NOTE ON COMPLETE COHOMOLOGY OF A QUASI-FROBENIUS ALGEBRA

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Let A be a quasi-Frobenius algebra over a field K. A has a complete (co)homology theory which may be established upon an augmented acyclic projective complex, i.e. a commutative diagram

$$(1) \qquad \cdots \to X_1 \to X_0 \to X_{-1} \to X_{-2} \to \cdots, \quad X_0 \xrightarrow{\varepsilon} A \xrightarrow{\iota} X_{-1}$$

of A-double-modules with exact horizontal row, projective X_p, and with epimorphic resp. monomorphic ε and ι. Negative-dimensional cohomology groups, over an A-double-module, are expected to be in close relationship with (ordinary positive-dimensional) homology groups. Indeed, in case A is a Frobenius algebra the cohomology groups $H^{-n}(A, M)$, $-n < -1$, over an A-double-module M may be identified, connecting homomorphisms taken into account, with the homology groups $H_{n-1}(A, M^*)$ over an A-double-module $M^* = (M, *)$ obtained from M by modifying its A-right-module structure with an automorphism $*$ of A belonging to the Frobenius algebra structure of A, and, moreover, the cohomology groups $H^0(A, M)$, $H^{-1}(A, M)$ are described explicitly in terms of commutation and norm-map, so to speak, defined by a certain pair of dual bases of A. In the present note we want to give the corresponding description of the 0- and negative-dimensional cohomology groups of a quasi-Frobenius algebra A. In doing so, we shall deal with a certain A-double-module $M^\S$ which is obtained from M by a certain construction but which is in general not A-left-isomorphic to M contrary to that M^* in case of a Frobenius algebra is A-left-isomorphic to M. Further, our construction will strongly rely upon the relationship of A with its core algebra A_0 which is a Frobenius algebra. In fact, the (co)homology theory of an algebra can, generally, be reduced to that of its core algebra, and this principle applies also to the complete (co)homology of a quasi-Frobenius algebra. However, description and construction in terms

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of a given quasi-Frobenius algebra itself, rather than of its Frobenius core, as those we shall obtain in the followings, are perhaps of some interest and use too.

1. A-double-module $M^{\S}$. Let A be a quasi-Frobenius algebra over a field K, and

(2) $$1=\sum_{\rho=1}^{r}\sum_{i=1}^{f(\rho)}e_i^{(\rho)}.$$

be a decomposition of its unit element 1 into mutually orthogonal primitive idempotents, where $e_i^{(\rho)}\approx e_j^{(\sigma)}$ if and only if $\rho=\sigma$. For each $\rho=1,\ldots,r$ there is in A a system of matrix units $c_{ij}^{(\rho)}$ $(c_{ij}^{(\rho)}c_{i'j'}^{(\rho)}=\delta_{ji'}c_{ij'}^{(\rho)})$ with $c_{ii}^{(\rho)}=e_i^{(\rho)}$. Put

(3) $$1_0=\sum_{\rho=1}^{r}e_1^{(\rho)},\qquad A_0=1_0A1_0.$$

A_0. the so-called core algebra (or basic algebra) of A, has 1_0 as its unit element, and is a Frobenius algebra. Let $*:x\to x^*$ $(x\in A_0)$ be an automorphism of A_0 belonging to its Frobenius algebra structure. Thus, if $(a_1,\ldots,a_k)$ is a K-basis of A_0, there is a non-singular parastrophic matrix $P=(\mu(a_\kappa a_\lambda))$ belonging to the basis (a_κ) such that for $x=\sum_\kappa a_\kappa\xi_\kappa$ $(\xi_\kappa\in K)$

(4) $$x^*=\sum_\kappa a_\kappa\xi_\kappa^*,\qquad(\xi_1^*,\ldots,\xi_k^*)=(\xi_1,\ldots,\xi_k)P'P^{-1};$$

there is a permutation π of $(1,\ldots,r)$ such that $e_1^{(\rho)*}\equiv e_1^{(\pi(\rho))}$ modulo the radical of A_0 for every $1,\ldots,r$, and by a suitable choice of $*$ (or of the decomposition (2) and matrix units $c_{ij}^{(\rho)}$ if we fix $*$) we may, and shall, assume

(5) $$e_1^{(\rho)*}=e_1^{(\pi(\rho))}\qquad\text{for every}\quad\rho=1,\ldots,r.$$

The basis $(b_1,\ldots,b_k)=(a_1,\ldots,a_k)(P')^{-1}$ is said to be dual to (a_κ) and has the property that the left regular representation of A_0 defined by (a_κ) coincides with the right regular representation defined by (b_κ) and, moreover, the product of $*$ with the left regular representation defined by (b_κ) coincides with the right regular representation defined by (a_κ).

Let M be a unitary A-double-module. Then $M_0=1_0M1_0$ is a unitary A_0-double-module. The map

(6) $$\nu_0:u\to\sum_\kappa a_\kappa ub_\kappa\qquad(u\in M_0)$$

is a K-endomorphism of M_0 and we have

(7) $$\nu_0(M_0) \subset M_0^{A_0} = \{u \in M_0 \mid xu = ux \quad \text{for all} \quad x \in A_0\},$$

(8) $$ux^* - xu \in \operatorname{Ker} \nu_0 \quad \text{for all} \quad u \in M_0,\ x \in A.$$

On denoting by $M_0^* = (M_0, *)$ the A_0-double-module which coincides with M_0 as A_0-left-module and whose A_0-right-module structure is defined by that ux $(u \in M_0^*,\ x \in A_0)$, under the structure of M_0^*, is ux^* under the old structure of M_0, we construct a new A-double-module

(9) $$M^\S = \sum_{\rho,\sigma=1}^{r} \sum_{i=1}^{f(\rho)} \sum_{j=1}^{f(\sigma)} c_{i1}^{(\rho)} e_1^{(\rho)} M_0^* e_1^{(\sigma)} c_{1j}^{(\sigma)},$$

where $c_{i1}^{(\rho)} e_1^{(\rho)} M_0^* e_1^{(\sigma)} c_{1j}^{(\sigma)}$ is the K-module consisting of all expressions $c_{i1}^{(\rho)} v c_{1j}^{(\sigma)}$ with $v \in e_1^{(\rho)} M_0^* e_1^{(\sigma)}$ (K-module structure inheriting that of $e_1^{(\rho)} M_0^* e_1^{(\sigma)}$), where the summations are formal direct ones, and where the A-double-module structure of $M^\S$ is defined by (the distributivity and) the relations: if $v \in e_1^{(\rho)} M_0^* e_1^{(\sigma)}$, $x \in e_{i'}^{(\rho')} A e_{j'}^{(\sigma')}$, then

(10) $$\begin{aligned} x c_{i1}^{(\rho)} v c_{1j}^{(\sigma)} &= \delta_{\sigma'\rho} \delta_{j'i} c_{i'1}^{(\rho')} ((c_{1i'}^{(\rho')} x c_{i1}^{(\rho)}) v) c_{1j}^{(\sigma)}, \\ c_{i1}^{(\rho)} v c_{1j}^{(\sigma)} x &= \delta_{\rho'\sigma} \delta_{i'j} c_{i1}^{(\rho)} (v (c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')})) c_{1j'}^{(\sigma')}. \end{aligned}$$

If in particular A is a Frobenius algebra, then (and only then) $f(\pi(\rho)) = f(\rho)$ for $\rho = 1, \ldots, r$. In this case the K-linear map:

(11) $$x (\in e_i^{(\rho)} A e_j^{(\sigma)}) \to c_{i1}^{(\pi(\rho))} (c_{1i}^{(\rho)} x c_{j1}^{(\sigma)})^* c_{1j}^{(\pi(\sigma))}$$

gives an automorphism of A and is readily seen to be a such belonging to the Frobenius algebra structure of A. Our module $M^\S$ is, in this case, obtained from M by retaining its A-left-module structure but modifying its A-right-module structure with this automorphism of A, and thus coincides with the module considered in [3] (with this choice of automorphism of A belonging to its Frobenius algebra structure).

Contrary to this Frobenius algebra case and contrary to that in particular M_0^* is A_0-left-isomorphic to M_0, our module $M^\S$ in general case is not, in general, A-left-isomorphic to M, as we wish to remark.

2. Map $\bar{\nu} : M^\S \to M$. For $u = c_{i1}^{(\rho)} v c_{1j}^{(\sigma)} \in e_i^{(\rho)} M^\S e_j^{(\sigma)}$ with $v \in e_1^{(\rho)} M_0^* e_1^{(\sigma)}$, consider v as the corresponding element of M_0, indeed of $e_1^{(\rho)} M_0 e^{(\pi(\sigma))}$, and construct $\nu_0(v)$ in M_0, with ν_0 given in (6). Put

(12) $$\bar{\nu}(u) = \delta_{ij} \sum_{\tau=1}^{r} \sum_{q=1}^{f(\tau)} c_{q1}^{(\tau)} \nu_0(v) c_{1q}^{(\tau)} \in M.$$

This defines a K-linear map $\bar{\nu}$ of $M^{\S}$ into M. We assert

(13) $$\bar{\nu}(M^{\S}) \subset M^{A} = \{u \in M \mid ux = xu \quad \text{for all} \quad x \in A\}.$$

Indeed, let $x \in e_{i'}^{(\rho')} A e_{j'}^{(\sigma')}$. Then, with u as above and with $i = j$, we have, on observing (7)

$$\bar{\nu}(u)x = c_{i'1}^{(\rho')} \nu_0(v) c_{1i'}^{(\rho')} x = c_{i'1}^{(\rho)} \nu_0(v) c_{1i'}^{(\rho')} x c_{j'1}^{(\rho')} c_{1j'}^{(\sigma')}$$
$$= c_{i'1}^{(\rho')} c_{1i'}^{(\rho')} x c_{j'1}^{(\sigma')} \nu_0(v) c_{1j'}^{(\sigma')} = x\bar{\nu}(u).$$

We have also

(14) $$ux - xu \in \operatorname{Ker} \bar{\nu} \quad \text{for all} \quad u \in M^{\S},\ x \in A.$$

To see this, let, again, $u = c_{i1}^{(\rho)} v c_{1j}^{(\sigma)} \in e_i^{(\rho)} M^{\S} e_j^{(\sigma)}$ with $v \in e_1^{(\rho)} M_0^* e_1^{(\sigma)}$ and $x \in e_{i'}^{(\rho')} A e_{j'}^{(\sigma')}$. Then we compute readily

$$\bar{\nu}(ux) = \delta_{\sigma\rho'} \delta_{ji'} \delta_{ij'} \sum_{\tau=1}^{r} \sum_{q=1}^{f(\tau)} c_{q1}^{(\tau)} \nu_0(v(c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')})) c_{1q}^{(\tau)},$$
$$\bar{\nu}(xu) = \delta_{\sigma'\rho} \delta_{j'i} \delta_{i'j} \sum_{\tau=1}^{r} \sum_{q=1}^{f(\tau)} c_{q1}^{(\tau)} \nu_0(c_{1i'}^{(\rho')} x c_{j'1}^{(\sigma')}) v) c_{1q}^{(\tau)}$$

(where in the first equality we operate $c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')} \in A_0$ on v as an element of M_0^* (and not as such of M_0) and then consider the result as an element of M_0 to form its image by ν_0). So $\bar{\nu}(ux - xu) = 0$ if $j \neq i'$ or $i \neq j'$. Let $j = i$ and $i = j'$. If $\sigma \neq \rho'$ and $\rho \neq \sigma'$, then $\bar{\nu}(ux - ux) = 0$ too. So, suppose $\sigma = \rho'$ but $\rho \neq \sigma'$ firstly. Then $\bar{\nu}(ux - ux) = \bar{\nu}(ux) = \sum_{\tau=1}^{r} \sum_{q=1}^{f(\tau)} c_{q1}^{(\tau)} \nu_0(v(c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')})) c_{1q}^{(\tau)}$ and here the argument of ν_0 is equal to $v(c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')}) - (c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')}) v$ since $\rho \neq \sigma'$. But $\nu_0(vy - yv) = 0$ for all $v \in M_0^*$, $y \in A$ ((8)). Thus $\bar{\nu}(ux - ux) = 0$. The same holds similarly in case $\sigma \neq \rho'$, $\rho = \sigma'$. Suppose finally $\sigma = \rho'$, $\rho = \sigma'$. Then $\bar{\nu}(ux - xu) = \sum_{\tau=1}^{r} \sum_{q=1}^{f(\tau)} c_{q1}^{(\tau)} \nu_0(v(c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')}) - (c_{1j}^{(\sigma)} x c_{j'1}^{(\sigma')}) v)) c_{1q}^{(\tau)}$ and this vanishes again by (8). This proves (14).

3. Cohomology groups. Having proved (13) and (14) we set

(15) $$H^0(M) = M^A / \bar{\nu}(M^{\S}),$$

(16) $$H^{-1}(M) = (\operatorname{Ker} \bar{\nu}) / (K\text{-submodule of } M^{\S} \text{ generated by the elements of form } ux - xu \text{ with } u \in M^{\S},\ x \in A).$$

Set further $H^{-n}(M) = H_{n-1}(A, M^{\S})$ for $-n \leqq -2$ and $H^n(M) = H^n(A, M)$ for $n \geqq 1$. With an exact sequence $0 \to P \to M \xrightarrow{\iota} Q \to 0$ of A-double-modules, we define $\tilde{\delta}_0, \tilde{\delta}_{-1}, \tilde{\delta}_{-2}$ to be the maps: $H^0(Q) \to H^1(P)$, $H^{-1}(Q) \to H^0(P)$, $H^{-2}(Q)$

$\to H^{-1}(P)$ induced by the maps: $u(\in M) \to$ (standard 1-cochain $x(\in A) \to ux - xu$), $\bar{\nu}$: $u(\in M^{\S}) \to \bar{\nu}(u)$, (standard 1-chain $x \otimes u(\in A \otimes M^{\S})) \to ux - xu$ $(\in M^{\S})$. The maps $\tilde{\delta}_p$ with $p > 0$ or < -2 are defined as the usual connecting homomorphisms of cohomology or homology groups. Then the groups $H^p(M)$ and the maps $\tilde{\delta}_p$, with varying M and exact sequence, (or, more precisely, covariant functors H^p and connecting homomorphisms $\tilde{\delta}_p$ ([2])) are easily seen to satisfy the axioms (I)-(IV) of cohomology groups (given in [1] for ordinary case and in [3], §6 for complete case) with a normalization axiom (V′) $H^1(M) = H^1(A, M)$, for example (cf. [3], §6). So we have: *the groups* $H^n(A, M)$ $(n \geqq 1)$, $H^{-n}(A, M) = H_{n-1}(A, M^{\S})$ $(-n \leqq -2)$ *and the groups* $H^0(A, M) = H^0(M)$, $H^{-1}(A, M) = H^{-1}(M)$ *in* (15), (16) *form, with connecting homomorphisms defined as above, the complete system of cohomology groups on the quasi-Frobenius algebra A in M.*

4. Case of a Frobenius algebra. Suppose that our quasi-Frobenius algebra A is in particular a Frobenius algebra, i.e. $f(\pi(\rho)) = f(\rho)$ for all $\rho = 1, \ldots, r$. We first consider a K-basis (a_κ) of the core $A_0 = 1_0 A 1_0$ such that each a_κ lies in some of the modules $e_1^{(\rho)} A_0 e_1^{(\sigma)}$. Let (b_κ) be a basis dual to (a_κ). We see readily, by the cited property of dual bases with respect to regular representations, that if $a_\kappa \in e_1^{(\rho)} A_0 e_1^{(\sigma)}$ then $b_\kappa \in e_1^{(\sigma)*} A_0 e_1^{(\rho)} = e_1^{(\pi(\sigma))} A_0 e_1^{(\rho)}$. So, we then construct the products $c_{i1}^{(\rho)} a_\kappa c_{1j}^{(\sigma)}$, $c_{j1}^{(\pi(\sigma))} b_\kappa c_{1i}^{(\rho)}$ $(i = 1, \ldots, f(\rho)$; $j = 1, \ldots, f(\sigma)$ $(= f(\pi(\sigma))))$. With $\kappa = 1, \ldots, k$, we order these two families of elements by the lexicographic order of (κ, i, j), for example, to obtain a pair of dual bases $(c_{i1}^{(\rho)} a_\kappa c_{1j}^{(\sigma)})$, $(c_{j1}^{(\pi(\sigma))} b_\kappa c_{1i}^{(\rho)})$ of A belonging to the Frobenius algebra automorphism defined by (11). With this last choice of automorphism our module $M^{\S}$ *is obtained directly from M by modifying its right-module structure with this automorphism* (*but retaining its left-module structure*), and with this choice of dual bases our map $\bar{\nu}$: $M^{\S} \to M$ *is readily seen to be the product of the thus existing trivial A-left-isomorphism* $M^{\S} \to M$ *and the K-endomorphism of M denoted by* σ *in* [3], §2.

After this observation with respect to the above specific dual bases of A, we consider the general case of an arbitrary pair of dual bases (a_κ), (b_κ) of the core A_0. By a K-linear transformation we can come to a basis with the above specific property that each member belongs to some $e_1^{(\rho)} A_0 e_1^{(\sigma)}$. The contragredient transformation turns (b_κ) to a dual to this basis (with respect to the same $*$). By the transition to this pair of dual bases the map ν_0 is left unchanged, and so is the expression in the right-hand side of (12). Applying then the above consideration to the newly constructed bases, we obtain that (in case A is a Frobenius algebra) *the above statements in italics concerning* $M^{\S}$ *and* $\bar{\nu}$ *are valid also with a given arbitrary pair of dual bases of the core* A_0 *and with a suitable dual bases of* A (*with respect to the automorphism of* A *given by* (11)) (Thus $M^{\S}$ coincides with M^* in [3] when our automorphism of A is denoted also by $*$, and $\bar{\nu}$ coincides with σ, in [3], up to a trivial transformation).

We repeat, however, that in case of a general quasi-Frobenius algebra the module $M^{\S}$ is not, in general, A, left-isomorphic to M and our rather complicated construction of $M^{\S}$ and $\bar{\nu}$ is rather inevitable.

5. Remarks. With a quasi-Frobenius algebra A we retain our notations as $e_i^{(\rho)}$, $c_{ij}^{(\rho)}$, A_0, (a_κ), (b_κ) and $*$. The A_0-double-module $A_0^\circ = \mathrm{Hom}_K(A_0, K)$ has a K-basis (β_κ) with $\beta_\kappa(b_\lambda) = \delta_{\kappa\lambda}$. By $a_\kappa \to \beta_\kappa$ we obtain an A_0-A_0-isomorphism of A_0 and $A_0^{\circ *} = (A_0^\circ, *)$ (cf. [3], §§2, 3). This is extended to an A-A-isomorphism of the modules $\sum c_{i1}^{(\rho)} e_1^{(\rho)} A_0 e_1^{(\sigma)} c_{1j}^{(\sigma)}$ and $\sum c_{i1}^{(\rho)} e_1^{(\rho)} A_0^{\circ *} e_1^{(\sigma)} c_{1j}^{(\sigma)}$ of the similar construction as of (9). Here the former module is nothing but A while the latter is (A-A-)isomorphic to $A^{\circ \S} = (\mathrm{Hom}_K(A, K))^{\S}$ as we readily see from the A_0-A_0-isomorphism $\sum e_1^{(\rho)} A^\circ e_1^{(\sigma)} \approx A_0^\circ$.

Now, let $0 \leftarrow A \leftarrow X_0 \leftarrow X_1 \leftarrow \dots$ be the standard (say) complex of A. From this we obtain an augmented acyclic projective (in fact free) complex $0 \leftarrow A_0 \leftarrow (X_0)_0 \leftarrow (X_1)_0 \leftarrow \dots$ $((X_n)_0 = 1_0 X_n 1_0 = \sum e_1^{(\rho)} X_n e_1^{(\sigma)})$ of A_0, which is, however, not the standard one. We obtain then, by dualization and (, $*$), an injective resolution $0 \to A_0^{\circ *} \to (X_0)_0^{\circ *} \to (X_1)_0^{\circ *} \to \dots$ of the A_0-double-module $A_0^{\circ *}$; the modules $(X_n)_0^{\circ *}$ are (A_0-A_0-)projective too. On observing $\sum c_{i1}^{(\rho)} e_1^{(\rho)} (X_n)_0^{\circ *} e_1^{(\sigma)} c_{1j}^{(\sigma)} \approx X_n^{\circ \S}$ we obtain further the exact sequence $0 \to A^{\circ \S} \to X_0^{\circ \S} \to X_1^{\circ \S} \to \dots$. Combining this with the standard complex of A, which we have started with, through the A-A-isomorphism of A and $A^{\circ \S}$ constructed above, we obtain an augmented acyclic projective complete complex (1) with $X_{-n-1} = X_n^{\circ \S}$, where, thus, ε is the original augmentation in the standard complex and ι is the product of our (A-A-)isomorphism $A \to A^{\circ \S}$ with the monomorphism $A^{\circ \S} \to X_0^{\circ \S}$. We see readily that this construction corresponds to our description of cohomology groups in **3**, and indeed gives a second derivation of the result there.

We remark here that we do not need to start with the standard complex of A; any projective resolution of the A-double-module A will do, except that the choice of X_0 in the standard complex makes the description of the 0- and -1-cohomology groups easy. (For instance we may use the resolution such that $0 \leftarrow A_0 \leftarrow (X_0)_0 \leftarrow (X_1)_0 \leftarrow \dots$ is the standard complex of A_0.) We note also that we then need not use the negative-dimensional part derived, by the above construction, from the positive-dimensional part, but may combine a given positive-dimensional part with a negative-dimensional part derived, by our construction, from another positive-dimensional part. Important is, however, that they are combined through our A-A-isomorphism $A \to A^{\circ\S}$.

A further remark is that another description of the cohomology groups $H(A, M)$, which is more economical than ours, is the one as those of the Frobenius core A_0 in M_0 i.e. $H^p(A, M) \approx H^p(A_0, M_0)$ (where the right-hand side is known in [3]). (This is verified either in axiomatic way or by complexes.)—It is indeed a general useful principle that the (co)homology theory of an algebra may be reduced to that of its core algebra.—But our description refers directly to A and A-double-modules, which is perhaps of use and interest too.

References

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