# AN INVESTIGATION ON THE LOGICAL STRUCTURE OF MATHEMATICS (IV)[0)]

## COMPENDIUM FOR DEDUCTIONS

SIGEKATU KURODA

In §1 the usage and conventions which are used in the deductions in UL are explained. In §§2-4 some metatheorems concerning the deductions are proved. Namely, in §2 the order of proof constituents in a proof is investigated; in §3 the applicability of composite proof constituents are proved on the basis of §2; in §4 the place of the "ordinarily used" principle of extensionality in a proof is specified. In §5 a sufficient condition of the mechanization of mathematics is given in such a manner that the mechanical and non-mechanical parts in solving mathematical problems are separated in accord with the usual way of thinking in mathematics.

### 1. Usage and conventions

(i) Usage of dots.

In order to show the articulation in a formula we use brackets and dots ., :, :., etc. The dots are used always in place of brackets. Whether they are used as left brackets, like (, {, [, etc. or as right brackets, like ), }, ], etc. can easily be seen by the context. For instance, the dots placed immediately after ⊃, ≡, $\forall x$ etc. are left brackets and those before ∧, ≡, etc. are right brackets.

The scope of dots used as left (right) brackets extends beyond the dots which consist of smaller number of dots and the dots consisting of the same number of dots and used as a left (right) brackets; but never extends beyond the dots which consist of greater number of dots, nor the dots consisting of the same number of dots and used as a right (left) bracket.

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0) (Added in proof) This Part (IV) is the continuation of my previous papers: Parts (I) and (II) are forthcoming in E. Artin-H. Hasse's jubilee volume of the Hamburger Abhandlungen and Part (III) is contained in this same volume. Further continuation will appear elsewhere.

When dots are used inside brackets, the scope of these dots can never extend beyond the innermost brackets between which the dots concerned are situated.

The dots used as right (left) brackets are omitted, if they should be placed at the end (beginning) of the formula or immediately before (after) a right (left) bracket.

The dots or brackets may also be omitted when the articulation can be seen by the power of connectivity of each logical symbol according to the usual convention.

Examples:

| | | |
|---|---|---|
| $A\rightarrow .\ B\rightarrow .\ C\rightarrow .\ D\rightarrow E$ | is read as | $A\rightarrow(B\rightarrow(C\rightarrow(D\rightarrow E)))$ |
| $A\rightarrow :\ B\rightarrow C.\ \rightarrow D\vee E$ | " | $A\rightarrow((B\rightarrow C)\rightarrow(D\vee E))$ |
| $\neg :\ (A\rightarrow .\ B\rightarrow C)\rightarrow .\ D\rightarrow E$ | " | $\neg((A\rightarrow(B\rightarrow C))\rightarrow(D\rightarrow E))$ |
| $\neg :\ \forall x .\ F(x)\vee\neg G(x).\ \rightarrow A$ | " | $\neg(\forall x(F(x)\vee\neg G(x))\rightarrow A)$ |

(ii) Ordinary and singular cut.

We call a cut *ordinary*, if one of the cut formulas is a fromula which has a proof; otherwise *singular*. Let the cut

(1) $$\frac{C \qquad \neg C}{}$$

be ordinary and the cut formula $C$ have a proof with $\tau$ as premises. When cut (1) is used in a proof $P$, we place the proof of the cut formula $C$ under the formula $C$. Thereby the premises of $P$ should be, if necessary, augmented so as to contain $\tau$.

The ordinary cut (1) in a proof $P$ is written as

$$\begin{array}{c} \vdots \quad \text{Cut } k \\ \hline \neg C \\ \vdots \end{array}$$

where $k$ denotes the number by which the formula $C$ is referred to: the proof of $C$ is omitted since the proof of $C$ is given previously in $k$.

(iii) Ordinary and singular use of defining formulas.

The proof constituent

[DA] $$\frac{}{m\in p \qquad F^m}$$

or

[DN] $$m\in p \qquad \neg F^m \ ,$$

in a proof $P$, which is associated to the defining formula

$$\forall u.\ u \in p \equiv F_{,}^{u}$$

is called *ordinary* or *singular*, according as the left formula $m \notin p$ or $m \in p$ of [DA] or [DN], respectively, is a bottom formula of $P$ or not.

If $P$ is a proof with the property (a)[1] that every $P$-formula is effective, then the left formula $m \notin p$ (or $m \in p$) of an ordinary $P$-constituent [DA] (or [DN]) has a cancelling partner $m \in p$ (or $m \notin p$) over the left $P$-bottom formula of [DA] (of [DN]). In this case we write the $P$-constituents [DA] and [DN] respectively as

$$\overset{(k)}{F^{m}} \qquad \text{and} \qquad \overset{(k)}{\neg F^{m}}$$

where $k$ is the number attached to the cancelling partner of the left $P$-bottom formula of [DA] and [DN], respectively.

(iv) Assignment in a proof.

Except the association of an ordinary [DA] and [DN] to a defining formula, explained in (iii), the association of a proof constituent to a formula and the cancelling pair of a proof string are shown by numbers as follows.

Namely, the number with brackets, say $(k)$, attached to the horizontal line of a proof constituent means that the constituent is associated to the formula to which the number $k$ without brackets is attached and which is over the constituent $(k)$. However, when the formula to which a constituent is associated lies closely over the constituent, the association is shown merely by the symbol "–" attached to the left of the formula.

The number with brackets, say $(k)$, which is under the bottom formula means that the bottom formula and the formula to which the number $k$ is attached are the cancelling pair of the string through the bottom formula $(k)$. If more than one number is found in brackets under a bottom formula, it means that the generalized cancelling property[2] is applied to the string through the bottom formula with respect to the formulas to which these numbers are attached. If the symbol I or = is found in brackets under a bottom formula, it

[1] See theorem 1, § 12, Part (II).

[2] See § 3.

means respectively that some proof constituents associated to the premise

(I) $$\forall xyz.\ x=y\wedge x\in z\rightarrow y\in z$$

or some ordinary cuts with the proved formulas[3] $=*3$, $=*4$, $=*5$ $*\mathrm{El}*7$, or $\mathrm{El}*9$ as cut formulas are abbreviated under the bottom formula.

(v) The eigen variables of a proof constituent are shown in the brackets attached to the horizontal line of the constituent. If the same eigen variables are found in the same proof string,[4] it means that the proof can be easily changed into a proof with the independent variable restriction by using the eigen variable, distinct each other.

(vi) When it is necessary to notice that some dependent variable, say $m$, is substituted for the bound variable $x$ of the formula of the form $\neg\forall xF^x$, it is shown by $[m]$ attached to the horizontal line of the constituent $[m]\frac{}{\neg F^m}$ associated to $\neg\forall xF^x$.

(vii) The symbol Spf attached to a proof formula means that the formula is superfluous in the proof in the sense of §20, Part (II). The symbol $*$Spf attached to a proof formula means that the formula is strongly superfluous in the proof and the proof is weakly irreducible in the sense of §20, Part (II). To the number of a formula with such a proof $*$ is attached, lik $*\wedge*1$.

The superfluousness of a proof formula is usually understood by taking the logical operators $\neg$, $\vee$, $\wedge$, $\forall$, $\exists$, $\rightarrow$, and $\equiv$ as primitive,[5] and by using, among others, characteristic properties of the ordered pair[6] as premises.[7] The characteristic properties and the specialization of definitions[6] used in a proof can be easily seen by the context of the proof given.

(viii) The premises of an assertion and of a top sequence of a proof are usually omitted, since they are clearly seen by the proof. If there is an ordinary cut in a proof, then the premises needed to prove the proved cut formula must

[3] See, Equality, Part (III).
[4] See, for instance, $\circ*6$, $\circ*9$, Part (III).
[5] See the end of the paragraph preceding the conclusion of Introduction in Part (II).
[6] See §21, Part (II).
[7] The characteriatic properties $\mathrm{El}*7$ and $\mathrm{El}*9$ of ordered pairs are deduced in the Section "Elementary Set," Part (III). In this section Spf and $*$Spf are used naturally without taking $\mathrm{El}*7$ and $\mathrm{El}*9$ as premises. The formulas $\mathrm{El}*7$ and $\mathrm{El}*9$ are used as premises in Part (III) in the deduction of the Section "Image of Operator" and the subsequent Sections.

be added to the premises of the proof considered.

**2. Order of proof constituents in a proof**

Tne procedures used in the transformations of proofs in chapter IV, Part (II), except in the proof of cut theorem, are:

(i) erasing and connecting method;

(ii) interchange of proof constituents;

(iii) homologous transformation (i.e. replacement of a formula by a formula homologous to it);

(iv) renaming of an eigen variable;

A transformation of a proof performed by a finite number of successive application of these procedures is called *elementary transformation of* a *proof.*

Let $P$ be a proof with the properties[8] (a), (b) and (d'), where (d') is a property similar to (d) but weaker than (d): namely,

(d') for any proof formula of the form $\forall x F^x$ there is in each string through $\forall x F^x$ at most one proof constituent $\overline{F^w}$ associated to $\forall x F^v$.

The lengths of all the proofs with the properties (a), (b), and (d'), to which $P$ can be transformed by an elementary transformation, has an upper bound, say $M$. We may take the number of all inhomologous $P$-formulas as $M$. The number $2^M$ is an upper bound of the numbers of strings in a proof of length $M$. Therefore, the number of such proofs are finite. Hence

THEOREM 1. *If there is a proof $P$ for an assertion $\tau \vdash H$, then we can construct a proof for $\tau \vdash H$ by arranging the $P$-constituents from above to below in any way, retaining the properties* (a), (b), *and* (d') *and not violating the proof properties.*

*Remark.* We can formulate a similar theorem with respect to a proof with the properties (a'), (b), and (d'), and with the primitive cancelling property.[9]

**3. Applicability of composite proof constituents**

Let $F$ be a formula. We put

$$F = \overline{\forall x_1 \ldots x_n}\ G \qquad (n \geqq 0)$$

[8] Property (a): Any formula under the top sequence is effective. Property (b): Any string of a proof contains no two homologous proof formulas, unless they are both the negation of some defining formulas. For the properties (a), (b), and (d), see §§ 12, 13, Part (II).

[9] See § 14, Part (II).

where $n$ is the number of all the consecutive prenex quantifiers $\neg\forall x_1\forall x_2 \ldots \forall x_n$ of $F$. If $F$ has no outermost $\neg\forall$ i.e. if $n=0$, we put $F=\neg G$. Put $G^* = G_{m_1,\ldots,m_n}^{x_1,\ldots,x_n}$ where $m_1, \ldots, m_n$ are any variables (when $n=0$ we consider $\neg G^*$ to be $F$ itself). Assume that $\neg G^*$ is imprimitive. Let $M$ be an (eventually void) species of proper subformulas[10] of $\neg G^*$ such that no two members of $M$ overlap in $\neg G^*$ and that any subformula of $\neg G^*$ of the form $\forall x K^x$ which is at a negative position in $\neg G^*$ is a subformula of one of the members of $M$.

We shall describe the way of constructing a figure of the form

(1)

from $\neg G^*$ and $M$, which shall be called a *composite proof constituent* associated to $F$.

First, we shall construct, by using $\neg G^*$ and $M$, a tree $T$ recursively as follows. Let $T_1$ be $\overline{\neg G^*}$. Assume that we have constructed the tree $T_k$ $(k \geqq 1)$. Let $B$ be a bottom formula of $T_k$ and let $A$ be a formula in the $T_k$-string through $B$ such that $A$ is neither primitive nor a bottom formula of the carrier[11] $C_{T_k}(Q, \neg G^*)$ for any member $Q$ of $M$. (The formula $\neg G^*$ of $T_1$ satisfies the above condition for $A$, in virtue of the assumption that $\neg G^*$ is imprimitive and $M$ consists of proper subformulas of $\neg G^*$.) Then $A$ is of the form $\forall x K^x$, $K \wedge L$, or $\neg \cdot K \wedge L$, since the case where $A$ is of the form $\neg\forall x K$ is excluded by the assumption on $A$ and by the definition of $M$. If the $T_k$-string through $B$ touches no proof constituent associated to $A$, then we put under $B$ a proof constituent $\overline{K \quad L}$, if $A$ is of the form $K \wedge L$; a proof constituent $\overline{K^w}$, if $A$ is of the form $\forall x K^x$, where $w$ is an independent variable, not occurring free in $T_k$ over $\overline{K^w}$; two proof constituents $\overline{\neg K}$ and $\overline{\neg L}$, if $A$ is of the form $\neg \cdot K \wedge L$. Let $T_{k+1}$ be the tree thus obtained. We perform the above procedures as far as possible. The procedures should clearly come to an end. Let $T$ be the tree thus obtained finally.

[10] By a proper subformula is meant here a subformula in the reduced sense, i.e. a subformula $B$ of $A$ is a proper subformula in the reduced sense, exactly if the reduced degree of $B$ is smaller than that of $A$. As usual, a subformula is considered in taking its position in a formula into account so that $M$ may contain identical formulas which are at different positions in $\neg G^*$.

[11] See § 19, Part (II).

Second, we shall construct from $T$ a figure $T_0$ of the form (1) as follows. Namely, from among the formulas contained in each $T$-string we collect all the formulas[12] to which no proof constituent is associated in $T$ and arrange them in a column under the horizontal line of (1). We get thus as many columns of $T_0$ as there are $T$-bottom formulas. We erase from $T_0$ all those columns which contain a cancelling pair. We erase also homologous formulas contained in the same $T_0$-column, leaving exactly one of them. Let the figure thus obtained be $E$. $E$ is determined thus by $F$ and $M$. This is a composite proof constituent to be associated to the formula $F$, unless $E$ is a prime proof constituent associated to $F$. The variables occurring free in $E$ but not free in $G^*$ are those variables which occur in $T$ as eigen variables. These variables are called the *eigen variables of* the *composite proof constituent* $E$.

Now, we shall explain why and how a composite proof constituent should be applicable in a proof.

By a *connected part in a proof* is meant a part of the proof consisting of those proof constituents any two of which can be connected by a way running successively through consecutive proof constituents belonging to the part. Any connected part of a proof has exactly one top constituent.

Let $P$ be a proof and $F$ a $P$-formula. To fix our idea, assume that $P$ has the properties (a'), (b) and (d') and the primitive cancelling property.[9] Then there is at least one derivative $\overline{\supset}G^*$ of $F$ in $P$, where $G^*$ is defined by $F$ as before with suitable $m_1, \ldots, m_n$. Assume further that $M$ is defined with respect to $F$, and $E$ with respect to $F$ and $M$, as before.

Now, by the remark in §2 we transform the part of $P$ which is under $F$ in such a form that all formulas of the carriers $C_P(Q, \overline{\supset}G^*)$ for all members $Q$ of $M$ and for all primitive subformulas of $\overline{\supset}G^*$ which occur outside any member of $M$ constitute a connected part of $P$. Let the connected part be $C$. In order to make this procedure always possible we have made in defining $M$ the assumption concerning the subformula of $\overline{\supset}G^*$ of the form $\forall x K^x$.[13] The

[12] That is, the formulas which are either primitive or a bottom formula of $C_T(Q, \overline{\supset}G^*)$ for a member $Q$ of $M$.

[13] Otherwise, it might happen that a subformula of the form $\forall x K^x$ at a negative position of $\overline{\supset}G^*$ would be contained in no member of $M$ so that the proof constituent $\overline{\overline{\supset}K^{*m}}$ associated to a bottom formula of the carrier $C_P(\forall x K^x, \overline{\supset}G^*)$ would belong to the tree $T$ constructed before. If so, $\overline{\supset}K^{*m}$ must be brought to the connected part $C$. But this might be impossible if $m$ dependeds on the eigen variable of a $P$-constituent which is under $\overline{\supset}\forall x K^x$ and over $\overline{\supset}K^{*m}$.

part $C$ is clearly a part of $T$ we have constructed before so that we can transform $P$ further in such a manner that $C$ becomes congruent in shape with $T$. To do this we may have to insert some superfluous $P$-constituents. Now, replace the connected part $T$ by $E$ and place in a natural manner under each column of $E$ the part of $P$ which is under a bottom formula of $C$. Erase the derivatives of $F$ which are between $F$ and $E$, and associate $E$ to $F$. We perform the same transformation for all derivatives $\nearrow G^*$, $\nearrow G^{**}$, . . . of $F$ in $P$.

We perform the same transformations successively for all $P$-formula $F^*$ which is congruent with $F$ up to variables occurring in them. Let $P_0$ be the figure obtained from $P$ after all these successive transformations. It is easily seen that $P_0$ has the proof properties if we adjoin $E$ as proof constituent besides the prime proof constituents.

Therefore $E$ can be used as composite proof constituent associated to $F$. The composite proof constituents defined in § 17, Part (II), are all special cases of the composite proof constituent treated generally in this § 3.

**4. Position of [I] in a proof**

We have seen that the composite proof constituent

[I] $$\frac{}{a=b \qquad a\in c \qquad a\notin c}$$

can be used, associated to the premise

(I) $$\forall xyz.\ x=y\wedge x\in z\rightarrow y\in z.$$

The proof constituent [I] in a proof, for which there is no proof constituent associated to $a=b$ of [I], is called *ordinary*, otherwise *singular*. By theorem 1, § 2, any ordinary proof constituent [I] in a proof can be brought at the bottom of the proof. Therefore

THEOREM 2. *If no singular* [I] *occurs in a proof, then the proof can be admissibly transformed to a proof in which the proof constituents* [I] *associated to the premise* (I) *are all used at the bottom of the proof strings.* Naturally, more than one [I] may be superposed at the bottom of a string.

In a proof it is useful to use the following generalized cancelling property. Namely, let $F^{x_1,\cdots,x_n}$ and $G^{y_1,\cdots,y_n}$ be formulas which depend on $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$, respectively. Let $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ be dependent or independent variables which can be substituted for $x_1,\ldots,x_n$ in $F$ and for

$y_1, \ldots, y_n$ in $G$, respectively.[14] Now, we formulate:

(Generalized cancelling property): *Any string of a proof contains either a pair of formulas, one of which is the negation of the other, or a pair of formulas* $F^{a_1,\cdots,a_n}$, $G^{b_1,\cdots,b_n}$ *and n inequalities* $a_1 \neq b_1, \ldots, a_n \neq b_n$ *such that* $F^{a_1,\cdots,a_n}$ *is the negation of* $G^{a_1,\cdots,a_n}$, $a_i$ *and* $b_i$ *being any variables.*

THEOREM 3. *The cancelling property can be replaced by the generalized cancelling property in the definition of proof.*

*Proof.* We can prove theorem 3 by the double induction on the degree of a formula and the order of a dependent variable. Namely, first, the generalized cancelling property can be reduced to the case in which $F$, and accordingly $G$, are primitive. Second, by using the formula (I) as premise, the latter case can be reduced to the case where $F$ is of the form $p = q$, and accordingly further to the case where $F$ is of the form $w \in p^{a_1,\cdots,a_n}$. By using the defining formula of $p^{a_1,\cdots,a_n}$ we can reduce this case to the case of variables of smaller order.

It is to be noticed that, when we transform a given proof with the generalized cancelling property by the method stated above, the dependent variable restriction is preserved. In fact, the dependent variables needed in the reduction, stated above, are the closure of the dependent variables occurring in the formulas $F^{a_1,\cdots,a_n}$ and $G^{b_1,\cdots,b_n}$, so that these variables are defined in the premises of the original proof.

From theorem 2 and 3 follows

THEOREM 4. *As far as we observe the proofs, in which no singular* [I] *is used, we can replace the cancelling property in the definition of proof by the generalized cencelling property and omit the formula* (I) *from the premises of proof.*

**5. Mechanization of mathematics**

Mechanization of mathematics means to give an *algorithm* to construct a proof, if any, for a given assertion in UL. Church (J. Symb. Logic 1, 1936) proved that there is no such algorithm in the quantification theory, and more-

[14] In order that $m$ can be substituted for $x$ in a formula $F^x$, it is necessary and sufficient that $m$ depends on no variable, say $y$, such that there is an $x$ in $F^x$ which is under the operator range of $\forall y$. See the condition of substitution of $m$ for $x$ in $U$ in § 8, Part (I) and foot note 8) there. The variables $x_1, \cdots, x_n$ and $y_1, \cdots, y_n$ are not necessarily the complete systems of variables of $F$ and $G$, respectively.

over, Church and Quine proved (J. Symb. Logic, 17, 1952) the same even in the quantification theory with a binary relation. Here we observe a sufficient condition for the mechanization of mathematics.

Before doing this, we observe free independent variables occurring in a proof. We exclude the free independent variables in the premises of the proof from the observation. Then, among other free independent variables in a proof, there are those occurring in the conclusion of the proof and those which are introduced to the proof as eigen variables. Such free independent variables in a proof are called *essential* for the proof. Any free independent variable in a proof other than mentioned above is called *trivial* for the proof. A trivial independent variable, say $l$, may eventually introduced into a proof when we substitute for a bound variable $x$ of a proof formula of the form $\neg\forall x F^x$ the variable $l$ or a variable $p$ depending on $l$, where $l$ is not essential for the proof. If there is more than one trivial variable in a proof, then we replace every trivial variable by the same trivial variable $l$. The resulting figure is a correct proof. For a moment, we reserve the letter $l$ for the unique trivial free variable, eventually occurring in a proof in this way, so that any free variable in a proof, different from those occurring in defining formulas, is essential or $l$.

Now, let $\tau \vdash H$ be an assertion and $L$ a natural number. In order to construct a proof without cut for $\tau \vdash H$ from above to below, the proof constituent to be placed in the next step under the bottom of a string of an incompleted proof must be associated to a formula $G$ among a finite number of formulas of the string. Thereby, the proof constituent is uniquely determined[15] if $G$ is not of the form $\neg\forall x F^x$. If $G$ is of the form $\neg\forall x F^x$, the variable $m$ of the proof constituent $\dfrac{\neg F^m}{}$ associated to $\neg\forall x F^x$ must be selected from among a finite number of variables, namely the essential independent variables occurring in the string, the trivial variable $l$, and the dependent variables which depend on these independent variables and are defined in $\sigma$. In this way, there are only a finite number of possibilities in any step of constructing a proof for $\sigma \vdash H$. In case of there being a proof without cut for $\sigma \vdash H$ of length not exceeding the given number $L$, we obtain a proof for $\sigma \vdash H$ after a finite number of trials, by the length of a proof being meant that as tree (see § 6, Part (I)). Thus

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[15] We associate the composite proof constituent $\dfrac{\neg A}{\neg B}$ to $G$, if $G$ is of the form $\neg . A \wedge B$.

THEOREM 5. *There is an algorithm to decide whether or not a proof without cut for an assertion* $\sigma \vdash H$ *of length less than a given number exists, and to find such proof, if it exists.*

Let $H$ be a conclusion and $\sigma$ the sequence of the defining formulas of the closure[16] of the variables occurring in $H$. If $H$ is proved from the premises $\sigma$, (I), the assertion $\sigma \vdash H$ may well be called *analytic*. Generally, some other sequence $\tau$ of defining formulas must be added as premises to prove $H$. The mathematician's idea is needed for collecting such premises sufficient to solve the problem $H$ and for predicting the length of the proof. When these two factors are determined the problem $H$ becomes a *mathematical conjecture*. Then, the proof can be performed mechanically, provided that the conjecture is true. It is practically convenient to collect the cuts, ordinary and singular, to be used in the proof. Thus, *the essential part in solving mathematical problems consists in the dependent variables (sets and concepts) and the cuts (including the theorems) to be used in the proof, and the length of the proof.*

*Mathematical Institute*
*Nagoya University*

16) See § 11, Part (I).

17) Although $Q$ is not a proof, it is clear what the elementary transformation of $Q$ means.