \circ \beta_p \circ P_p^h = \binom{k+h-1}{h} P_p^{k+h} \circ \beta_p + \binom{k+h-1}{k} \beta_p \circ P_p^{k+h},$$

$$\text{for} \quad 0 \leqq k \leqq p-1,$$

where $$\binom{a}{b} = \begin{cases} \text{binomial coefficient reduced mod } p, \text{ if } a, b \geqq 0, \\ 0, \text{ if } a<0 \text{ or } b<0, \end{cases}$$

(cf. Cartan [5]).

B. Eilenberg-MacLane complexes

Let n be an integer $\geqq 1$, and Π be an abelian group. An arcwise connected space X is called the *Eilenberg-MacLane complex* $K(\Pi, n)$ if all its homotopy groups of dimension >0 are zero except for $\pi_n(X) = \Pi$. All these spaces have the same homotopy type, and among them there exists a simplicial complex. Moreover,

1) If Π is an abelian group of *finite type*, there exists a simplicial complex $K(\Pi, n)$ whose q-skelton is a *finite* complex (cf. Thom [18], p. 36).

We denote the cohomology ring of $K(\Pi, n)$ with coefficients in G by the notation $H^*(\Pi, n; G)$; the group $H^n(G, n; G)$ possesses a *fundamental class* which we will denote by ι_n.

2) For any cohomology class $u \in H^n(X, G)$ of a topological space X, there exists a mapping $f: X \to K(G, n)$ such that $u = f^*(\iota_n)$.

The cohomology of complexes $K(Z, n)$ and $K(Z_p, n)$ has been determined by Cartan [4] and Serre [14]. Here let us recall some of their results.

3) The cohomology ring $H^*(Z_2, n; Z_2)$ is generated by the Steenrod squarings of the fundamental class $\iota_n \in H^n(Z_2, n; Z_2)$ and their cup products; for $h < n$ (stable part of $H^*(Z_2, n; Z_2)$), a *base* of the group $H^{n+h}(Z_2, n; Z_2)$ is given by the sequences of iterated squarings of ι_n : $Sq^I(\iota_n)$, where $I = (i_1, i_2, \ldots, i_r)$ with $\sum_m i_m = h$ and $i_s \geqq 2i_{s+1}$, for $1 \leqq s \leqq r-1$.

We have an analogous result for $H^*(Z, n; Z_p)$.

4) For $h < n$, a *base* of $H^{n+h}(Z, n; Z_p)$ is given by the sequences of iterated reduced powers (squarings if $p=2$) $St_p^I(\iota_n)$, where $I = (a_1, a_2, \ldots, a_r)$ satisfying the following conditions:[1]

$$
\begin{aligned}
& a_i \equiv 0 \text{ or } 1, \bmod 2p-2, \text{ for } 0 \leqq i \leqq r,\\
& a_i \geqq p a_{i+1}, \text{ for } 1 \leqq i \leqq r-1,\\
& a_r \geqq 2p-2,\\
& \sum_i a_i = h.
\end{aligned}
$$

C. Serre's C-theory

Let C_p be the class of finite abelian groups whose p-primary components are zero, where p is a prime number. We shall often use the following theorem in §2.

[1] We denote by ι_n also the fundamental class of $K(Z, n)$ reduced mod p.

THEOREM. *Let two spaces A and B be connected and simply arcwise connected, $f: A \to B$ be a mapping which maps $\pi_2(A)$ onto $\pi_2(B)$ and n be a positive integer. Suppose the homology groups $H_i(A, Z)$ and $H_i(B, Z)$ are finite type in all dimensions. Then the following two properties are equivalent:*

a) $f_*: \pi_i(A) \to \pi_i(B)$ *is C_p-isomorphism for $i < n$, and C_p-onto for $i \leqq n$.*

b) $f^*: H^i(B, Z_p) \to H^i(A, Z_p)$ *is an isomorphism for $i \leqq n$, and onto for $i < n$.*

(Serre [13], Chapitre III, Théorème I and Proposition 2).

Let G and H be abelian groups. We denote "G is C-isomorphic to H" by $G \cong H \bmod C$.

D. Some formulas in Grassmann manifold

We donote by $\hat{G}_n$ the Grassmann manifold of oriented n-spaces in a Euclidean s-space R^s where s is sufficiently large. It is well known that $\hat{G}_n$ is the classifying space $B_{SO(n)}$ associated with the rotation group $SO(n)$.

1) We know that the cohomology ring $H^*(\hat{G}_n, Z_2)$ is a polynomial algebra generated by $W^2, W^3, \ldots, W^n$, where W^i is the i-dimensional *Stiefel-Whitney class* (Borel [2]).

2) Let n be *even*, $n = 2m$, and p be an odd prime. The cohomology ring $H^*(\hat{G}_n, Z_n)$ is a polynomial algebra generated by the *Pontrjagin classes* mod p and the *Euler-Poincaré class* mod p:

$$P^4, P^8, \ldots, P^{4m-4}, X^n$$

(Borel-Serre [3]).

3) The following formula, which gives the Steenrod squarings of W^i, was introduced by Wu [20]:

$$Sq^i W^j = \sum_{t=0}^{i} \binom{j-i+t-1}{t} W^{i-t} W^{j+t}, \qquad (i \leqq j),$$

with the following conventions

$$\binom{a}{b} = \begin{cases} \text{binomial coefficient reduced mod 2, if } a \geqq b > 0, \\ 1, \text{ if } b = 0, \\ 0, \text{ if } b \neq 0,\ a < b, \end{cases}$$

and $W^1 = 0$.

4) It is often useful to consider the Pontrjagin classes mod p and the

Euler-Poincaré class mod p of the universal bundle $E_{SO(n)} \to B_{SO(n)}$ as the symmetric functions[2] of m variables $x_1, x_2, \ldots, x_m$ of degree 2:

$$P^{4i} = \sum x_1^2 x_2^2 \ldots x_i^2, \qquad 1 \leqq i \leqq m-1,$$
$$X^n = x_1 x_2 \ldots x_m, \qquad n = 2m,$$

where $x_1, x_2, \ldots, x_m$ are generators of $H^*(B_T, Z_p)$ (T is the maximal torus of the rotation group $SO(n)$). The introduction of the variables x_i leads us to the following formulas:

i) $P_p^k(X^n) = X^n \sum x_1^{2h} x_2^{2h} \ldots x_k^{2h}$, where $h = \frac{p-1}{2}$,

ii) $P_p^k(P^{4i}) = P_p^k(\sum x_1^2 \ldots x_i^2)$

$$= \sum_{2r+s=k} 2^s \sum x_1^{2p} x_2^{2p} \ldots x_r^{2p} \cdot x_{r+1}^{p+1} \ldots x_{r+s}^{p+1} \cdot x_{r+s+1}^2 \ldots x_i^2$$

(Borel-Serre [3]).

5) Here we adopt Hirzebruch's definition of Pontrjagin classes (Hirzebruch [8], p. 67). Then we have

$$\rho_2(P^{4i}) = (W^{2i})^2,$$

where $\rho_2 : H^*(\hat{G}_n, Z) \to H^*(\hat{G}_n, Z_2)$ is the reduction mod 2.

§ 2. Homotopy groups of Thom complex $M(SO(n))$

In this section we shall calculate the stable homotopy groups of the Thom complex $M(SO(n))$ associated with the rotation group $SO(n)$.

A. Thom complexes

We know that any $(n-1)$-sphere bundle over a finite complex whose structure group is the rotation group $SO(n)$ is induced from the universal sphere bundle $p : E'_{SO(n)} \to B_{SO(n)}$. We denote by $A_{SO(n)}$ the mapping cylinder of the projection p; this is a manifold with boundary $E'_{SO(n)}$; we denote by $A'_{SO(n)}$ the complement $A_{SO(n)} - E'_{SO(n)}$ of the boundary in $A_{SO(n)}$.

We call the complex obtained from $A_{SO(n)}$ by the identification of its boundary $E_{SO(n)}$ to a point a *Thom complex associated with the rotation group* $SO(n)$; we shall denote it by $M(SO(n))$. Then $M(SO(n))$ is the Alexandroff's compactification of $A'_{SO(n)}$.

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2) In the present paper, we denote a symmetric function by its initial term preceded with $\sum$.

The homotopy groups and the cohomology rings of Thom complex $M(SO(n))$ have been studied by Thom [18].

1) The homotopy groups $\pi_{n+k}(M(SO(n)))$ are independent on n, for $n>k$ (Thom [18], Théorème II.7).

Owing to 1), hereafter we assume n to be *sufficiently large* and *even* without loss of generality.

2) $\pi_1(M(SO(n)))=0$.

3) The cohomology ring $H^*(M(SO(n)), Z_2)$ of $M(SO(n))$ is isomorphic to the ideal generated by the n-dimensional Stiefel-Whitney class W^n in the polynomial algebra $H^*(\hat{G}_n, Z_2)=Z_2[W^2, W^3, \ldots, W^n]$.

4) Let p be an odd prime. The cohomolohy ring $H^*(M(SO(n)), Z_p)$ of $M(SO(n))$ is isomorphic to the ideal generated by the Euler-Poincaré class mod p X^n in the polynomial algebra $H^*(\hat{G}_n, Z_p)=Z_p[P^4, P^8, P^{4m-4}, X^n]$, where $n=2m$.

5) The cohomology ring $H^*(M(SO(n)), Z)$ of $M(SO(n))$ with integer coefficient is isomorphic to the ideal generated by the Euler-Poincaré class X^n in the algebra $H^*(\hat{G}_n, Z)$ (Thom [18], Chapitre II, §5).

Henceforth we identify the cohomology rings of $M(SO(n))$ and the above-mentioned ideals of the cohomology rings of Grassmann manifold respectively.

6) The stable homotopy groups $\pi_{n+k}(M(SO(n)))$ are finite if $k \not\equiv 0$, mod 4; the free components of the stable homotopy groups $\pi_{n+4j}(M(SO(n)))$ are of rank $\pi(j)$, where $\pi(j)$ is the number of partition of j (Thom [18], Théorème IV.15).

To our purpose, therefore, it is sufficient to calculate the p-primary components of $\pi_{n+k}(M(SO(n)))$ for each prime p.

B. 2-primary components of $\pi_{n+k}(M(SO(n)))$

We will calculate the 2-primary components of $\pi_{n+k}(M(SO(n)))$ for $8 \leqq k \leqq 12$. Let Y_2 be the product of the Eilenberg-MacLane complexes:

$$Y_2=K(Z, n)\times K(Z, n+4)\times K(Z_2, n+5)\times(K(Z, n+8))^2$$
$$\times(K(Z_2, n+9))^2\times K(Z_2, n+10)\times K(Z_2, n+11)\times(K(Z, n+12))^3.$$

Let

$$f_1: M(SO(n))\to K(Z, n), \qquad f_2: M(SO(n))\to K(Z, n+4),$$
$$f_3: M(SO(n))\to K(Z_2, n+5), \qquad f_4: M(SO(n))\to K(Z, n+8),$$
$$f_5: M(SO(n))\to K(Z, n+8), \qquad f_6: M(SO(n))\to K(Z_2, n+9),$$

$f_7 : M(SO(n)) \to K(Z_2, n+9)$, $f_8 : M(SO(n)) \to K(Z_2, n+10)$,
$f_9 : M(SO(n)) \to K(Z_2, n+11)$, $f_{10} : M(SO(n)) \to K(Z, n+12)$,
$f_{11} : M(SO(n)) \to K(Z, n+12)$, $f_{12} : M(SO(n)) \to K(Z, n+12)$,

be mappings defined by

$$
\begin{aligned}
&f_1^*(\iota_n) = X^n, && f_2^*(\iota_{n+4}) = X^n P^4,\\
&f_3^*(\iota_{n+5}) = W^n W^3 W^2, && f_4^*(\iota_{n+8}) = X^n P^8,\\
&f_5^*(\iota'_{n+8}) = X^n (P^4)^2, && f_6^*(\iota_{n+9}) = W^n W^5 W^4,\\
&f_7^*(\iota'_{n+9}) = W^n W^3 (W^2)^3, && f_8^*(\iota_{n+10}) = W^n W^6 W^4,\\
&f_9^*(\iota_{n+11}) = W^n W^5 (W^2)^3, && f_{10}^*(\iota_{n+12}) = X^n P^{12},\\
&f_{11}^*(\iota'_{n+12}) = X^n P^8 P^4, && f_{12}^*(\iota''_{n+12}) = X^n (P^4)^3,
\end{aligned}
$$

respectively, where ι_{n+i}, ι'_{n+i} and ι''_{n+12} are the fundamental classes of the corresponding Eilenberg-MacLane complexes.

We define $F : M(SO(n)) \to Y_2$ by $f \circ d$ where d is the diagonal map $M(SO(n)) \to (M(SO(n)))^{12}$ and $f = \prod_{i=1}^{12} f_i : (M(SO(n)))^{12} \to Y_2$.

Let us calculate the homomorphism F^* induced by F

$$F^* : H^*(Y_2, Z_2) \to H^*(M(SO(n)), Z_2).$$

We consider F^* for the dimension $i \leqq n+13$. It has been verified by Thom [18] that F^* is an isomorphism of $H^i(Y_2, Z_2)$ onto $H^i(M(SO(n)), Z_2)$ for $i \leqq n+7$. We can verify further that $F^* : H^i(Y_2, Z_2) \to H^i(M(SO(n)), Z_2)$ is an isomorphism into for $i \leqq n+13$, and a homomorphism onto for $i < n+13$ by continuing the analogous calculation as Thom ([18], Chapitre II, §8; §1.**B**, §1.**D**, §2.**A**). From this we deduce by Serre's C-theory (§1.**C**) that $F_* : \pi_i(M(SO(n))) \to \pi_i(Y_2)$ are C_2-isomorphisms for $i < n+13$ and C_2-onto for $i \leqq n+13$. Thus we have:

$$
\begin{aligned}
&\pi_{n+8}(M(SO(n))) \cong Z+Z, \text{ mod } C_2,\\
&\pi_{n+9}(M(SO(n))) \cong Z_2+Z_2, \text{ mod } C_2,\\
&\pi_{n+10}(M(SO(n))) \cong Z_2, \text{ mod } C_2,\\
&\pi_{n+11}(M(SO(n))) \cong Z_2, \text{ mod } C_2,\\
&\pi_{n+12}(M(SO(n))) \cong Z+Z+Z, \text{ mod } C_2.
\end{aligned}
$$

Consequently we obtain

PROPOSITION 1. *The 2-primary components of the stable homotopy groups* $\pi_{n+k}(M(SO(n)))$, *for* $8 \leqq k \leqq 12$, *are*:

$$\begin{array}{cccccc} & 0, & Z_2+Z_2, & Z_2, & Z_2, & 0, \\ \textit{for} & k=8, & 9, & 10, & 11, & 12, \end{array}$$

respectively.

C. p-primary components of $\pi_{n+k}(M(SO(n)))$

Let p be an odd prime. The calculations of the p-primary components of the stable homotopy groups $\pi_{n+k}(M(SO(n)))$ are a little more complicated than 2-primary components.

There exists an aspherical fibre space A with the fibre $K(Z, n+2p-2)$ over the base $K(Z, n+2p-1)$ (Serre [12]). Consider a mapping φ of the complex $K(Z, n)$ into $K(Z, n+2p-1)$ such that $\varphi^*(\iota_{n+2p-1}) = \alpha_p \circ St_p^{2p-2}(\iota_n)$. We denote by $L(n, p)$ the fibre space induced from the fibre space A by the mapping φ. So $L(n, p)$ is the fibre space over the complex $K(Z, n)$ of fibre $K(Z, n+2p-2)$; non zero homotopy groups of $L(n, p)$ are only π_n and π_{n+2p-2}, both isomorphic to Z. The Eilenberg-MacLane invariant $\mathbf{k} \in H^{n+2p-1}(Z, n; Z)$ associated with $L(n, p)$ is $\alpha_p \circ St_p^{2p-2}(\iota_n)$.

Now we consider the cohomology mod p of the complex $L(n, p)$ in dimension $< n+4p-4$. It is necessaly to discuss the spectral sequence relative to the fibering of $L(n,p)$ over $K(Z, n)$. By the construction of $L(n,p)$, ι_{n+2p-2} is mapped by the transgression τ (explicitly by d_{n+2p-1}) to the class $St_p^{2p-1}(\iota_n)$. It follows that in total degree $\leqq n+4p-5$, $E_\infty^{r,s}$ are zero except for the following terms:

$$E_\infty^{0,0} = E_2^{0,0} = H^0(Z, n; Z_p),$$
$$E_\infty^{n,0} = E_2^{n,0} = H^n(Z, n; Z_p),$$
$$E_\infty^{n+2p-2,0} = E_2^{n+2p-2,0} = H^{n+2p-2}(Z, n; Z_p).$$

Consequently *the cohomology groups $H^i(L(n, p), Z_p)$, for $i \leqq n+4p-5$, admit only the following generators*:

in dimension n, $\bar{\iota}_n = p^*(\iota_n)$,
in dimension $n+2p-2$, $St_p^{2p-2}(\bar{\iota}_n) = p^*(St_p^{2p-2}(\iota_n))$,

where $p : L(n, p) \to K(Z, n)$ is the projection (cf. Serre [12], p. 456).

Let Y_p be the product of the complex $L(n, p)$ and the Eilenberg-MacLane complexes:

$$Y_p = L(n, p) \times K(Z, n+4) \times \dots \times (K(Z, n+4h))^{\pi(h)} \times \dots \times (K(Z, n+2p-6))^{\pi((p-3)/2)} \times (K(Z, n+2p-2))^{\pi((p-1)/2)-1}.$$

Let us define a mapping $G : \dot{M}(SO(n)) \to Y_p$. There exists a mapping g of $M(SO(n))$ into $K(Z, n)$ such that $g^*(\iota_n) = X^n$. And there exists a mapping s of the $(n+2p-2)$-skeleton of $K(Z, n)$ into $L(n, p)$ such that $s_* : \pi_i(K(Z, n)^{n+2p-2}) \to \pi_i(L(n, p))$ is an isomorphism onto for $i \leqq n+2p-3$. So we have the mapping $s \circ g$ of the $(n+2p-2)$-skeleton of $M(SO(n))$ into $L(n, p)$ such that $(s \circ g)(\tau_n) = X^n$. The obstruction to the extension of the mapping s is given by $\alpha_p \circ St_p^{2p-2}(\iota_n)$, and therefore, the obstruction to the extension of the mapping $s \circ g$ is

$$c(s \circ g) = g^*(c(s)) = \alpha_p \circ St_p^{2p-2}(X^n).$$

Since $H^*(M(SO(n)), Z)$ has no p-torsion, we have $c(s \circ g) = 0$. So we can extend the mapping $s \circ g$ to a mapping $\bar{g} : M(SO(n)) \to L(n, p)$ such that $\bar{g}^*(\tau_n) = X^n$.

Now we introduce in the free base of $H^{n+4h}(M(SO(n)), Z)$ the lexicographic order using the dimension of Pontrjagin classes:

$$\begin{gathered} X^n P^{4h_1} P^{4h_2} \ldots P^{4h_r} > X^n P^{4k_1} P^{4k_2} \ldots P^{4k_s}, \\ \text{if} \quad h_1 \geqq h_2 \geqq \ldots \geqq h_r, \ k_1 \geqq k_2 \geqq \ldots \geqq k_s, \\ h_1 = k_1, \ \ldots, \ h_t = k_t, \ h_{t+1} > k_{t+1}. \end{gathered}$$

For example in $H^{n+12}(M(SO(n)), Z)$, we have: $X^n P^{12} > X^n P^8 P^4 > X^n (P^4)^3$. We denote by $K_i(Z, n+4h)$ the i-th copy of $K(Z, n+4h)$ in the corresponding factor of the product space Y_p, and by ι^i_{n+4h} the fundamental class of $K_i(Z, n+4h)$. There exist mappings

$$\begin{gathered} g^i_h : M(SO(n)) \to K_i(Z, n+4h), \quad 0 < h \leqq \frac{p-1}{2}, \quad 1 \leqq i \leqq \pi(h), \\ (h, i) \neq \left(\frac{p-1}{2}, \ \pi\left(\frac{p-1}{2}\right)\right), \end{gathered}$$

such that $(g^i_h)^*(\iota^i_{n+4h}) =$ the i-th element of the free base of $H^{n+4h}(M(SO(n)), Z)$.

Let $d : M(SO(n)) \to (M(SO(n)))^m$, $m = \sum_{h=1}^{(p-1)/2} \pi(h)$, be the diagonal map and $\bar{G} : (M(SO(n)))^m \to Y_p$ be

$$\begin{gathered} \bar{G} = g \times \Pi g^i_h, \quad 0 < h \leqq \frac{p-1}{2}, \quad 1 \leqq i \leqq \pi(h), \\ (h, i) \neq \left(\frac{p-1}{2}, \ \pi\left(\frac{p-1}{2}\right)\right). \end{gathered}$$

We define $G : M(SO(n)) \to Y_p$ by the composed mapping $\bar{G} \circ d$.

We will calculate the homomorphism G^* induced by G

$$G^* : H^{n+j}(Y_p, Z_p) \to H^{n+j}(M(SO(n)), Z_p), \quad \text{for} \quad j \leqq 2p+1.$$

In the first place we consider $G^*(St_p^{2p-2}(\bar{\iota}_n))$:

$$G^*(St_p^{2p-2}(\bar{\iota}_n)) = St_p^{2p-2}(G^*(\bar{\iota}_n)) = St_p^{2p-2}(X^n)$$
$$= X^n \sum (x_1^2)^{(p-1)/2} = X^n \{(\sum x_1^2)^{(p-1)/2} + Q(x_1^2, x_2^2, \ldots, x_{n/2}^2)\},$$

where $Q(x_1^2, x_2^2, \ldots, x_{n/2}^2)$ is a polynomial of elementary symmetric functions of x_i^2 which does not contain the term $(\sum x_1^2)^{(p-1)/2}$ (cf. § 1. D). Now we have the next table:

$j = 0,$ $\quad G^*(\bar{\iota}_n) = X^n,$

$j = 4h,\ 0 < h < \dfrac{p-1}{2},$ $\quad G^*(\iota^i_{n+4h}) =$ the i-th base element of $H^{n+4h}(M(SO(n)), Z_p)$, for $1 \leqq i \leqq \pi(h)$,

$j = 2p-2,$ $\quad G^*(\iota^i_{n+2p-2}) =$ the i-th base element of $H^{n+2p-2}(M(SO(n)), Z_p)$, for $1 \leqq i < \pi\left(\dfrac{p-1}{2}\right)$,

$\quad G^*(St_p^{2p-2}(\bar{\iota}_n)) = X^n(P^1)^{(p-1)/2} +$ linear combination of other base elements.

We know that the classes of $H^k(Y_p, Z_p)$ and $H^k(M(SO(n)), Z_p)$, which appear in the table, form bases of $H^k(Y_p, Z_p)$ and of $H^k(M(SO(n)), Z_p)$ respectively for $k \leqq n + 2p + 1$; therefore we can easily verify that $G^* : H^k(Y_p, Z_p) \to H^k(M(SO(n)), Z_p)$ is an isomorphism onto for $k \leqq n+2p+1$. By Serre's C-theory we have that $G^* : \pi_k(M(SO(n))) \to \pi_k(Y_p)$ is C_p-isomorphism for $k < n+2p+1$ and C_p-onto for $k \leqq n+2p+1$. Thus we have:

$$\pi_{n+k}(M(SO(n))) \cong 0, \text{ mod } C_p, \text{ if } k \not\equiv 0 \ (4),\ k \leqq 2p,$$
$$\pi_{n+4h}(M(SO(n))) \cong \underbrace{Z + Z + \ldots + Z}_{\pi(h)}, \text{ mod } C_p, \text{ if } 2h < p.$$

Consequently we obtain

PROPOSITION 2. *The p-primary components of the stable homotopy groups $\pi_{n+k}(M(SO(n)))$ are zero if $k \leqq 2p$, where p is an odd prime.*

As a special case

PROPOSITION 3. *Let p be a prime $\geqq 7$. The p-primary components of the stable homotopy groups $\pi_{n+k}(M(SO(n)))$ are zero for $k \leqq 14$.*

D. 5-primary components of $\pi_{n+k}(M(SO(n)))$

Now we consider two Postnikov complexes $L(n, 5)$ and $L(n+4, 5)$ defined above. As is mentioned in §2.C, the cohomology groups $H^i(L(n, 5), Z_5)$ for $i \leqq n+15$ admit only the following generators:

in dimension n, $\quad \bar{\iota}_n = p_1^*(\iota_n)$,
in dimension $n+8$, $\quad St_5^8(\bar{\iota}_n) = p_1^*(St_5^8(\iota_n))$,

where $p_1 : L(n, 5) \to K(Z, n)$ is the projection of $L(n, 5)$. And we know also that the cohomology groups $H^i(L(n+4, 5), Z_5)$ for $i \leqq n+15$ admit only the following generators:

in dimension $n+4$, $\quad \bar{\iota}_{n+4} = p_2^*(\iota_{n+4})$,
in dimension $n+12$, $\quad St_5^8(\bar{\iota}_{n+4}) = p_2^*(St_5^8(\iota_{n+4}))$,

where $p_2 : L(n+4, 5) \to K(Z, n+4)$ is the projection.

Let Y_5 be the product of $L(n, 5)$, $L(n+4, 5)$ and the Eilenberg-MacLane complexes:

$$Y_5 = L(n, 5) \times L(n+4, 5) \times K(Z, n+8) \times (K(Z, n+12))^2.$$

Let us define a mapping $H : M(SO(n)) \to Y_5$. By the same method as in §2.C, we can find a mapping h_1 of $M(SO(n))$ into $L(n, 5)$ such that $h_1^*(\bar{\iota}_n) = X^n$, and a mapping h_2 of $M(SO(n))$ into $L(n+4, 5)$ such that $h_2^*(\bar{\iota}_{n+4}) = X^n P^4$. On the other hand, there exist mappings

$$\begin{aligned} h_3 &: M(SO(n)) \to K(Z, n+8), && \text{such that} \quad h_3^*(\iota_{n+8}) = X^n P^8, \\ h_4 &: M(SO(n)) \to K(Z, n+12), && \text{such that} \quad h_4^*(\iota_{n+12}) = X^n P^8 P^4, \\ h_5 &: M(SO(n)) \to K(Z, n+12), && \text{such that} \quad h_5^*(\iota'_{n+12}) = X^n (P^4)^3. \end{aligned}$$

We define $H : M(SO(n)) \to Y_5$ to be $h \circ d$, where $d : M(SO(n)) \to M(SO(n)))^5$ is the diagonal map, and

$$h = \prod_{i=1}^{5} h_i : (M(SO(n)))^5 \to Y_5.$$

We will calculate the homomorphism H^* induced by H,

$$H^* : H^{n+i}(Y_5, Z_5) \to H^{n+i}(M(SO(n)), Z_5), \quad \text{for} \quad i \leqq 15.$$
$$i = 0, \quad H^*(\bar{\iota}_n) = X^n,$$
$$i = 4, \quad H^*(\bar{\iota}_{n+4}) = X^n P^4,$$

$$i=8,\quad H^*(St_5^8\bar{\iota}_n)=3X^nP^4+X^n(P^4)^2,$$
$$H^*(\iota_{n+8})=X^nP^8,$$
$$i=12,\quad H^*(St_5^8\bar{\iota}_{n+4})=X^nP^{12}+2X^nP^8P^4+3X^n(P^4)^3,$$
$$H^*(\iota_{n+12})=X^nP^8P^4,$$
$$H^*(\iota'_{n+12})=X^n(P^4)^3, \qquad \text{(cf. §1.D)}.$$

For the present section $2p-2=8$, so the classes of $H^*(Y_5, Z_5)$ and $H^*(M(SO(n)), Z_5)$ written in the table form bases of $H^i(Y_5, Z_5)$ and $H^i(M(SO(n)), Z_5)$. Therefore we can easily verify that $H^*: H^i(Y_5, Z_5) \to H^i(M(SO(n)), Z_5)$ is an isomorphism onto for $i \leqq n+15$. By the C-theory we have that $H_*: \pi_i(M(SO(n))) \to \pi_i(Y_5)$ is C_5-isomorphism for $i<n+15$, and C_5-onto for $i \leqq n+15$; namely

$$\pi_{n+8}(M(SO(n))) \cong Z+Z, \text{ mod } C_5,$$
$$\pi_{n+12}(M(SO(n))) \cong Z+Z+Z, \text{ mod } C_5,$$
$$\pi_{n+i}(M(SO(n))) \cong 0, \text{ mod } C_5, \text{ for } 8<i<12,\ 12<i<15.$$

Thus we have

PROPOSITION 4. *The* 5-*primary components of the stable homotopy groups* $\pi_{n+i}(M(SO(n)))$, *for* $8 \leqq i \leqq 14$, *are all zero.*

E. 3-primary components of $\pi_{n+k}(M(SO(n)))$

The calculation of the 3-primary components of the stable homotopy groups $\pi_{n+k}(M(SO(n)))$ is the most complicated.

In the first place we consider the cohomology groups mod 3 of the Postnikov complex $L(n, 3)$ in the dimension $\leqq n+15$. We discuss the spectral sequence mod 3 relative to the fibering of $L(n, 3)$ over $K(Z, n)$. By the construction of $L(n, 3)$ the fundamental class ι_{n+4} of fibre is mapped by the transgression τ (explicitly d_{n+5}) onto the class $St_3^5(\iota_n)$. As the reduced powers commute with the transgression, the calss $St_3^4(\iota_{n+5})$ of fibre is mapped by τ to the class $St_3^4 \circ St_3^5(\iota_n)=St_3^9(\iota_n)$ and the class $St_3^8(\iota_{n+5})$ to the class $St_3^8 \circ St_3^5(\iota_n) = St_3^{13}(\iota_n)$ (cf. §1.A). It follows that in total degree $\leqq n+15$, $E_\infty^{r,s}$ are zero except for the following terms:

$$E_\infty^{0,0}=E_2^{0,0}=H^0(Z, n;\ Z_3), \qquad E_\infty^{n,0}=E_2^{n,0}=H^n(Z, n;\ Z_3),$$
$$E_\infty^{n+4,0}=E_2^{n+4,0}=H^{n+4}(Z, n;\ Z_3), \qquad E_\infty^{n+8,0}=E_2^{n+8,0}=H^{n+8}(Z, n;\ Z_3),$$
$$E_\infty^{0,n+9}=E_2^{0,n+9}=H^{n+9}(Z, n+4;\ Z_3), \qquad E_\infty^{n+12,0}=E_2^{n+12,0}=H^{n+12}(Z, n;\ Z_3),$$
$$E_\infty^{0,n+13}=E_2^{0,n+13}=H^{n+13}(Z, n+4;\ Z_3).$$

Consequently the cohomology groups $H^i(L(n, 3), Z_3)$, for $i \leqq n+15$, admit only the following generetors:

in dimension n,	$\bar{\iota}_n = p_1^*(\iota_n)$,
in dimension $n+4$,	$St_3^4(\bar{\iota}_n)$,
in dimension $n+8$,	$St_3^8(\bar{\iota}_n)$,
in dimension $n+9$,	$(i_1^*)^{-1} \circ St_3^5(\iota_{n+4})$,
in dimension $n+12$,	$St_3^{12}(\bar{\iota}_n)$,
in dimension $n+13$,	$(i_1^*)^{-1} \circ St_3^9(\iota_{n+1})$,

where p_1 is the projection of $L(n, 3)$ into $K(Z, n)$ and $i_1 : K(Z, n+4) \to L(n, 3)$ is the injection of fibre in the total space (cf. Serre [12], p. 456).

We need a certain class to construct the fibre space over $L(n, 3)$ which is indispensable to our purpose.

LEMMA 1. *There exists an integral cohomology class* $u \in H^{n+9}(L(n, 3), Z)$ *such that*

i) $\rho_3(u) = (i_1^*)^{-1} \circ St_3^5(\iota_{n+4})$,

ii) $3^m u = 0$, *for a certain integer* m,

where ρ_3 *is the reduction* mod 3.

Proof. The following diagram is commutative:

$$\begin{array}{ccc} H^{n+9}(Z, n+4;\ Z) & \xleftarrow{i_1^*} & H^{n+9}(L(n, 3), Z) \\ \downarrow{\scriptstyle \rho_3} & & \downarrow{\scriptstyle \rho_3} \\ H^{n+9}(Z, n+4;\ Z_3) & \underset{\cong}{\xleftarrow{i_1^*}} & H^{n+9}(L(n, 3), Z_3). \end{array}$$

So it is sufficient to show that $St_3^5(\iota_{n+4}) \in H^{n+9}(Z, n+4;\ Z_3)$ is the image by $\rho_3 \circ i_1^*$ of a class $u \in H^{n+9}(L(n, 3), Z)$ satisfying the condition ii). Now we factorize the homomorphism i_1^* with use of the spectral sequence associated with the fibering $L(n, 3)$:

$$\begin{aligned} H^{n+9}(Z, n+4;\ Z) &= E_2^{0, n+9} = E_{n+10}^{0, n+9} \xleftarrow{\lambda} E_{n+11}^{0, n+9} \approx E_\infty^{0, n+9} \\ &= D^{0, n+9}/D^{1, n+8} = D^{0, n+9}/E_\infty^{n+9, 0} \xleftarrow{\mu} D^{0, n+9} = H^{n+9}(L(n, 3), Z), \end{aligned}$$

where λ is an isomorphism into and μ is a homomorphism onto (cf. Serre [12], p. 456). The term $E_{n+11}^{0, n+9}$ is the subgroup of $E_{n+10}^{0, n+9}$ consisting of d_{n+10}-cocycles:

$$E_{n+10}^{0, n+9} \xrightarrow{d_{n+10}} E_{n+10}^{n+10, 0} \approx E_2^{n+10, 0} = H^{n+10}(Z, n;\ Z).$$

We know $H^{n+10}(Z, n; Z)$ have no component but 2-primary one, and therefore for any element x of $E_{n+10}^{0,n+9}$, $2^h x$ is a d_{n+10}-cocycle and belongs to $E_{n+11}^{0,n+9}$ for a certain integer h. Consequently we can easily find an integer $i>0$ and an element u' such that

a) $2^i \cdot \alpha_3 \circ St_3^4(\iota_{n+4}) = \lambda(u'), \quad u' \in E_{n+11}^{0,n+9}$,

b) $2^i \equiv 1, \quad \bmod 3$.

We know that $H^{n+9}(Z, n+4; Z)$ has no components but 2- and 3-primary ones, and $H^{n+9}(L(n, 3), Z)$ has no components but 2-, 3-, and 5-primary ones. Since $3\lambda(u')=0$, we have $3u'=0$, therefore we can find a class $u \in H^{n+9}(L(n, 3), Z)$ such that

i) $\mu(u) = u'$,

ii) $3^m u = 0$, for a certain integer m,

Thus the lemma is proved.

Let A be an aspherical fibre space over the complex $K(Z, n+9)$ of fibre $K(Z, n+8)$. There exists a mapping φ of the complex $L(n, 3)$ in $K(Z, n+9)$ such that $\varphi^*(\iota_{n+9}) = u$. We denote by K_3 the fibre space induced from the fibre space A by the mapping φ. The non-zero homotopy groups of K_3 are only π_n, π_{n+4} and π_{n+8}, all isomorphic to Z.

Now we consider the spectral sequence mod 3 relative to the fibering of K_3 over $L(n, 3)$. By the construction of K_3 the fundamental class ι_{n+8} of fibre is mapped by the transgression τ (explicitly d_{n+9}) onto the class $(i_1^*)^{-1} \circ St_3^5(\iota_{n+4})$, which is the generator of $H^{n+9}(L(n, 3), Z_3)$ (Lemma 1).

$$\begin{array}{ccc} H^{n+9}(L(n,3), Z_3) & \xrightarrow{St_3^4} & H^{n+13}(L(n,3), Z_3) \\ \downarrow{\scriptstyle i_1^*} & & \downarrow{\scriptstyle i_1^*} \\ H^{n+9}(Z, n+4; Z_3) & \xrightarrow{St_3^4} & H^{n+13}(Z, n+4; Z_3). \end{array}$$

This diagram is commutative and the vertical homomorphisms are both isomorphisms onto. Consequently $St_3^4(\iota_{n+8})$ is mapped by the transgression to

$$\begin{aligned} St_3^4 \circ (i_1^*)^{-1} \circ St_3^5(\iota_{n+4}) &= (i_1^*)^{-1} \circ St_3^4 \circ St_3^5(\iota_{n+8}) \\ &= (i_1^*)^{-1} \circ St_3^9(\iota_{n+8}). \end{aligned}$$

It follows that in total degree $\leqq n+15$, $E_\infty^{r,s}$ are zero except for the following terms:

$$E_\infty^{0,0}=E_2^{0,0}=H^0(L(n,3),Z_3), \qquad E_\infty^{n,0}=E_2^{n,0}=H^n(L(n,3),Z_3),$$
$$E_\infty^{n+4,0}=E_2^{n+4,0}=H^{n+4}(L(n,3),Z_3),$$
$$E_\infty^{n+8,0}=E_2^{n+8,0}=H^{n+8}(L(n,3),Z_3),$$
$$E_\infty^{n+12,0}=E_2^{n+12,0}=H^{n+12}(L(n,3),Z_3),$$
$$E_\infty^{0,n+13}=E_2^{0,n+13}=H^{n+13}(Z,n+8;Z_3).$$

Consequently the cohomology groups $H^i(K_3, Z_3)$ for $i \leqq n+15$ admit only the following generators:

in dimension n, $\quad \bar{\bar{\iota}}_n = p_2^*(\bar{\iota}_n)$,
in dimension $n+4$, $\quad St_3^4(\bar{\bar{\iota}}_n)$,
in dimension $n+8$, $\quad St_3^8(\bar{\bar{\iota}}_n)$,
in dimension $n+12$, $\quad St_3^{12}(\bar{\bar{\iota}}_n)$,
in dimension $n+13$, $\quad (i_2^*)^{-1}\circ St_3^5(\iota_{n+8})$,

where p_2 is the projection of the fibre space K_3 and $i_2 : K(Z, n+8) \to K_3$ is the injection of fibre in the total space K_3.

To eliminate the cohomology groups of dimension $n+13$, we will construct a fibre space over K_3 of fibre $K(Z, n+12)$.

LEMMA 2. *There exists an integral cohomology class $v \in H^{n+13}(K_3, Z)$ such that*

i) $3^m v = 0$, *for an integer m,*
ii) $\rho_3(v) = (i_2^*)^{-1}\circ St_3^5(\iota_{n+8})$.

Proof. We know that $H^{n+14}(L(n,3), Z)$ has no components but 2-primary one, that $H^{n+13}(Z, n+8; Z)$ has no components but 2- and 3-primary ones, and that $H^{n+13}(K_3, Z)$ has no components but 2-, 3-, 5- and 7-primary ones. Therefore we can prove Lemma 2 by the same method as Lemma 1.

Let B be an aspherical fibre space over $K(Z, n+13)$ of fibre $K(Z, n+12)$. There exists a mapping ψ of the complex K_3 in $K(Z, n+13)$ such that $\psi^*(\iota_{n+13}) = v$. We denote by K_4 a fibre space induced from the fibre space B by the mapping ψ. The non zero homotopy groups of K_4 are only π_n, π_{n+4}, π_{n+8} and π_{n+12}, all isomorphic to Z.

Now we consider the spectral sequence mod 3 relative to the fibering of K_4 over K_3. By the construction of K_4 the fundamental class ι_{n+12} of fibre is mapped by the transgression τ (explicitly d_{n+13}) onto the class $(i_2^*)^{-1}\circ St_3^5(\iota_{n+8})$

(Lemma 2). It follows that in total degree $\leqq n+15$, $E_\infty^{r,s}$ are zero except for the following terms:

$$E_\infty^{0,0}=H^0(K_3, Z_3), \qquad E_\infty^{n,0}=H^n(K_3, Z_3),$$
$$E_\infty^{n+4,0}=H^{n+4}(K_3, Z_3), \qquad E_\infty^{n+8,0}=H^{n+8}(K_3, Z_3),$$
$$E_\infty^{n+12,0}=H^{n+12}(K_3, Z_3).$$

Consequently the cohomology groups $H^i(K_4, Z_3)$ for $i\leqq n+15$ admit only the following generators:

in dimension n, $\quad \tilde{\tau}_n=p_3^*(\bar{\tau}_n)$,
in dimension $n+4$, $\quad St_3^4(\tilde{\tau}_n)$,
in dimension $n+8$, $\quad St_3^8(\tilde{\tau}_n)$,
in dimension $n+12$, $\quad St_3^{12}(\tilde{\tau}_n)$,

where p_3 is the projection of K_4.

On the other hand, as is shown in §2. C, the cohomology groups $H^i(L(n+8, 3), Z_3)$ of the Postnikov complex $L(n+8, 3)$ for $i\leqq n+15$ admit only the following generators:

in dimension $n+8$, $\quad \bar{\tau}_{n+8}=p_4^*(\iota_{n+8})$,
in dimension $n+12$, $\quad St_3^4(\bar{\tau}_{n+8})$,

where p_4 is the projection of the fibre space $L(n+8, 3)$ over $K(Z, n+8)$.

Let Y_3 be the product of K_4, $L(n+8, 3)$ and $K(Z, n+12)$. Let us define a mapping $F: M(SO(n))\to Y_3$. As is shown in §2. C, we can find a mapping f_1 of $M(SO(n))$ into $L(n, 3)$ such that $f_1^*(\bar{\tau}_n)=X^n$. We know that there exists a mapping t of the $(n+8)$-skeleton of $L(n, 3)$ into K_3 such that $t_*: \pi_i(L(n, 3)^{n+8})\to\pi_i(K_3)$ is an isomorphism onto for $i<n+8$. So we obtain the mapping $t\circ f_1$ of the $(n+8)$-skeleton of $M(SO(n))$ into K_3 such that $(t\circ f_1)^*(\bar{\tau})=X^n$. The obstruction to the extension of the mapping $t\circ f_1$ is given by $f_1^*(u)$. Since $H^*(M(SO(n)), Z)$ has no 3-torsion, we have $f_1^*(u)=0$ (see Lemma 1). Now we have the mapping f_2 of $M(SO(n))$ into K_3 such that $f_2^*(\bar{\tau}_n)=X^n$. In virture of Lemma 2, we can find a mapping f_3 of $M(SO(n))$ into K_4 such that $f_3^*(\tilde{\tau}_n)=X^n$ by the same method as above. Similarly we can find a mapping f_4 of $M(SO(n))$ into $L(n+8, 3)$ such that $f_4^*(\bar{\tau}_{n+8})=X^n(P^4)^2$. On the other hand there exists a mapping f_5 of $M(SO(n))$ into $K(Z, n+12)$ such that $f_5^*(\iota_{n+12})=X^n(P^4)^3$. Now we define a mapping $F: M(SO(n))\to Y_3$ to be the composition

of the diagonal map d and $f_3 \times f_4 \times f_5$:

$$M(SO(n)) \xrightarrow[d]{} (M(SO(n)))^3 \xrightarrow[f_3 \times f_4 \times f_5]{} Y_3.$$

We will calculate the homomorphism F^* induced by F

$$F^* : H^i(Y_3, Z_3) \to H^i(M(SO(n)), Z_3), \quad \text{for} \quad i \leqq n+15.$$

$$\begin{array}{ll} i=n, & F^*(\tilde{\iota}_n) = X^n, \\ i=n+4, & F^*(St_3^4(\tilde{\iota}_n)) = X^n P^4, \\ i=n+8, & F^*(St_3^8(\tilde{\iota}_n)) = X^n P^8, \quad F^*(\tilde{\iota}_{n+8}) = X^n (P^4)^2, \\ i=n+12, & F^*(St_3^{12}(\tilde{\iota}_n)) = X^n P^{12}, \\ & F^*(St_3^4(\tilde{\iota}_{n+8})) = X^n P^8 P^4 + 2X^n (P^4)^3, \\ & F^*(\iota_{n+12}) = X^n (P^4)^3. \end{array}$$

For the present case $2p-2=4$, so the classes of $H^i(Y_3, Z_3)$ and $H^i(M(SO(n)),Z_3)$ written above form bases of $H^i(Y_3, Z_3)$ and $H^i(M(SO(n)), Z_3)$ for $i \leqq n+15$. Therefore we can verify that $F^* : H^i(Y_3, Z_3) \to H^i(M(SO(n)), Z_3)$ is an isomorphism onto for $i \leqq n+15$. By Serre's C-theory, we obtain that $F_* : \pi_i(M(SO(n))) \to \pi_i(Y_3)$ is C_3-isomorphism for $i < n+15$ and C_3-onto for $i \leqq n+15$. This implies

$$\begin{array}{l} \pi_{n+8}(M(SO(n))) \cong Z+Z, \quad \text{mod } C_3, \\ \pi_{n+12}(M(SO(n))) \cong Z+Z+Z, \quad \text{mod } C_3, \\ \pi_{n+i}(M(SO(n))) \cong 0, \quad \text{mod } C_3 \text{ for } 8 < i < 12,\ 12 < i < 15. \end{array}$$

Thus we have

PROPOSITION 5. *The 3-primary components of the stable homotopy groups* $\pi_{n+i}(M(SO(n)))$, *for* $8 \leqq i \leqq 14$, *are all zero.*

F. Results

Here we state the results obtained in this section §2.

THEOREM 1. i) *The stable homotopy groups* $\pi_{n+i}(M(SO(n)))$ *are for* $8 \leqq i \leqq 14$:

$$\begin{array}{lll} \pi_{n+8} = Z+Z, & \pi_{n+9} = Z_2+Z_2, & \pi_{n+10} = Z_2, \\ \pi_{n+11} = Z_2, & \pi_{n+12} = Z+Z+Z, & \pi_{n+13} = 2\text{-group}, \\ \pi_{n+14} = 2\text{-group}. & & \end{array}$$

ii) *Let* p *be an odd prime. For* $i \leqq 2p$, $\pi_{n+i}(M(SO(n)))$ *has no* p*-primary components.*

§ 3. The groups of cobordism Ω^k

In this section we determine the groups of cobordism Ω^k for $8 \leqq k \leqq 12$ and discuss some problems related to the cobordism classes.

All manifolds considered are to be compact, orientable and differentiable, unless otherwise stated.

A. Thom algebra

Here we state briefly the definition of Thom algebra Ω and the central results of Thom [18] concerning Ω.

We define the sum $V^n + W^n$ of two disjoint oriented manifolds V^n and W^n of the same dimension as the union of V^n and W^n. The sum is oriented in natural way. For an oriented manifold V^n an oriented manifold $-V^n$ is defined as follows: $-V^n$ is identical with V^n as manifold and has the orientation opposite to that of V^n. The product $V^n \times W^m$ of two oriented manifolds V^n, W^m of any dimensions is the oriented cartesian product,

An oriented manifold V^n is *bounded*, when there exists an oriented manifold with boundary, X^{n+1}, whose oriented boundary (with the orientation and differentiable structure induced from X^{n+1}) is identical with the given oriented manifold V^n. Two oriented manifolds V^n and W^n are called "*cobordantes*" when $V^n + (-W^n)$ is bounded. This is an equivalence relation and compatible with the operation $+$, $-$ and $\times$ defined above. The equivalence classes of n-dimensional oriented manifolds form an additive group Ω^n under the operation $+$ and $-$, and its null element is the class of bounded manifolds. We call Ω^n *the group of cobordism of dimension* n. The direct sum $\Omega = \sum_{n=0}^{\infty} \Omega^n$ becomes an anticommutative graded algebra under the operation $+$, $-$ and $\times$ defined above.

The groups of cobordism Ω^k are related to the stable homotopy groups of Thom complex $M(SO(n))$ by Thom [18].

THEOREM. *The groups of cobordism Ω^k of dimension k are isomorphic to the stable homotopy groups of Thom complex $M(SO(n))$:*

$$\Omega^k = \pi_{n+k}(M(SO(n))), \quad \text{for} \quad k < n.$$

The following results of Thom [18] are founded on this theorem.

1) The groups Ω^k are finite for $k \not\equiv 0$, mod 4. The group Ω^{4i} is the direct sum of $\pi(i)$ free cyclic groups and a finite group.

2) For $k < 8$, the groups Ω^k are:

$$\Omega^0 = Z, \qquad \Omega^1 = \Omega^2 = \Omega^3 = 0,$$
$$\Omega^4 = Z, \qquad \Omega^5 = Z_2, \qquad \Omega^6 = \Omega^7 = 0.$$

B. The groups of cobordism Ω^k

Thom's theorem and Theorem 1 give

THEOREM 2. i) *For $8 \leqq k \leqq 14$, the groups Ω^k are:*

$$\Omega^8 = Z + Z, \qquad \Omega^9 = Z_2 + Z_2, \qquad \Omega^{10} = Z_2, \qquad \Omega^{11} = Z_2,$$
$$\Omega^{12} = Z + Z + Z, \quad \Omega^{13} = 2\text{-group}, \quad \Omega^{14} = 2\text{-group}.$$

ii) *Let p be an odd prime. The p-primary components of Ω^k are zero for $k \leqq 2p$.*

Generators of Ω^k.

We denote by $PC(i)$ the complex i-dimensional projective space, and by $P(m, n)$ the $(m+2n)$-manifold defined by Dold [6]. We know that $P(m, n)$ is orientable if and only if $m \not\equiv n$, mod 2 or $m = 0$ (cf. Dold [6], C). We denote by $[V^n]$ the element of Ω which contains an n-manifold V^n.

i) Generators of Ω^8 are given by $[PC(4)]$ and $[PC(2) \times PC(2)]$.

ii) Generators of Ω^9 are given by $[P(1, 4)]$ and $[P(1, 2) \times PC(2)]$, where $[P(1, 2)]$ is the generator of $\Omega^5 = \mathfrak{N}^5 = Z_2$ given by Wu [19].

iii) The generator of Ω^{10} is $[P(1, 2) \times P(1, 2)]$.

iv) The generator of Ω^{11} is $[P(3, 4)]$.

v) Generators of Ω^{12} are given by $[PC(6)]$, $[PC(4) \times PC(2)]$ and $[(PC(2))^3]$.

As is shown above, Ω^8 and Ω^{12} are free groups, therefore i) and v) are trivial (Thom [18], Chapitre IV, §8); ii)-iv) are the direct consequences of Theorem 2 and Dold [6] (Satz 3 and H).

Remark. Among the groups of cobordism Ω^k there exist no free groups but Ω^4, Ω^8 and Ω^{12}, because for $i \geqq 4$, Ω^{4i} has 2-torsion $[V^{4i}]$, where

$$V^{4i} = PC(2i-8) \times P(3, 4) \times P(1, 2).$$

Now we consider the problem of Steenrod: What algebraic conditions are necessary and sufficient for an orientable manifold V^n to be bounded? (Eilenberg [7], Problem 26). Rohlin and Thom have given partial answers for this problem ([10], [11], [18], Chapitre IV, §8);

i) V^n is always bounded for $n=1, 2, 3, 6, 7$;

ii) V^4 is bounded if and only if its index $\tau(V^4)$ is zero.

iii) V^5 is bounded if and only if its Stiefel-Whitney number W^3W^2 is zero.

We apply Theorem 2 to this problem.

THEOREM 3. i) *An 8-manifold is bounded if and only if all its Pontrjagin numbers are zero.*

ii) *A 9-manifold is bounded if and only if both its Stiefel-Whitney numbers $W^3(W^2)^3$ and W^7W^2 are zero.*

iii) *A 10-manifold is bounded if and only if its Stiefel-Whitney number W^6W^4 is zero.*

iv) *An 11-manifold is bounded if and only if its Stiefel-Whitney number $(W^3)^3W^2$ is zero.*

v) *A 12-manifold is bounded if and only if all its Pontrjagin numbers are zero.*

LEMMA 1. *The Stiefel-Whitney class*[3] *of the manifold $P(m, n)$ is given by*

$$W(m, n) = (1+c)^m(1+c+d)^{n+1}, \quad \textit{with} \quad c^{m+1}=0, \quad d^{n+1}=0,$$

where c and d are 1- and 2-dimensional cohomology classes mod 2 *respectively, and they generate $H^*(P(m, n), Z_2)$* (Dold [6], Satz 2).

LEMMA 2. *The Chern class*[3] *of the complex projective space $PC(n)$ is equal to $(1+g)^{n+1}$, where g is the generator of $H^2(PC(n), Z)$. The Pontrjagin class*[3] *of $PC(n)$ is equal to $(1+g^2)^{n+1}$* (Hirzebruch [8], Satz 4.10.2).

Proof of Theorem 3. For any 8-manifold V^8, we can describe $[V^8] = a[PC(4)]+b[PC(2)\times PC(2)]$, where a and b are certain integers. As the Pontrjagin numbers are additive, a and b are uniquely determined by Lemma 2:

$$a = \frac{1}{5}(P^4P^4-2P^8), \qquad b = \frac{1}{9}(5P^8-P^4P^4),$$

where P^4P^4 and P^8 denote the corresponding Pontrjagin numbers of V^8. Therefore i) is proved. Using Lemma 1 and Lemma 2 we can prove ii)-v) by the same method as above.

Remark. The selections of Stiefel-Whitney numbers described in ii) and iv)

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3) Precisely, we must say the Stiefel-Whitney polynomial with the variable $t=1$ (cf. Wu [21], p. 41) or total Stiefel-Whitney class.

are not unique. It is evident from the proof that in iv), for example, we can take $W^5(W^2)^3$ instead of $(W^3)^3W^2$. However, in iii) no other Stiefel-Whitney numbers than W^6W^4 give a sufficient condition, because W^6W^4 is the unique Stiefel-Whitney number that is not zero for $P(1,2)\times P(1,2)$.

Especially we observe

COROLLARY 1. *For $n\leqq 12$, an n-sphere S^n is bounded, even if it admits any differentiable structures.*[4]

Proof. For $n\neq 8, 12$, it is trivial. For $n=8, 12$, we know the following index formulas (Hirzebruch [8]):

$$45\tau = 7P^8 - P^4P^4,$$
$$945\tau = 62P^{12} - 13P^8P^4 + 2(P^4)^3,$$

where ΠP^{4i} are the corresponding Pontrjagin numbers. Thus the corollary is the direct consequence of Theorem 3.

COROLLARY 2. *For $n\leqq 11$, $n\neq 8$, the cobordism classes of n-manifolds V^n are topological invariants, i.e., they are independent of their differentiable structures.*[4]

Proof. We know that the Stiefel-Whitney classes $W^i(V^n)$ of an n-dimensional manifold are topological invariants of V^n (Thom [17]). Therefore the corollary is the immediate consequence of Theorem 3.

Remark. If the Pontrjagin classes $P^{4k}(V^n)$ of an n-dimensional manifold are topological invariants, Corollary 2 holds for $n=8$ and 12.

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